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EXTREMAL PROPERTIES OF THE SUCCESSIVE DERIVATIVES OF POLYNOMIALS AND RATIONAL FUNCTIONS

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If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $|p(x)| \leqq 1$ in the unit interval $[-1,1]$, then by a theorem of S. N. Bernstein [1]

$$(1) \qquad |p'(x)| \leqq n(1-x^2)^{-\frac{1}{2}}$$

for $-1<x<1$. For the n-th Tchebycheff polynomial this becomes an equality at certain points of the interval. However, this formula is not useful if $1-x^2$ is small. The following result due to A. Markoff [5] is complementary to the above estimate.

MARKOFF'S THEOREM. *If $p(z)$ is a rational polynomial of degree n such that $|p(x)| \leqq 1$ in the unit interval $[-1,1]$, then in the same interval*

$$(2) \qquad |p'(x)| \leqq n^2 .$$

Since the Tchebycheff polynomial is extremal in both of the above cases we cannot hope to get sharper estimates by restricting ourselves to real valued polynomials. But if we assume in addition that $p(z)$ does not vanish in the unit circle $|z|<1$, then we do get a refinement of (1). Thus Erdös [3, Theorem 2] proved the following

THEOREM A. *Let $p(z)$ be a real valued rational polynomial of degree n having no root in the interior of the unit circle. If $|p(x)| \leqq 1$ in the unit interval $-1 \leqq x \leqq 1$ and $0<c<1$ then for $-1+c<x<1-c$,*

$$(3) \qquad |p'(x)| < (4/c^2)n^{\frac{1}{2}}$$

for $n>n_0$.

This estimate is very much better than (1) and is best possible in the sense that $n^{\frac{1}{2}}$ cannot be replaced by any function tending to infinity more slowly.

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It is natural to ask for the result corresponding to (2) when $p(z)$ satisfies the hypotheses of Theorem A. Erdös [3, Theorem 1] has given an answer to this question in the special case when the zeros of the polynomial are all real. He proves:

THEOREM B. *Let $p(x)$ be a polynomial of degree n satisfying the inequality $|p(x)| \leqq 1$ for $-1 \leqq x \leqq 1$. If $p(x)$ has only real roots none of which lies in the interval $-1, +1$, then for $-1 \leqq x \leqq 1$,*

$$|p'(x)| < \tfrac{1}{2}en . \tag{4}$$

This is the best possible result.

We shall suppose $p(z)$ to satisfy the hypotheses of Theorem A, and determine a bound for $|p'(x)|$ over the closed interval $-1 \leqq x \leqq 1$. Since $p'(z)$ may have zeros inside the unit circle it is of interest to determine a bound for $|p''(x)|$ and in general for $|p^{(k)}(x)|$ where $p^{(k)}(x)$ is the k-th derivative of $p(x)$ with respect to x.

THEOREM 1. *Let $p(z)$ be a real valued rational polynomial of degree n having no root in the circular region $|z| < (1-1/n^2)^{\frac{1}{2}}$. If $|p(x)| \leqq 1$ in the unit interval $-1 \leqq x \leqq 1$, then for $n > 1$ and $-1 \leqq x \leqq 1$ we have*

$$|p^{(k)}(x)| < A(n,k)\, k!\, n^k , \tag{5}$$

where $A(n,k)$ is a constant which depends only on n,k, and is less than 1.7 for small k and sufficiently large n.

The polynomial

$$p(x) = \frac{n}{2^n(1-1/n)^{n-1}}(x-1)(x+1)^{n-1}$$

satisfies all the hypotheses of the theorem and

$$|p'(1)| = \frac{n}{2(1-1/n)^{n-1}} \approx \tfrac{1}{2}en ,$$

$$|p''(1)| = \frac{n(n-1)}{2(1-1/n)^{n-1}} \approx \tfrac{1}{2}en^2 ,$$

$$|p'''(1)| = \frac{3n(n-1)(n-2)}{8(1-1/n)^{n-1}} \approx \tfrac{3}{8}en^3 ,$$

etc., where the sign $\approx$, meaning asymptotically equal to, refers to $n \to \infty$; k being fixed, in (5), n^k cannot be replaced by any function tending to infinity more slowly.

Our next result is an analogue of Theorem 1 for the L^p norm.

In an attempt to generalize Markoff's theorem (loc. cit.) Potapov [6] has proved the following

THEOREM C. *If $p(z)$ is a rational polynomial of degree n, then there exists a constant B such that*

$$(6)\qquad \left(\int_{-1}^{1} |p'(x)|^{\delta}\,dx\right)^{1/\delta} \leqq Bn^2\left(\int_{-1}^{1} |p(x)|^{\delta}\,dx\right)^{1/\delta},$$

for every $\delta \geqq 1$.

We prove

THEOREM 2. *If $p(z)$ is a real valued rational polynomial of degree n having no root in the interior of the unit circle, then for every $\delta \geqq 1$,*

$$(7)\qquad \left(\int_{-1}^{1} |p^{(k)}(x)|^{\delta}\,dx\right)^{1/\delta} \leqq C(k)\,k!\,n^k\left(\int_{-1}^{1} |p(x)|^{\delta}\,dx\right)^{1/\delta},$$

with $C(k)$ depending only on k.

Finally, we prove a result which is a refinement as well as an extension of Theorem A. Malik [4] has proved the following

THEOREM D. *Let $f(z)$ be a rational function which is the quotient of two real valued polynomials $p(z)$ and $q(z)$ of degrees m and n respectively. If $f(z)$ has neither zeros nor poles inside the unit circle and $|f(x)| \leqq 1$ for $-1 < x < 1$, then for $-1+c < x < 1-c$,*

$$(8)\qquad |f'(x)| < c^{-1}\{2(m+n)\}^{1/2}$$

$$(9)\qquad |f''(x)| < c^{-2}\{2(m+n)\}^{5/4}$$

for $m > m_0$ and $n > n_0$.

We note that the bound for $|f''(x)|$ can be considerably improved. In fact, we prove the following

THEOREM 3. *Let $\lambda = \max(m,n) > 1$. Then under the conditions of Theorem D,*

$$(10)\qquad |f^{(k)}(x)| \leqq D(\lambda)\,k!\,\lambda^{k/2}\,c^{-k}$$

where $D(\lambda) < 2$ for $\lambda \geqq 6$. If $\lambda = 1$, then

$$|f^{(k)}(x)| < k!\,(c^{-k} + \tfrac{1}{2}c^{-k+1}).$$

(10) gives a better estimate for $|f''(x)|$ than (9).

Now consider

$$f(x) = e^{-\frac{1}{2}}(x^2-1)^n(1+x)^{[n^{\frac{1}{2}}]}$$

where $[n^{\frac{1}{2}}]$ denotes the greatest integer not exceeding $n^{\frac{1}{2}}$. This is a polynomial of degree $m=2n+[n^{\frac{1}{2}}]$. Writing $x=\alpha n^{-\frac{1}{2}}$ we have

$$|f(x)| = e^{-\frac{1}{2}}(1-\alpha^2/n)^n(1+\alpha/n^{\frac{1}{2}})^{[n^{\frac{1}{2}}]} < e^{-\frac{1}{2}}e^{-\alpha^2+\alpha} < 1 .$$

Also

$$|f''(0)| > e^{-\frac{1}{2}}(n+[n^{\frac{1}{2}}]) ,$$
$$|f'''(0)| > e^{-\frac{1}{2}}[n^{\frac{1}{2}}](5n+3[n^{\frac{1}{2}}]-2) ,$$

etc. It follows that for small k the bound given by (10) is correct (except for a constant multiplier) for $m \to \infty$.

We start with the proof of Theorem 3.

PROOF OF THEOREM 3. It is enough to establish the conclusion for $0 \leqq x < 1$. To get it for $-1 < x < 0$ we may consider the function $f(-x)$. Suppose $\lambda > 1$. Let $(z_\mu)_{\mu=1}^m$, $(\xi_\nu)_{\nu=1}^n$ denote respectively the zeros and poles of $f(z)$. If $z_\mu = x_\mu + iy_\mu$ is complex, then $\bar{z}_\mu$ will also be a zero of $f(z)$.

Putting

$$p = a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi ,$$
$$q = (1-a)\lambda^{-\frac{1}{2}}\sin\varphi ,$$

we have for $0 \leqq a < 1$ and $0 \leqq \varphi < 2\pi$

$$\left|\frac{\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}-z_\mu\right)\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}-\bar{z}_\mu\right)}{\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-z_\mu\right)\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-\bar{z}_\mu\right)}\right|$$
$$= \left|\frac{\left(p-x_\mu+(q-y_\mu)i\right)\left(p-x_\mu+(q+y_\mu)i\right)}{(p-x_\mu-y_\mu i)(p-x_\mu+y_\mu i)}\right|$$
$$= \left(\frac{\{(p-x_\mu)^2+q^2+y_\mu^2-2qy_\mu\}\{(p-x_\mu)^2+q^2+y_\mu^2+2qy_\mu\}}{\{(p-x_\mu)^2+y_\mu^2\}^2}\right)^{\frac{1}{2}}$$
$$\leqq 1+\frac{q^2}{(p-x_\mu)^2+y_\mu^2}$$
$$= 1+\frac{(1-a)^2\lambda^{-1}\sin^2\varphi}{\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-x_\mu\right)^2+y_\mu^2}$$
$$\leqq 1+\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2} . \tag{11}$$

In case z_μ $(=x_\mu)$ is real, then

$$\left|\frac{a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}-z_\mu}{a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-z_\mu}\right| = \left\{1+\frac{1}{\lambda}\frac{(1-a)^2\sin^2\varphi}{\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-x_\mu\right)^2}\right\}^{\frac{1}{2}}$$

$$\leqq \left\{1+\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right\}^{\frac{1}{2}}. \tag{12}$$

Hence for $0\leqq a<1$ and $0\leqq\varphi<2\pi$ we have

$$|p(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi})| \leqq \left\{1+\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right\}^{m/2} |p(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi)|. \tag{13}$$

On the other hand, if $\zeta_\nu=\xi_\nu+i\eta_\nu$ is complex, then with the same notation as before for $0\leqq a<1$ and $0\leqq\varphi<2\pi$

$$\left|\frac{\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}-\zeta_\nu\right)\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}-\bar{\zeta}_\nu\right)}{\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-\zeta_\nu\right)\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-\bar{\zeta}_\nu\right)}\right|$$

$$\geqq \left(1+2\frac{q^2}{(p-\xi_\nu)^2+\eta_\nu^2}-4\frac{q^2\eta_\nu^2}{\{(p-\xi_\nu)^2+\eta_\nu^2\}^2}\right)^{\frac{1}{2}}$$

$$\geqq \left(1-2\frac{q^2}{(p-\xi_\nu)^2+\eta_\nu^2}\right)^{\frac{1}{2}}$$

$$= \left(1-2\frac{(1-a)^2\lambda^{-1}\sin^2\varphi}{\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-\xi_\nu\right)^2+\eta_\nu^2}\right)$$

$$\geqq \left(1-2\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right)^{\frac{1}{2}}, \tag{14}$$

and if ζ_ν $(=\xi_\nu)$ is real, then

$$\left|\frac{a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}-\zeta_\nu}{a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi-\zeta_\nu}\right| \geqq 1 \tag{15}$$

so that

$$\left|q\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}\right)\right| \geqq \left\{1-\frac{2\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right\}^{n/4} \left|q\left(a+(1-a)\lambda^{-\frac{1}{2}}\cos\varphi\right)\right|. \tag{16}$$

From (13) and (16) we get for $0\leqq a<1$ and $0\leqq\varphi<2\pi$

$$\left|f\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}\right)\right| \leqq \left\{1+\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right\}^{m/2}\left\{1-\frac{2\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right\}^{-n/4}. \tag{17}$$

If k is an integer, then

$$f^{(k)}(a) = \frac{k!}{2\pi i}\int_{|w-a|=(1-a)|\lambda^{-\frac{1}{2}}}\frac{f(w)}{(w-a)^{k+1}}dw,$$

and from (17) we get

$$|f^{(k)}(a)| \leqq \frac{k!}{2\pi}\frac{\lambda^{k/2}}{(1-a)^k}\int_0^{2\pi}\left(1+\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right)^{m/2}\left(1-\frac{2\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right)^{-n/4} d\varphi\,,$$

from which (10) follows.

If $\lambda=1$, then there are three different possibilities.

(i) $m=1$ and $n=0$. In this case $f(x)$ is of the form $c_1(x-c_2)$ where $|c_2|\geqq 1$ and $|c_1|\leqq 1/(1+|c_2|)$. Consequently $|f'(x)|=|c_1|\leqq \frac{1}{2}$.

(ii) $m=0$ and $n=1$. The function $f(x)$ has the form $c_3(x-c_4)^{-1}$ where $|c_3|\leqq |c_4|-1$ and

$$\begin{aligned}|f^{(k)}(x)| &\leqq k!\,|c_3|(|c_4|-|x|)^{-k-1}\\ &\leqq k!\,(|c_4|-1)(|c_4|-1+c)^{-k-1}\\ &\leqq k!\,c^{-k}k^k(k+1)^{-k-1}\,.\end{aligned}$$

(iii) $m=1$ and $n=1$. Here $f(x)$ has the form $c_5(x-x_1)(x-\xi_1)^{-1}$, where

$$|c_5| < (|\xi_1|-1)(|x_1|+1)^{-1}$$

and

$$\begin{aligned}|f^{(k)}(x)| &\leqq |c_5|\,|x_1-\xi_1|\,k!\,|x-\xi_1|^{-k-1}\\ &< (|\xi_1|-1)(|x_1|+1)^{-1}(|x_1|+|\xi_1|)\,k!\,(|\xi_1|+c-1)^{-k-1}\\ &\leqq k!\,(c^{-k}+\tfrac{1}{2}c^{-k+1})\,.\end{aligned}$$

This completes the proof of the theorem.

Remark. Quite a few theorems in function theory which were first proved for functions with only real zeros have been found to be true for functions whose zeros satisfy a condition of closeness to the real axis (see for example [2, chapters 8 and 10]). If we suppose that $f(z)$ is the quotient of two polynomials of degrees m and n respectively which are not necessarily real valued but have their zeros in the angular regions defined by

$$|y| \leqq A\lambda^{-\frac{1}{2}}(|x|-1)$$

where A is a positive constant, then we can verify as above that for $0\leqq a<1$ and $0\leqq\varphi<2\pi$

$$\left|f\left(a+(1-a)\lambda^{-\frac{1}{2}}e^{i\varphi}\right)\right| \leqq \left(1+\frac{2A}{\lambda}\,|\sin\varphi|+\frac{\sin^2\varphi}{(\lambda^{\frac{1}{2}}-1)^2}\right)^{m/2}\left(1-\frac{2A}{\lambda}\,|\sin\varphi|\right)^{-n/2}.$$

Hence in this case also, we have for $-1+c\leqq x\leqq 1-c$,

$$f^{(k)}(x) = O(\lambda^{\frac{1}{2}}/c)^k \quad \text{as } \lambda\to\infty \text{ and } k \text{ is fixed}\,.$$

Proof of Theorem 1. If $p(z_\nu)=0$, where $z_\nu=x_\nu+iy_\nu$ is complex, then $(z-z_\nu)(z-\bar{z}_\nu)$ is a factor of $p(z)$ and for $0\leqq\varphi<2\pi$

$$\left|\frac{(1+n^{-1}e^{i\varphi}-z_\nu)(1+n^{-1}e^{i\varphi}-\bar z_\nu)}{(1-n^{-1}-z_\nu)(1-n^{-1}-\bar z_\nu)}\right|$$

$$=\frac{[\{x_\nu^2+(1+n^{-1}\cos\varphi)^2-2x_\nu(1+n^{-1}\cos\varphi)+y_\nu^2+n^{-2}\sin^2\varphi\}^2-4n^{-2}y_\nu^2\sin^2\varphi]^{\frac12}}{x_\nu^2+y_\nu^2+(1-n^{-1})^2-2x_\nu(1-n^{-1})}$$

$$\leqq\frac{x_\nu^2+(1+n^{-1}\cos\varphi)^2-2x_\nu(1+n^{-1}\cos\varphi)+y_\nu^2+n^{-2}\sin^2\varphi}{x_\nu^2+y_\nu^2+(1-n^{-1})^2-2x_\nu(1-n^{-1})}$$

$$=1+\frac{2}{n}\frac{(1-x_\nu)(1+\cos\varphi)}{x_\nu^2+y_\nu^2+(1-n^{-1})^2-2x_\nu(1-n^{-1})}\,. \tag{18}$$

It is clear that if $x_\nu \geqq 1$, then the right hand side of (18) does not exceed 1. For $x_\nu < 1$ it is at most $1+(1+\cos\varphi)/(n-1)$. For this we have to show that

$$\frac{1-x_\nu}{x_\nu^2+y_\nu^2+(1-n^{-1})^2-2x_\nu(1-n^{-1})}\leqq\frac{1}{2(1-n^{-1})}$$

or

$$x_\nu^2+y_\nu^2 \geqq 1-n^{-2}$$

which is true by hypothesis. If the zero $z_\nu(=x_\nu)$ is real, then

$$\left|\frac{1+n^{-1}e^{i\varphi}-x_\nu}{1-n^{-1}-x_\nu}\right|=\left\{1+\frac{2}{n}\frac{(1-x_\nu)(1+\cos\varphi)}{x_\nu^2-2x_\nu(1-n^{-1})+(1-n^{-1})^2}\right\}^{\frac12}$$

$$\leqq\left(1+\frac{1+\cos\varphi}{n-1}\right)^{\frac12}.$$

Summing up we obtain for $0\leqq\varphi<2\pi$,

$$|p(1+n^{-1}e^{i\varphi})| \leqq \left(1+\frac{1+\cos\varphi}{n-1}\right)^{n/2}|p(1-n^{-1})|$$

$$\leqq\left(\frac{n+\cos\varphi}{n-1}\right)^{n/2}. \tag{19}$$

If k is an integer such that $1\leqq k\leqq n$, then

$$p^{(k)}(1)=\frac{k!}{2\pi i}\int\limits_{|w-1|=n^{-1}}\frac{p(w)}{(w-1)^{k+1}}\,dw\,,$$

and from (19) we get

$$|p^{(k)}(1)| \leqq \frac{k!}{2\pi i}n^k\int_0^{2\pi}\left(\frac{n+\cos\varphi}{n-1}\right)^{n/2}d\varphi\,. \tag{20}$$

If $0<a\leqq 1$, then the polynomial $p(ax)$ satisfies all the hypotheses of the theorem, and it follows from (20) that

$$(21)\qquad |p^{(k)}(a)| \leqq k!\,(n/a)^k \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{n+\cos\varphi}{n-1}\right)^{n/2} d\varphi\,.$$

Thus for $0<b\leqq x\leqq 1$ we have

$$(22)\qquad |p^{(k)}(x)| \leqq k!\,(n/b)^k \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{n+\cos\varphi}{n-1}\right)^{n/2} d\varphi = k!\,D(n/b)^k\,,$$

where

$$D = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{n+\cos\varphi}{n-1}\right)^{n/2} d\varphi\,.$$

Let $b=1-n^{-\frac{1}{2}}$. Then we get

$$(23)\qquad |p^{(k)}(x)| \leqq k!\,D(1-n^{-\frac{1}{2}})^{-k}n^k$$

for $1-n^{-\frac{1}{2}}\leqq x\leqq 1$.

Besides, in the same way as for Theorem 3 we get

$$|p^{(k)}(x)| < k!\left(\frac{n^{\frac{1}{2}}}{1-a}\right)^k \frac{1}{2\pi}\int_0^{2\pi}\left\{1+\frac{(1-a)^2\sin^2\varphi}{\left((n^2-1)^{\frac{1}{2}}-na-(1-a)n^{\frac{1}{2}}\right)^2}\right\}^{n/2} d\varphi$$

for $a\geqq 0$ provided

$$a+(1-a)n^{-\frac{1}{2}} < (1-n^{-2})^{\frac{1}{2}}\,.$$

Thus for $0\leqq a\leqq 1-n^{-\frac{1}{2}}$ we obtain

$$(24)\qquad |p^{(k)}(x)| < k!\,n^k \frac{1}{2\pi}\int_0^{2\pi}(1+n^{-\frac{3}{2}}\sin^2\varphi)^{n/2}\,d\varphi\,.$$

(23) and (24) together give the desired result. Note that for $n>1$,

$$D < \left(1+\frac{1}{n-1}\right)^{\frac{1}{2}(n+1)}\cdot 1.0635\,.$$

PROOF OF THEOREM 2. If $z_\nu=x_\nu+iy_\nu$ is complex and $p(z_\nu)=0$, then $(z-z_\nu)(z-\bar{z}_\nu)$ is a factor of $p(z)$, and for $0\leqq a<1$, $0\leqq\varphi<2\pi$

$$\left|\frac{(a+i(1-a)n^{-\frac{1}{2}}-z_\nu)(a+i(1-a)n^{-\frac{1}{2}}-\bar{z}_\nu)}{(a-z_\nu)(a-\bar{z}_\nu)}\right| \leqq \left\{1+\frac{1}{n}\,\frac{(1-a)^2}{(a-x_\nu)^2+y_\nu^2}\right\}^{\frac{1}{2}} \leqq (1+n^{-1})^{\frac{1}{2}}\,.$$

Hence

$$\left|p\left(a+i(1-a)n^{-\frac{1}{2}}\right)\right| \leqq (1+n^{-1})^{n/2}\,|p(a)|\;.$$

In fact, if $w=u+iv$ is a point on the boundary $\mathscr{C}$ of the rhombus having end points at -1, $+1$, $-i/n^{\frac{1}{2}}$ and $i/n^{\frac{1}{2}}$, then

$$|p(w)| \leqq (1+n^{-1})^{n/2}|p(u)|\;. \tag{25}$$

But for $-1<x<1$,

$$p^{(k)}(x) = \frac{k!}{2\pi i}\int\limits_{\mathscr{C}} \frac{p(w)}{(w-x)^{k+1}}\,dw\;.$$

Therefore by (25)

$$\begin{aligned}|p^{(k)}(x)| &\leqq \frac{k!}{\pi}(1+n^{-1})^{\frac{1}{2}(n+1)}\frac{(n+1)^{\frac{1}{2}(k+1)}}{(1-x)^{k+1}}\int\limits_{-1}^{1}|p(u)|\,du \\ &\leqq \frac{k!}{\pi}(1+n^{-1})^{\frac{1}{2}(n+1)}\frac{(n+1)^{\frac{1}{2}(k+1)}}{(1-x)^{k+1}}\,2^{1-1/\delta}\left(\int\limits_{-1}^{1}|p(u)|^{\delta}\,du\right)^{1/\delta}\end{aligned}$$

by Minkowski's inequality. Consequently

$$\begin{aligned}&\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}|p^{(k)}(x)|^{\delta}\,dx \\ &\quad\leqq \left\{\frac{k!}{\pi}(1+n^{-1})^{\frac{1}{2}(n+1)}\right\}^{\delta}2^{\delta-1}(n+1)^{\frac{1}{2}(k+1)\delta}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\frac{dx}{(1-x)^{(k+1)\delta}}\int\limits_{-1}^{1}|p(u)|^{\delta}\,du \\ &\quad< \{C_1(k)\}^{\delta}\,n^{k\delta}\int\limits_{-1}^{1}|p(u)|^{\delta}\,du\end{aligned} \tag{26}$$

with $C_1(k)$ depending only on k.

While proving Theorem 1 we have actually shown that for $0<x\leqq 1$

$$|p^{(k)}(x)| < \left(1+\frac{1}{n-1}\right)^{\frac{1}{2}(n+1)} 1.0635\,k!\,(n/x)^{k}|p(x-x/n)|\;.$$

Hence

$$\int\limits_{\frac{1}{2}}^{1}|p^{(k)}(x)|^{\delta}\,dx < \{C_2(x)\}^{\delta}n^{k\delta}\int\limits_{-1}^{1}|p(x)|^{\delta}\,dx\;. \tag{27}$$

Similarly

$$\int\limits_{-1}^{-\frac{1}{2}}|p^{(k)}(x)|^{\delta}\,dx < \{C_3(k)\}^{\delta}n^{k\delta}\int\limits_{-1}^{1}|p(x)|^{\delta}\,dx\;. \tag{28}$$

From (26), (27) and (28) the result follows.

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