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AN EXTREMAL PROBLEM RELATED TO BERNSTEIN'S APPROXIMATION PROBLEM

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1.

Let $K(x)$ be a continuous function defined on $(-\infty,\infty)$ with $K(x) \geqq 1$ and such that

$$\lim_{|x|\to\infty} \frac{x^n}{K(x)} = 0, \qquad n=0,1,2,\ldots . \tag{1.1}$$

Let further C_K be the space of all complex-valued continuous functions $f(x)$ defined on $(-\infty,\infty)$ and such that

$$\lim_{|x|\to\infty} \frac{f(x)}{K(x)} = 0 \tag{1.2}$$

with the norm

$$\|f\|_{C_K} = \sup_{-\infty<x<\infty} \frac{|f(x)|}{K(x)} ,$$

L_K^2 the space of all complex-valued measurable functions defined a.e. on $(-\infty,\infty)$ and such that

$$\int_{-\infty}^{\infty} \left|\frac{f(x)}{K(x)}\right|^2 dx < \infty \tag{1.3}$$

with the norm

$$\|f\|_{L_K^2} = \left\{ \int_{-\infty}^{\infty} \left|\frac{f(x)}{K(x)}\right|^2 dx \right\}^{\frac{1}{2}} ,$$

and $\mathscr{P}$ the class of all polynomials $P(x)$ with complex coefficients.

From (1.1) it follows that $\mathscr{P} \subset C_K$ and $\mathscr{P} \subset L_K^2$. Bernstein's approximation problem is to determine conditions on the function $K(x)$ under which $\mathscr{P}$ is dense in C_K respectively in L_K^2. This problem has been treated by many authors; for a survey of known results, see Mergelyan [5], Ahiezer [2] and Pollard [7].

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The purpose of this paper is to solve by elementary methods a certain extremal problem closely related to Bernstein's approximation problem. For the sake of completeness we also show how, starting from the solution of this extremal problem, one can derive with elementary methods some known results concerning Bernstein's approximation problem.

I wish to express my gratitude to Professor Lennart Carleson for his generously given advice and never failing interest.

2.

The above-mentioned extremal problem and its solution can be formulated as the following

THEOREM 1. *Let $p_n(x)$ be a given polynomial of degree n with complex coefficients and without real zeros. Let $\mathcal{P}_{n-1}$ be the class of all polynomials $P(x)$ of degree at most $n-1$ with complex coefficients. Then, for an arbitrary non-real z,*

(2.1)

$$\inf_{P(x)\in\mathcal{P}_{n-1}} \int_{-\infty}^{\infty} \left|\frac{(x-z)^{-1}-P(x)}{p_n(x)}\right|^2 dx = \frac{\pi}{|\operatorname{Im} z|}\exp\left(-\frac{2|\operatorname{Im} z|}{\pi}\int_{-\infty}^{\infty}\frac{\log|p_n(x)|}{|x-z|^2}\,dx\right).$$

PROOF. Let

$$p_n(x) = c\prod_{\nu=1}^{n}(x-c_\nu)$$

and let $c_0=z$. Obviously, we may assume that

$$\operatorname{Im} c_\nu > 0, \qquad \nu=0,1,\ldots,n.$$

Because of the continuity in c_ν, $\nu=0,1,\ldots,n$, of both of the members in (2.1) it is sufficient to consider the case when all the numbers c_ν, $\nu=0,1,\ldots,n$, are different. We have

$$\frac{(x-z)^{-1}-P(x)}{p_n(x)} = \frac{1}{c}\left(\frac{1}{\prod_{\nu=0}^{n}(x-c_\nu)} - \sum_{\nu=1}^{n}\frac{z_\nu}{x-c_\nu}\right) = \frac{1}{c}\left(\varphi(x)-\sum_{\nu=1}^{n} z_\nu f_\nu(x)\right)$$

for certain numbers z_ν, $\nu=1,2,\ldots,n$, and with

$$\varphi(x) = \frac{1}{\prod_{\nu=0}^{n}(x-c_\nu)}$$

and

$$f_\nu(x) = \frac{1}{x-c_\nu}, \qquad \nu=1,2,\ldots,n.$$

Using the scalar product

$$(f,g) = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$$

in the Hilbert space $L^2(-\infty,\infty)$ the left-hand side in (2.1) can be written

$$\min_{\{z_\nu\}} \frac{1}{|c|^2}\left\|\varphi - \sum_{\nu=1}^{n} z_\nu f_\nu\right\|^2 = \frac{\delta^2}{|c|^2},$$

and it is well known that

$$\delta^2 = \frac{G(\varphi,f_1,f_2,\ldots,f_n)}{G(f_1,f_2,\ldots,f_n)},$$

where $G(f_1,f_2,\ldots,f_n)$ is Gram's determinant

$$G(f_1,f_2,\ldots,f_n) = |(f_\nu,f_\mu)|_{\nu,\mu=1}^{n}.$$

We obtain easily

$$(f_\nu,f_\mu) = \int_{-\infty}^{\infty} \frac{dx}{(x-c_\nu)(x-\bar{c}_\mu)} = \frac{2\pi i}{c_\nu-\bar{c}_\mu}, \qquad \nu,\mu=1,2,\ldots,n.$$

Using the notation

$$a_j = \frac{1}{\prod_{\nu\neq j}^{0,n}(c_j-c_\nu)}, \qquad j=0,1,\ldots,n,$$

we find

$$(\varphi,f_\mu) = \int_{-\infty}^{\infty} \frac{dx}{(x-\bar{c}_\mu)\prod_{\nu=0}^{n}(x-c_\nu)} = 2\pi i \sum_{j=0}^{n} \frac{a_j}{c_j-\bar{c}_\mu}.$$

By means of the decomposition into partial fractions

$$\varphi(x) = \sum_{j=0}^{n} \frac{a_j}{x-c_j}$$

we obtain

$$(\varphi,\varphi) = \int_{-\infty}^{\infty} \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{a_j\bar{a}_k}{(x-c_j)(x-\bar{c}_k)}\,dx = 2\pi i \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{a_j\bar{a}_k}{c_j-\bar{c}_k}.$$

The determinant $G(f_1,f_2,\ldots,f_n)$ now becomes a so-called Cauchy determinant and can easily be calculated explicitly (see e.g. Achiezer [1]). We find

$$\frac{1}{(2\pi i)^n} G(f_1, f_2, \ldots, f_n) = \begin{vmatrix} \frac{1}{c_1-\bar{c}_1} & \frac{1}{c_1-\bar{c}_2} & \cdots & \frac{1}{c_1-\bar{c}_n} \\ \frac{1}{c_2-\bar{c}_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{c_n-\bar{c}_1} & \cdots & \cdots & \frac{1}{c_n-\bar{c}_n} \end{vmatrix}$$

$$= \frac{\prod_{1 \leqq \nu < \mu \leqq n} (c_\mu - c_\nu)(\bar{c}_\nu - \bar{c}_\mu)}{\prod_{\nu,\mu=1}^{n} (c_\nu - \bar{c}_\mu)} .$$

Moreover,

$$\frac{1}{(2\pi i)^{n+1}} G(\varphi, f_1, f_2, \ldots, f_n) = \begin{vmatrix} \sum_{j=0}^{n} \sum_{k=0}^{n} \frac{a_j \bar{a}_k}{c_j - \bar{c}_k} & \sum_{j=0}^{n} \frac{a_j}{c_j - \bar{c}_1} & \cdots & \sum_{j=0}^{n} \frac{a_j}{c_j - \bar{c}_n} \\ \sum_{k=0}^{n} \frac{\bar{a}_k}{c_1 - \bar{c}_k} & \frac{1}{c_1 - \bar{c}_1} & \cdots & \frac{1}{c_1 - \bar{c}_n} \\ \vdots & \vdots & & \vdots \\ \sum_{k=0}^{n} \frac{\bar{a}_k}{c_n - \bar{c}_k} & \frac{1}{c_n - \bar{c}_1} & \cdots & \frac{1}{c_n - \bar{c}_n} \end{vmatrix} .$$

To calculate this determinant we first subtract the $(j+1)$-th row multiplied by a_j from the first row, for $j = 1, 2, \ldots, n$. In the determinant so obtained we factor out a_0 from the first row and then subtract the $(k+1)$-th column multiplied by $\bar{a}_k$ from the first column, for $k = 1, 2, \ldots, n$. Finally, we factor out $\bar{a}_0$ from the first column and obtain

$$\frac{1}{(2\pi i)^{n+1}} G(\varphi, f_1, f_2, \ldots, f_n) = |a_0|^2 \begin{vmatrix} \frac{1}{c_0-\bar{c}_0} & \frac{1}{c_0-\bar{c}_1} & \cdots & \frac{1}{c_0-\bar{c}_n} \\ \frac{1}{c_1-\bar{c}_0} & \frac{1}{c_1-\bar{c}_1} & \cdots & \frac{1}{c_1-\bar{c}_n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{c_n-\bar{c}_0} & \frac{1}{c_n-\bar{c}_1} & \cdots & \frac{1}{c_n-\bar{c}_n} \end{vmatrix}$$

$$= \frac{1}{\prod_{\nu=1}^{n} (c_0 - c_\nu)(\bar{c}_0 - \bar{c}_\nu)} \cdot \frac{\prod_{0 \leqq \nu < \mu \leqq n} (c_\mu - c_\nu)(\bar{c}_\nu - \bar{c}_\mu)}{\prod_{\nu,\mu=0}^{n} (c_\nu - \bar{c}_\mu)} .$$

Hence we have

$$\delta^2 = \frac{2\pi i}{(c_0-\bar{c}_0)\prod_{\nu=1}^{n}|c_0-\bar{c}_\nu|^2} = \frac{\pi}{(\operatorname{Im} z)\prod_{\nu=1}^{n}|z-\bar{c}_\nu|^2}.$$

Now, if we observe that

$$\frac{\operatorname{Im} z}{\pi}\int_{-\infty}^{\infty}\frac{\log|x-c_\nu|}{|x-z|^2}\,dx = \log|z-\bar{c}_\nu|, \qquad \nu=1,2,\ldots,n,$$

and

$$\frac{\operatorname{Im} z}{\pi}\int_{-\infty}^{\infty}\frac{dx}{|x-z|^2} = 1,$$

(2.1) follows.

From Theorem 1 we easily obtain

THEOREM 2. *Let $H(x)$ be an entire function of the type*

(2.2) $$H(x) = \sum_{\nu=0}^{\infty} b_\nu x^{2\nu}; \quad b_0>0; \quad b_\nu \geqq 0,\ \nu=1,2,\ldots .$$

Then for an arbitrary non-real z

(2.3) $$E(z) \equiv \inf_{\mathscr{P}}\int_{-\infty}^{\infty}\frac{|(x-z)^{-1}-P(x)|^2}{H(x)}\,dx$$

$$= \frac{\pi}{|\operatorname{Im} z|}\exp\left(-\frac{|\operatorname{Im} z|}{\pi}\int_{-\infty}^{\infty}\frac{\log H(x)}{|x-z|^2}\,dx\right),$$

where the meaning of the last member is to be 0 if

$$\int_{-\infty}^{\infty}\frac{\log H(x)}{|x-z|^2}\,dx = \infty .$$

PROOF. Let

$$H_n(x) = \sum_{\nu=0}^{n} b_\nu x^{2\nu}.$$

By Theorem 1

(2.4) $$E_n(z) \equiv \inf_{\mathscr{P}_{n-1}}\int_{-\infty}^{\infty}\frac{|(x-z)^{-1}-P(x)|^2}{H_n(x)}\,dx$$

$$= \frac{\pi}{|\operatorname{Im} z|}\exp\left(-\frac{|\operatorname{Im} z|}{\pi}\int_{-\infty}^{\infty}\frac{\log H_n(x)}{|x-z|^2}\,dx\right).$$

Furthermore,

$$E(z) = \lim_{n\to\infty} E_n(z) . \tag{2.5}$$

For given an $\varepsilon > 0$, there exists a polynomial $P_\varepsilon(x)$ such that

$$\int_{-\infty}^{\infty} \frac{|(x-z)^{-1} - P_\varepsilon(x)|^2}{H(x)} dx < E(z) + \varepsilon$$

and for sufficiently large n evidently

$$\int_{-\infty}^{\infty} \frac{|(x-z)^{-1} - P_\varepsilon(x)|^2}{H_n(x)} dx < \int_{-\infty}^{\infty} \frac{|(x-z)^{-1} - P_\varepsilon(x)|^2}{H(x)} dx + \varepsilon ,$$

whence

$$E(z) \leqq E_n(z) < E(z) + 2\varepsilon .$$

Finally, (2.3) follows from (2.4) and (2.5).

3.

We now return to Bernstein's approximation problem. Let $M_K \equiv M_{K(x)}$ be the class of all complex polynomials $p(x)$ for which

$$|p(x)| \leqq K(x), \qquad -\infty < x < \infty .$$

The following theorem holds true.

THEOREM 3. *If*

$$\sup_{M_K} \int_{-\infty}^{\infty} \frac{\log |p(x)|}{1+x^2} dx = \infty , \tag{3.1}$$

then $\mathscr{P}$ *is dense in* L_K^2.

PROOF. Since the class C_0 of continuous functions on $(-\infty, \infty)$ with compact support is dense in L_K^2 it is sufficient to approximate an arbitrary function $g(x)$ in C_0 with an arbitrary degree of accuracy. And to every $\varepsilon > 0$ there exists a linear combination $r(x)$ of functions of the type

$$\frac{1}{x-z}, \qquad \operatorname{Im} z \neq 0 ,$$

such that

$$\int_{-\infty}^{\infty} |g(x) - r(x)|^2 dx < \varepsilon .$$

(This can be proved by elementary methods; see e.g. Shohat and Tamarkin [8] or Achiezer [1].) Hence a fortiori

$$\int_{-\infty}^{\infty}\left|\frac{g(x)-r(x)}{K(x)}\right|^2 dx < \varepsilon .$$

Consequently it is sufficient to prove that

$$\inf_{\mathscr{P}}\int_{-\infty}^{\infty}\left|\frac{(x-z)^{-1}-P(x)}{K(x)}\right|^2 dx = 0 . \tag{3.2}$$

From (3.1) there follows the existence of a sequence of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ in M_K such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\log|p_n(x)|}{1+x^2}dx = \infty .$$

We have for $n=1,2,\ldots$

$$|p_n(x)|^2 < 1+|p_n(x)|^2 = |q_n(x)|^2, \qquad -\infty<x<\infty ,$$

where $q_n(x)$ is a polynomial satisfying $|q_n(x)|\geqq 1$ and

$$|p_n(x)| < |q_n(x)| \leqq 2^{\frac{1}{2}}K(x), \qquad -\infty<x<\infty .$$

Because for an arbitrary complex z

$$|x-z| \leqq |x-i|(1+|i-z|),$$

we conclude

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\log|q_n(x)|}{|x-z|^2}dx = \infty, \quad \operatorname{Im} z \neq 0 . \tag{3.3}$$

Finally, since

$$\inf_{\mathscr{P}}\int_{-\infty}^{\infty}\left|\frac{(x-z)^{-1}-P(x)}{K(x)}\right|^2 dx \leqq 2\inf_{\mathscr{P}_{n-1}}\int_{-\infty}^{\infty}\left|\frac{(x-z)^{-1}-P(x)}{q_n(x)}\right|^2 dx , \tag{3.4}$$

(3.2) follows from (3.4), (2.1) and (3.3).

4.

We can now easily prove the following theorem (compare Pollard [7], Ahiezer [2]):

THEOREM 4. *Suppose $K(x)$ is non-decreasing as $|x|$ increases. Then a necessary and sufficient condition for $\mathscr{P}$ to be dense in C_K is that*

$$\sup_{M_K}\int_{-\infty}^{\infty}\frac{\log|p(x)|}{1+x^2}dx = \infty . \tag{3.1}$$

REMARK. The condition that $K(x)$ be non-decreasing as $|x|$ increases is needed only to guarantee that

$$\max_{|\xi-\frac{1}{2}x|\leqq\frac{1}{2}} K(\xi) \leqq BK(x), \qquad -\infty<x<\infty,$$

for some constant B. If, for example, $K(x)\equiv 1$ for $|x|\leqq 1$, then we can take $B=1$.

PROOF. *The sufficiency.* As in the proof of Theorem 3 it is sufficient to prove that an arbitrary continuous function $g(x)$ with compact support can be approximated with an arbitrary degree of accuracy. Let, for $0<a<1$,

$$g_a(x) = \big(1/(2a)\big)\int_{-a}^{a} g(x+t)\,dt\,.$$

Then

$$g(x)-g_a(x) = \big(1/(2a)\big)\int_{-a}^{a} [g(x)-g(x+t)]\,dt\,.$$

Given $\varepsilon>0$, since $g(x)$ is uniformly continuous on $(-\infty,\infty)$, we can choose a so small that

$$|g(x)-g_a(x)| < \tfrac{1}{2}\varepsilon, \qquad -\infty<x<\infty\,.$$

From (3.1) it follows that

$$\sup_{M_{K(\frac{1}{2}x)}} \int_{-\infty}^{\infty} \frac{\log|p(x)|}{1+x^2}\,dx = \infty\,.$$

By Theorem 3, then, there exists a polynomial $P(x)$ such that

$$\int_{-\infty}^{\infty} \left|\frac{g(x)-P(x)}{K(\frac{1}{2}x)}\right|^2 dx < \frac{a^2\varepsilon^2}{2B^2}$$

(for the definition of B see the remark above). Let

$$P_a(x) = \big(1/(2a)\big)\int_{-a}^{a} P(x+t)\,dt\,.$$

Then $P_a(x)$ is a new polynomial and we have

$$\begin{aligned}|g_a(x)-P_a(x)|^2 &\leqq \frac{1}{4a^2}\left\{\int_{-a}^{a} |g(x+t)-P(x+t)|\,dt\right\}^2 \\ &\leqq \frac{1}{4a^2}\int_{-\infty}^{\infty}\left|\frac{g(y)-P(y)}{K(\frac{1}{2}y)}\right|^2 dy \int_{-1}^{1} K^2(\tfrac{1}{2}x+\tfrac{1}{2}t)\,dt \\ &< \frac{\varepsilon^2}{4B^2}\max_{|\xi-\frac{1}{2}x|\leqq\frac{1}{2}} K^2(\xi)\,,\end{aligned}$$

whence

$$|g_a(x)-P_a(x)| < \tfrac{1}{2}\varepsilon\, K(x), \qquad -\infty<x<\infty .$$

Finally

$$\|g-P_a\|_{C_K} \leqq \|g-g_a\|_{C_K}+\|g_a-P_a\|_{C_K} < \varepsilon ,$$

which shows the sufficiency of the condition (3.1).

The necessity. Suppose that $\mathscr{P}$ is dense in C_K. Then to an arbitrary $\varepsilon>0$ there exists a polynomial $Q(x)$ such that

$$\left|\frac{1}{x-i}-Q(x)\right| < \varepsilon^{\frac{1}{2}}K(x), \qquad -\infty<x<\infty .$$

Choose an $A>0$ so large that

$$\int\limits_{|x|\geqq A} \frac{|(x-i)^{-1}-Q(x)|^2}{(|Q(x)|^2+1)(1+x^2)}\,dx<\varepsilon .$$

From the assumption that $\mathscr{P}$ is dense in C_K there follows easily (see Pollard [6]) the existence of a sequence of polynomials $\{P_n(x)\}_{n=1}^{\infty}$ with

$$|P_n(x)| \leqq 2K(x), \qquad -\infty<x<\infty, \quad n=1,2,\ldots ,$$

and

$$\lim_{n\to\infty} P_n(x) = K(x)$$

uniformly on every compact set. Then choose an N so large that

$$|P_N(x)| > \tfrac{1}{2}K(x), \qquad -A < x < A ,$$

and let

$$|Q(x)|^2+|P_N(x)|^2+1 = |R_\varepsilon(x)|^2 ,$$

where $R_\varepsilon(x)$ is a polynomial. Now we have

$$(4.1)\quad \int\limits_{-\infty}^{\infty} \frac{|(x-i)^{-1}-Q(x)|^2}{|R_\varepsilon(x)|^2(1+x^2)}\,dx$$

$$\leqq \varepsilon \int\limits_{-A}^{A} \frac{K^2(x)}{|P_N(x)|^2(1+x^2)}\,dx + \int\limits_{|x|\geqq A} \frac{|(x-i)^{-1}-Q(x)|^2}{(|Q(x)|^2+1)(1+x^2)}\,dx$$

$$< 4\varepsilon \int\limits_{-\infty}^{\infty} \frac{dx}{1+x^2}+\varepsilon = (4\pi+1)\varepsilon .$$

On the other hand, by (2.1)

$$(4.2)\qquad \int\limits_{-\infty}^{\infty} \frac{|(x-i)^{-1}-Q(x)|^2}{|R_\varepsilon(x)|^2(1+x^2)}\,dx \geqq \pi e^{-I_\varepsilon/\pi} ,$$

where

$$I_\varepsilon = \int_{-\infty}^{\infty} \frac{\log |R_\varepsilon(x)|^2}{1+x^2}\,dx + \int_{-\infty}^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx .$$

Hence

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\log |R_\varepsilon(x)|}{1+x^2}\,dx = \infty .$$

Since

$$|Q(x)| \leqq |Q(x)-(x-i)^{-1}|+|(x-i)^{-1}| < \varepsilon^{\frac{1}{2}}K(x)+1 < 2K(x),$$

we have

$$|R_\varepsilon(x)|^2 < 4K^2(x)+4K^2(x)+1 \leqq 9K^2(x),$$

that is, $\frac{1}{3}R_\varepsilon(x) \in M_K$, and therefore (3.1) holds.

5.

Using Theorem 2 we can also derive the following result concerning Bernstein's approximation problem in the case when $K(x)$ is even and $\log K(x)$ is a convex function of $\log|x|$ (see Carleson [3]).

THEOREM 5. *Suppose that $K(x)$ is even and $\log K(x)$ is a convex function of $\log|x|$. Then a necessary and sufficient condition for $\mathscr{P}$ to be dense in C_K and in L_K^2 is that*

$$\int_{-\infty}^{\infty} \frac{\log K(x)}{1+x^2}\,dx = \infty . \tag{5.1}$$

PROOF. From Theorem 2 it follows (compare the proof of Theorem 3) that if $g(x) \in C_0$ and if, for a function $H(x)$ of the type (2.2),

$$\int_{-\infty}^{\infty} \frac{\log H(x)}{1+x^2}\,dx = \infty ,$$

then

$$\inf_{\mathscr{P}} \int_{-\infty}^{\infty} \frac{|g(x)-P(x)|^2}{H(x)}\,dx = 0 .$$

On the other hand, by the same theorem, if

$$\int_{-\infty}^{\infty} \frac{\log H(x)}{1+x^2}\,dx < \infty ,$$

then

$$\inf_{\mathscr{P}} \int_{-\infty}^{\infty} \frac{|(x-i)^{-1}-P(x)|^2}{H(x)} dx > 0 .$$

Now by a lemma by Y. Domar [4], to every even function $K(x) \geqq 1$ such that $\log K(x)$ is a convex function of $\log|x|$ there exists a function $H(x)$ of the type (2.2) such that

$$K(x) \leqq H(x) \leqq x^5 K(x)$$

for all sufficiently large $|x|$. (The elementary proof of this lemma proceeds using only simple properties of convex functions.) From this Theorem 5 follows in what concerns L_K^2.

If $\mathscr{P}$ is dense in L_K^2, we conclude as in the proof of Theorem 4 that $\mathscr{P}$ is dense also in C_K. On the other hand, suppose that $\mathscr{P}$ is dense in C_K and let

$$K_1(x) = K(x)\,(1+x^2)^{\frac{1}{2}} .$$

Then $\log K_1(x)$ is convex in $\log|x|$ and we have for $g(x) \in C_0$ and $P(x) \in \mathscr{P}$

$$\|g(x)-P(x)\|_{L^2_{K_1}} \leqq \pi^{\frac{1}{2}} \|g(x)-P(x)\|_{C_K} .$$

Hence $\mathscr{P}$ is dense in $L^2_{K_1}$ and (5.1) follows.

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