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A NOTE ON FORMAL AND ANALYTIC CAUCHY PROBLEMS FOR SYSTEMS WITH CONSTANT COEFFICIENTS

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1. Introduction.

Let $x=(x_1,\ldots,x_n)$ be indeterminates and put $x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ where $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multiindex with integral non-negative components. Let E be a finite-dimensional normed vector space over the complex numbers $\mathbf{C}$ and let $F_n(E)$ be all formal power series

$$f(x) = \sum f_\alpha x^\alpha \tag{1}$$

with coefficients in E, topologized by the semi-norms

$$|f|_k = \sum_{|\alpha|\leqq k} |f_\alpha|, \qquad k = 0,1,\ldots ,$$

where $|\alpha|=\alpha_1+\ldots+\alpha_n$ and $|\cdot|$ denotes the norm in E. If there exists a formal power-series $g(x)$ such that $f(x)=x^\beta g(x)$, we write $f(x)=O(x^\beta)$. Differentiation of formal power-series is defined in the usual way.

Let $A_n(E)$ be the set of power-series (1) which converge when $x_1,\ldots,x_n$ are complex numbers and $|x|=|x_1|+\ldots+|x_n|$ is sufficiently small. Let $A_n(E,r)$ be the set of $f\in A_n(E)$ which converge when $|x|<r$, equipped with the norms

$$\|f\|_\varrho = \max |f(x)|, \qquad |x|\leqq \varrho<r .$$

We say that a linear mapping $T\colon A_n(E_1)\to A_m(E_2)$ is continuous if there exist numbers r_1 and $r_2>0$ such that $T\colon A_n(E_1,r_1)\to A_m(E_2,r_2)$ is continuous. This definition extends in a natural way to direct sums (see (5) below).

Let t and $x=(x_1,\ldots,x_n)$ be indeterminates or complex numbers as the case may be, put $D_t=\partial/\partial t$ and $D_x=(D_1,\ldots,D_n)$, $D_k=\partial/\partial x_k$, and let $P(D_t,D_x)$ be a $l\times l$ matrix whose elements are complex polynomials in D_t,D_x. Put

$$\Delta_L f = \{D_t^s f(0,x)\}, \qquad 0\leqq s<L ,$$

where $f\in F_{n+1}(E)$ or $A_{n+1}(E)$. Clearly, $\Delta_L f=0$ if and only if $f=O(t^L)$. We shall be interested in the following Cauchy problems: given

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find
$$V, W \in F_{n+1}(\mathbf{C}^l) \quad \text{or} \quad A_{n+1}(\mathbf{C}^l),$$
$$U \in F_{n+1}(\mathbf{C}^l) \quad \text{or} \quad A_{n+1}(\mathbf{C}^l),$$

respectively, such that

$$P(D_t, D_x)\, U(t,x) = V(t,x), \qquad U(t,x) - W(t,x) = O(t^L). \tag{2}$$

We shall refer to these problems as Cauchy's problem for formal power-series and analytic functions respectively. In general these problems have no solution unless we impose some consistency conditions on V and W which should be identically true when $V = PU$ and $W - U = O(t^L)$ for some U. Also, if L is small we cannot expect the solution to be unique. In any case, if (2) combined with some consistency condition on V and W has a unique solution U depending continuously on V and W, then the operator
$$P \oplus \Delta_L$$
defined by
$$(P \oplus \Delta_L)U = PU \oplus \Delta_L U,$$
where $U \in F_{n+1}(E)$ or $A_{n+1}(E)$, has a continuous inverse. We shall prove

THEOREM 1. *If*

$$P \oplus \Delta_L\colon\ F_{n+1}(\mathbf{C}^l) \to F_{n+1}(\mathbf{C}^l) \oplus F_n(\mathbf{C}^{Ll}) \tag{3}$$

has a continuous inverse, then
$$p(D_t, D_x) = \det P(D_t, D_x)$$
has the form
$$p(D_t, D_x) = c D_t^N + \sum_{j=0}^{N-1} p_j(D_x) D_t^j, \tag{4}$$
where $c \neq 0$ is a constant and the p_j are polynomials. Conversely, if (4) *holds, then* (3) *has a continuous inverse.—If*

$$P \oplus \Delta_L\colon\ A_{n+1}(\mathbf{C}^l) \to A_{n+1}(\mathbf{C}^l) \oplus A_n(\mathbf{C}^{Ll}) \tag{5}$$

has a continuous inverse, then (4) *holds and*

$$\operatorname{degree} p_j \leqq N - j \tag{6}$$

so that N is the degree of p. Conversely, if this condition holds, then (5) *has a continuous inverse.*

The converse parts of this theorem are consequences of the following more precise statement.

THEOREM 2. *Let $P'(D_t, D_x)$ be the $l \times l$ matrix of minors of $P(D_t, D_x)$ so that*

$$P'(D_t,D_x)P(D_t,D_x) = P(D_t,D_x)P'(D_t,D_x) = p(D_t,D_x)I$$

where I is the $l \times l$ unit matrix. Let M be the degree of P with respect to D_t and suppose that

(7) $$L \geqq M+N,$$

(8) $$PW-V = O(t^{L-M}),$$

(9) $$pIW-P'V = O(t^{L-N}).$$

Then if (4) *holds, Cauchy's problem* (2) *for formal power-series has a unique solution U such that the mapping*

$$F_{n+1}(\mathbf{C}^l) \oplus F_n(\mathbf{C}^{Ll}) \ni V \oplus \Delta_L W \to U \in F_{n+1}(\mathbf{C}^l)$$

is continuous.—If, in addition, (6) *holds, then Cauchy's problem* (2) *for analytic functions has a unique solution U such that the mapping*

$$A_{n+1}(\mathbf{C}^l) \oplus A_n(\mathbf{C}^{Ll}) \ni V \oplus \Delta_L W \to U \in A_{n+1}(\mathbf{C}^l)$$

is continuous.

Remark. If

$$P = QD_t^M + \ldots, \qquad M = \operatorname{degree} P, \quad \det Q \neq 0,$$

then $N=lM$ so that (9) is a consequence of (8) and, by virtue of (8), $\Delta_L W$ is a continuous linear function of $\Delta_M W$ and $\Delta_{L-M} V$. Hence, in this case, (2) can be stated as

$$PU = V, \qquad U-W = O(t^M)$$

without consistency conditions. In the general case, this is not possible.

2. Proof of Theorem 1.

Put $\zeta x = \zeta_1 x_1 + \ldots + \zeta_n x_n$ where $\zeta_1,\ldots,\zeta_n$ are complex numbers. The equation $PU=0$ has exponential solutions

(1) $$U = U_0 e^{\tau t+\zeta x}, \qquad U_0 \in \mathbf{C}^l, \quad U_0 \neq 0,$$

if and only if

(2) $$p(\tau,\zeta) = 0.$$

Let us first consider the case of formal power-series. If $P \oplus \Delta_L$ has a continuous inverse then there exist an integer K and a number C such that

$$|U|_L \leqq C|\Delta_L U|_K.$$

In particular,

$$|U_0|(1+|\tau|^L) \leqq C_1|U_0|(1+|\tau|)^{L-1}(1+|\zeta|)^K,$$

where C_1 is another constant. In view of (1) and (2) this means that

$$(3) \qquad p(\tau,\zeta) = 0 \quad \Rightarrow \quad |\tau| \leqq C_2(1+|\zeta|)^K$$

with a third constant C_2. Let us now write p in the form

$$(4) \qquad p(\tau,\zeta) = \sum_0^N p_j(\zeta)\,\tau^j, \qquad p_N(\zeta) \not\equiv 0 .$$

The polynomials p_j cannot have a common zero ζ since this contradicts (3). Assume that $p_N(\zeta)$ has a zero ζ_0 and let ζ tend to ζ_0 in such a way that $p_N(\zeta) \neq 0$. Then at least one quotient $p_j(\zeta)/p_N(\zeta)$, $j<N$, tends to infinity so that at least one zero $\tau=\tau(\zeta)$ of $p(\tau,\zeta)=0$ tends to infinity, and this again contradicts (3) so that $p_N(\zeta)$ is a constant. This proves the first direct part of Theorem 1.

Next take the analytic case. If $P \oplus \Delta_L$ has a continuous inverse, then there exist numbers ϱ_1, ϱ_2 and $C>0$ such that

$$\|U\|_{\varrho_1} \leqq C\,\|\Delta_L U\|_{\varrho_2} .$$

In particular,

$$|U_0|e^{\varrho_1(|\tau|+|\zeta|)/(n+1)} \leqq |U_0|\,C_1(1+|\tau|)^L e^{\varrho_2|\zeta|} ,$$

where C_1 is another constant. In view of (1) and (2) this means that

$$p(\tau,\zeta) = 0 \quad \Rightarrow \quad |\tau| \leqq C_2(1+|\zeta|)$$

with a third constant C_2. Hence p has the form (4) and

$$p_j(\zeta) = O\big((1+|\zeta|)^{N-j}\big)$$

which shows that degree $p_j \leqq N-j$. This finishes the proof of the second direct part of Theorem 1. The converse parts follow from Theorem 2.

3. Proof of Theorem 2.

It follows from (1.4) that the formal Cauchy problem

$$(1) \qquad pIU - P'V = 0, \qquad U-W = O(t^N)$$

has a unique solution $U \in F_{n+1}(\mathbf{C}^l)$. We shall see that

$$(2) \qquad PU = V, \qquad U-W = O(t^L) ,$$

i.e. that U solves the formal Cauchy problem (1.2). In fact, by (1.9),

$$pI(U-W) = pIU - P'V + P'V - pIW = O(t^{L-N})$$

so that, by (1.4),

$$U - W = O(t^L). \tag{3}$$

Put $Z = PU - V$. Then

$$pIZ = P(pIU - P'V) = 0,$$

and (3) and (1.8) show that

$$Z = P(U - W) + PW - V = O(t^{L-M}).$$

Since $L - M \geqq N$, this shows that $Z = 0$ so that we have (1.2).—If U is a solution of (1.2) with $V = 0$, $\Delta_L W = 0$, then $pIU = P'PU = 0$ and $U = O(t^N)$ so that $U = 0$. This shows that a solution is unique. It follows from (1) that U is a continuous function of $P'V$ and $\Delta_N W$ and hence a continuous function of V and $\Delta_L W$. This proves Theorem 2 in the formal case. The proof in the analytic case is the same since, by the Cauchy–Kowalevski theorem, (1) has a unique holomorphic solution U if p has the properties (1.4) and (1.6). That $P \oplus \Delta_L$ has a continuous inverse follows from the proof of the same theorem, reasoning as in the formal case.

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