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A GENERALIZATION OF THE $C(X)$-CHARACTERIZATIONS OF TOPOLOGICAL SPACES

AUDUN HOLME

1. Introduction.

If $C(X)$ denotes some kind of algebraic system of continuous complex-valued functions on X, we have a number of well-known theorems which very roughly can be expressed in the following way: If $C(X)$ and $C(Y)$ are isomorphic, then X and Y are homeomorphic, where X and Y belong to some suitably restricted class of topological spaces. It is enough to cite classical instances proved by Gelfand-Kolmogoroff, Stone, Milgram and Kaplansky respectively. It was shown in [1] how to obtain a general theorem of this kind which for instance contained the theorems of Gelfand-Kolmogoroff and Stone as very special cases. Since the lattice, semigroup and ring of all real-valued functions on X are equivalent for determining the topology on X ([8]) it is not surprising that a great part of the above-mentioned situations may be given a unified treatment. The purpose of the present note is to show how the x-ideals of [1] may be used in order to prove a general theorem, which includes the corresponding theorem of [1], as well as for instance the theorems of Milgram [5] and Kaplansky [6].

The examples given at the end of this paper do not present a complete list of applications of the theorem, and do not present the results in their most general form. In particular, generalizations of Examples 3 and 4 may be found in [8].

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2. Preliminaries.

A commutative semigroup S is said to be equipped with an x-system if there is defined an operation $A \to A_x$ on the subsets of S such that

$$A \subseteq A_x, \qquad A \subseteq B_x \Rightarrow A_x \subseteq B_x, \qquad AB_x \subseteq B_x \cap (AB)_x .$$

The subsets of the form A_x are called the x-ideals of S (or only the ideals of S, when no confusion can arise.) This generalizes the concept of an

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ideal in a commutative ring, a semigroup ideal in a commutative semigroup, and l-ideal in a distributive lattice L, L being considered as a semigroup under $\wedge$. Furthermore the family of convex, lattice-closed subgroups in a lattice-ordered abelian group G forms an x-system if G is considered as a semi-group under the operation $a \circ b = |a| \wedge |b|$. The x-system on S is said to be of finite character if the set-theoretic union of any chain of x-ideals is again an x-ideal. Two semigroups S and T, each with x-systems denoted respectively by y and z are said to be (y,z)-isomorphic if there exists a semigroup-isomorphism φ of S onto T such that $\varphi(A_y) = (\varphi(A))_z$ and $\varphi^{-1}(B_z) = (\varphi^{-1}(B))_y$ for $A \subseteqq S$, $B \subseteqq T$. For the theory of x-ideals and further special cases, we refer to [1].

3. Characteristic semigroups of functions.

Let X be a topological space, and let $S(X)$ denote a commutative semigroup (with respect to some operation) of functions from X into a set T. Let $S(X)$ be equipped with an x-system, and denote by $\mathscr{I}$ some family of x-ideals in $S(X)$.

We shall say that $S(X)$ is a *characteristic semigroup of functions with respect to* $\mathscr{I}$ if the following conditions are satisfied:

(1) To every $A_x \in \mathscr{I}$ there is associated one and only one element a in X. We write $A_x \sim a$. Put

$$\mathscr{A}(B) = \{A_x \in \mathscr{I};\ \exists\, b \in B,\ A_x \sim b\}$$

for $B \subseteqq X$.

(2) $\mathscr{A}(\{a\})$ is non-empty for every $a \in X$.

For two ideals A_x and B_x in $\mathscr{I}$ there exists an element a in X such that

(3) $A_x \sim a$ and $B_x \sim a$ if and only if there exists an ideal $C_x \in \mathscr{I}$ such that $C_x \subseteqq A_x \cap B_x$.

We write $A_x \approx B_x$ if $A_x \sim a$ and $B_x \sim a$ for some $a \in X$. Clearly this defines an equivalence relation in $\mathscr{I}$. If A_x, B_x, C_x and a satisfy (3), then $C_x \sim a$.

Now, let $S(X_i)$ be a characteristic semigroup of functions from X_i into T with respect to the family $\mathscr{I}_i$ of x_i-ideals in $S(X_i)$; $i = 1,2$. If φ: $S(X_1) \to S(X_2)$ is an (x_1,x_2)-isomorphism of $S(X_1)$ onto $S(X_2)$ such that $\varphi(\mathscr{I}_1) = \mathscr{I}_2$, then there exists a bijective transformation Φ: $X_1 \to X_2$. In fact, for $a_1 \in X_1$, denote by $[A_{x_1}]$ the equivalence class of all ideals in $\mathscr{I}_1$ associated to a_1. By (3) the equivalence classes of $\mathscr{I}_1$ and $\mathscr{I}_2$ are in 1–1 correspondence by φ, in particular $[A_{x_1}]$ is transferred by φ to some $[A_{x_2}]$ in $\mathscr{I}_2$. There exists by the definition of $\approx$ an a_2 in X_2 such that $[A_{x_2}]$ is the totality of ideals in $\mathscr{I}_2$ associated to a_2. Put $\Phi(a_1) = a_2$.

The conditions (1), (2) and (3) do not deal with the topology on X, and of course a characteristic semigroup of functions in the above sense does not determine the topological space X up to homeomorphism. To establish the correspondence between the topology on X and the algebraic structure on $S(X)$, it is necessary to add some new conditions. The following two conditions (4) and (5), or their duals, (6) and (7), seem to be the appropriate ones:

(4) Let F be closed in X, $a \notin F$ and let A_x be an ideal in $\mathscr{I}$ associated to a. Then $A_x \not\supseteq \bigcap \mathscr{A}(F)$.[1])

(5) For every a in X it is possible to choose an ideal $R_x(a)$ associated to a such that the following implication holds for every $f \in S(X)$ and every $B \subseteq X$:

$$f \in R_x(b) \text{ for every } b \in B \Rightarrow f \in R_x(b) \text{ for every } b \in \bar{B}\,.$$

(6) Let F be closed in X, $a \notin F$, and let A_x be an ideal in $\mathscr{I}$ associated to a. Then $A_x \not\subseteq \bigcup \mathscr{A}(F)$.

(7) For every a in X it is possible to choose an ideal $R_x(a)$ associated to a such that the following implication holds for every $f \in S(X)$ and every $B \subseteq X$:

$$f \notin R_x(b) \text{ for every } b \in B \Rightarrow f \notin R_x(b) \text{ for every } b \in \bar{B}\,.$$

Lemma. *Let $S(X)$ be a characteristic semigroup of functions from X into T with respect to $\mathscr{I}$. Let $B \subseteq X$. If $S(X)$ satisfies* (4) *and* (5), *then*

$$a \in \bar{B} \Leftrightarrow \exists\, A_x \sim a, \qquad A_x \supseteq \bigcap \mathscr{A}(B)\,, \tag{8}$$

and if $S(X)$ satisfies (6) *and* (7), *then*

$$a \in \bar{B} \Leftrightarrow \exists\, A_x \sim a, \qquad A_x \subseteq \bigcup \mathscr{A}(B)\,. \tag{9}$$

Proof. Assume $a \in \bar{B}$. We note that $R_x(b) \in \mathscr{A}(B)$ for every $b \in B$. If (4) and (5) are satisfied,

$$f \in \bigcap \mathscr{A}(B) \Rightarrow f \in R_x(b) \text{ for every } b \in B \Rightarrow f \in R_x(a)\,.$$

On the other hand, if (6) and (7) are satisfied, then

$$f \notin \bigcup \mathscr{A}(B) \Rightarrow f \notin R_x(b) \text{ for every } b \in B \Rightarrow f \notin R_x(a)\,.$$

Thus $\Rightarrow$ is proved in (8) and (9). Conversely, assume $a \notin \bar{B}$. The re-

[1] We put $\bigcap \mathscr{A}$ for $\bigcap_{A_x \in \mathscr{A}} A_x$ and $\bigcup \mathscr{A}$ for $\bigcup_{A_x \in \mathscr{A}} A_x$.

maining part of (8) then follows from (4), the remaining part of (9) from (6).

We may now prove the following

THEOREM. *Let $S(X_i)$ be characteristic semigroups of functions from X_i into T with respect to the family $\mathscr{I}_i$ of x_i-ideals in $S(X_i)$, $i=1,2$. If $\varphi\colon S(X_1)\to S(X_2)$ is an (x_1,x_2)-isomorphism of $S(X_1)$ onto $S(X_2)$, such that $\varphi(\mathscr{I}_1)=\mathscr{I}_2$, and if $S(X_i)$ satisfies either* (4) *and* (5) *for $i=1,2$ or satisfies* (6) *and* (7) *for $i=1,2$, then X_1 and X_2 are homeomorphic.*

PROOF. We shall show that Φ as defined above is a homeomorphism under the assumptions of the theorem, i.e., that for every $B_1\subseteq X_1$, $\Phi(\bar{B})=\overline{\Phi(B_1)}$. We first note that

$$\mathscr{A}(\Phi(B_1)) = \varphi(\mathscr{A}(B_1)) . \tag{10}$$

This follows at once by the definition of Φ. In fact, for $E_{x_2}\in\mathscr{A}(\Phi(B_1))$ there exists $D_{x_1}\in\mathscr{I}_1$ such that $E_{x_2}=\varphi(D_{x_1})$. Let $E_{x_2}\sim\Phi(b_1)$, $b_1\in B_1$, and let $D_{x_1}\sim a_1$. Then $\Phi(a_1)=\Phi(b_1)$, and $a_1=b_1$ since Φ is bijective. Thus $D_{x_1}\in\mathscr{A}(B_1)$. On the other hand, let $D_{x_1}\in\mathscr{A}(B_1)$. By the definition of Φ,

$$E_{x_2} = \varphi(D_{x_1}) \sim \Phi(b_1), \quad \text{where} \quad D_{x_1}\sim b_1\in B_1 .$$

Assume now that (4) and (5) are satisfied. Then by the lemma

$$\begin{aligned} a_1\in\bar{B}_1 \;&\Leftrightarrow\; \exists\, A^{(1)}_{x_1}\sim a_1, \quad A^{(1)}_{x_1}\supseteq \bigcap\mathscr{A}(B_1) \;\Leftrightarrow\; \exists\, A^{(1)}_{x_1}\sim a_1 ,\\ &\varphi(A^{(1)}_{x_1}) \supseteq \varphi\big(\bigcap\mathscr{A}(B_1)\big) = \bigcap\varphi(\mathscr{A}(B_1)) \;\Leftrightarrow\; \exists\, A^{(2)}_{x_2}\sim\Phi(a_1) ,\\ &A^{(2)}_{x_2}\supseteq \bigcap\mathscr{A}(\Phi(B_1)) \;\Leftrightarrow\; \varphi(a_1)\in\overline{\Phi(B_1)} . \end{aligned}$$

Finally assume that (6) and (7) are satisfied. Then

$$\begin{aligned} a_1\in\bar{B}_1 \;&\Leftrightarrow\; \exists\, A^{(1)}_{x_1}\sim a_1, \quad A^{(1)}_{x_1}\subseteq \bigcup\mathscr{A}(B_1) \;\Leftrightarrow\; \exists\, A^{(1)}_{x_1}\sim a_1 ,\\ &\varphi(A^{(1)}_{x_1}) \subseteq \varphi\big(\bigcup\mathscr{A}(B_1)\big) = \bigcup\varphi(\mathscr{A}(B_1)) \;\Leftrightarrow\; \exists\, A^{(2)}_{x_2}\sim\Phi(a_1) ,\\ &A^{(2)}_{x_2}\subseteq \bigcup\mathscr{A}(\Phi(B_1)) \;\Leftrightarrow\; \Phi(a_1)\in\overline{\Phi(B_1)} . \end{aligned}$$

This completes the proof.

4. Special cases.

A characteristic semigroup of functions satisfying (4) and (5) represents a generalization of the concept of a characteristic semigroup of functions with an x-system of finite character introduced in [1]. We show below that the finite character assumption is redundant.

Let X denote a compact Hausdorff-space, and $S(X)$ a semigroup (with respect to some operation) of complex continuous functions defined on X. The semigroup $S(X)$ is equipped with an x-system. In [1] $S(X)$ is referred to as a characteristic semigroup of functions if the following two conditions are satisfied:

(11) To every closed $F \subseteq X$ and $a \notin F$ there exists $h \in S(X)$ such that $h(a) \neq 0$ and $h(b)=0$ for every $b \in F$.

(12) A subset in $S(X)$ is a maximal x-ideal if and only if it is of the form $R_x(a)=\{f \in S(X);\ f(a)=0\}$ for some $a \in X$.

Put $R_x(a) \sim a$. Here the correspondence between the maximal x-ideals and X given by $\sim$ is 1–1 by (11), and (1), (2) and (3) follow. Condition (4) follows by (11), (5) is satisfied since the functions in $S(X)$ are continuous. We get the following (Theorem 30 in [1])

Corollary. *If for two compact Hausdorff spaces X and Y, semigroups $S(X)$ and $S(Y)$ of continuous, complex functions satisfying* (11) *and* (12) *with an x,y-system, respectively, are (x,y)-isomorphic, then X and Y are homeomorphic.*

In the next two examples we turn to characteristic semigroups of functions for more general classes of topological spaces than the above compact Hausdorff spaces.

Example 1. (Hewitt [4]). Denote by $C(X)$ the ring of all continuous, real functions defined on a topological space X. (Pointwise operations in $C(X)$). Assume that X is completely regular and real compact, i.e., that X is a completely regular space such that every free maximal ideal in $C(X)$ is hyperreal ([2]). Denote by $\mathscr{I}$ the set of all real ideals in $C(X)$. Since X is real compact, clearly

$$\mathscr{I} = \{M(a)\}_{a \in X}, \quad \text{where} \quad M(a) = \{f \in C(x);\ f(a)=0\}.$$

Put $M(a) \sim a$. As above, (1), (2) and (3) are satisfied. Since X is completely regular, (4) is satisfied, and since the functions in $C(X)$ are continuous, (5) follows. Clearly the image by an isomorphism of a real ideal is again a real ideal, and we conclude by the theorem that a completely regular real compact space is determined to within homeomorphism by the ring of continuous, real functions defined on it.

Example 2. (Pursell [7]). Let $R(X)$ be a ring of functions from the regular space X to a field K (Pursell assumes only that K is a division ring) such that

(13) $Z(f)$ is closed for every $f \in R(x)$.

(14) F closed in X and $a \notin F \Rightarrow \exists f \in R(X)$ such that $a \notin Z(f)$ and $Z(f)$ contains a neighbourhood of F.

(15) If $f \in R(X)$ does not vanish on the closed set F, then there exists a function $g \in R(X)$ such that $f(a)g(a)=1$ for $a \in F$.

(16) For each $x \in X$ there exists a function $f \in R(X)$ such that $Z(f)=\{x\}$.

Under these conditions, the maximal fixed ideals in $R(X)$ may be given an algebraic characterization (see [7]). This means that if $R(X)$ and $R(Y)$ are rings of functions from the regular spaces X, Y into the fields K and K', and if $R(X)$, $R(Y)$ satisfy (13)–(16), then the family

$$\mathscr{I} = \{\{f \in R(X);\ f(a)=0\}\}, \quad a \in X ,$$

is preserved by an isomorphism of $R(X)$ into $R(Y)$ in the sense that

$$\varphi(\mathscr{I}) = \{\{f \in R(Y);\ f(b) = 0\}\}, \quad b \in Y .$$

With $\sim$ defined as in Example 1, conditions (1), (2) and (3) are obvious, (4) follows from (14), and (5) is equivalent to

$$Z(f) \supseteq B \Rightarrow Z(f) \supseteq \bar{B} \qquad \text{for} \qquad B \subseteq X,\ f \in R(X) .$$

This follows from (13). We conclude that X and Y are homeomorphic.

Example 3. (Milgram [6]). Let X be a compact Hausdorff space, and let $S(X)$ be the semigroup of all continuous real functions defined on X, under pointwise multiplication. An 0-ideal I in $S(X)$ is a semigroup-ideal in $S(X)$ which satisfies:

(17) To each $f \in I$ there corresponds a g in $S(X)$, $g \neq 0$, such that $gf=0$. (0 denotes the zero functions.)

(18) For f_1, f_2 in I there exists e_{12} in $S(X)$ such that $e_{12}f_1=f_1$ and $e_{12}f_2=f_2$.

The set of 0-ideals is preserved under a semigroup isomorphism. Furthermore, there is a 1–1 correspondence between the closed subsets of X and the 0-ideals in $S(X)$, the 0-ideal $I(F)$ corresponding to $F \subseteq X$ being the collection of the functions f in $S(X)$ vanishing on some neighbourhood V_f of F. Clearly the maximal 0-ideals are those corresponding to points. Now, let $S(X)$ be equipped with the x-system of the semi-group-ideals, let $\mathscr{I}$ denote the set of all maximal 0-idelas, and put, for $I \in \mathscr{I}$, $I \sim a$ if I corresponds to a in the above sense. Clearly (1), (2) and (3) are satisfied. If F is closed and $a \notin F$, there exists a closed neighbourhood V of a such that $V \cap F = \emptyset$. By Urysohn's lemma we find $h \in S(X)$ such that $h(b)=0$ for $b \in V$, $h(c)=1$ for $c \in F$, and (6) is satisfied. Finally, if $f \in S(X)$ vanishes on a neighbourhood V of $a' \in \bar{A}$, then there exists an interior point a in V such that $a \in A$, and f vanishes on a neighbourhood of a. Thus (7) is satisfied. We thus conclude that a compact Hausdorff space is determined to within homeomorphism by the semigroup of all continuous real functions defined on it.

EXAMPLE 4. (Kaplansky [5]). Denote by $L(X)$ the lattice of all continuous real functions on the compact Hausdorff space X. Choose some $f_0 \in L(X)$ and denote by $\mathscr{I}$ the set of all proper, prime l-ideals in $L(X)$ which contain f_0. Put $P_l \sim a$ if $f \in P_l$ and $g(a) < f(a)$ imply $g \in P_l$. Lemma 3 of [4] expresses that (1) is satisfied, Lemma 4 and 5 that (3) is satisfied. (2) follows by the fact that for every $a \in X$,

$$P_l(a) = \{f \in L(X);\ f(a) \leqq f_0(a)\} \in \mathscr{I}$$

and $P_l(a) \sim a$. Let F be a closed subset of X, $a \notin F$, and assume that $Q_l \sim a$, $Q_l \in \mathscr{I}$. Then since $Q_l \neq L(X)$, $f(a) < M$ for every $f \in Q_l$ for some $M < \infty$. Since X is compact, $f_0(b) > m$ for every $b \in F$ for some $m > -\infty$. There exists $h \in L(X)$ such that $h(a) = M$, $h(b) = m$ for every $b \in F$, thus $h \notin Q_l$ and $h \in \bigcap \mathscr{A}(F)$ and (4) is satisfied. (5) follows since every $f \in L(X)$ is continuous. If X and Y are two compact Hausdorff spaces such that $L(X)$ and $L(Y)$ are isomorphic as lattices under $\varphi: L(X) \to L(Y)$, choose $f_0 \in L(X)$ and define $\mathscr{I}$ as above. Then

$$\varphi(\mathscr{I}) = \{P_l \in L(Y);\ P_l \text{ proper prime } l\text{-ideal and } \varphi(f_0) \in P_l\}\,.$$

We conclude that $L(Y)$ is a characteristic semigroup of functions, with respect to $\varphi(\mathscr{I})$, which satisfies (4) and (5), and X and Y are homeomorphic.

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UNIVERSITY OF OSLO, NORWAY