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A NOTE ON THE RADICAL OF THE SECOND CONJUGATE ALGEBRA OF A SEMIGROUP ALGEBRA

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1. Introduction.

Let S be a discrete semigroup, $\ell_1(S)$ its semigroup algebra, $m(S)$ the space of bounded real functions on S with usual sup norm and $m(S)^*$ the conjugate Banach space of $m(S)$. Then, as known, $m(S)^*$ becomes a Banach algebra under convolution as multiplication (see Day [2, pp. 526–530]).

The semigroup S is said to be left amenable if there exists a non-negative φ in $m(S)^*$, of norm 1, such that $\varphi(f_s)=\varphi(f)$ for any f in $m(S)$ and s in S, where $(f_s)(t)=f(st)$ for any s,t in S and f in $m(S)$. As known, any abelian semigroup, any finite or any solvable or any locally finite group are left amenable. For these and more examples of left amenable semigroups see Day [2, pp. 515–518]. We refer also to Day [2] for any notation used in this paper.

Let $\operatorname{rad} m(S)^*$ denote the radical of the algebra $m(S)^*$ and $\dim\operatorname{rad} m(S)^*$ denote its algebraic dimension, $\dim\operatorname{rad} m(S)^*=0$ denoting the semi-simplicity of $m(S)^*$.

Let S_n denote the semigroup whose elements are $\{e_1,\ldots,e_n\}$, the multiplication in S_n being defined by $e_ie_j=e_j$ for any $1\leqq i,j\leqq n$. If G is a semigroup then $G\times S_n$ will denote the usual direct product of G and S_n.

Our main result is:

THEOREM 1. *Let S be a left cancellation left amenable semigroup. Then* $\dim\operatorname{rad} m(S)^*=N$, $0\leqq N<\infty$, *if and only if $S=G\times S_n$, where G is any finite group whose order k and n satisfy $N=(n-1)k$. (Hence* $\dim\operatorname{rad} m(S)^*$ $<\infty$ *implies that S is finite.)*

COROLLARY 1. If S is a left amenable group then $\dim\operatorname{rad} m(S^*)$ is either 0 or ∞ and $\dim\operatorname{rad} m(S)^*=0$ if and only if S is a finite group.

For the case where S is the additive group of integers, it has been proved by Civin and Yood [1] that $m(S)^*$ has an infinite dimensional radical. This has been conjectured by them to hold for any infinite abe-

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lian group (see [1, p. 853]). Corollary 1, which is an immediate consequence of Theorem 1, has been proved by the first author in [3, p. 48]. It shows that the above conjecture holds true even for any left amenable infinite group.

COROLLARY 2. Let S be a left amenable left cancellation semigroup. Then $\dim \operatorname{rad} m(S)^* = p$ where p is a prime if and only if S is either S_{p+1} or $S = G \times S_2$ where G is the cyclic group of order p. (Use the fact that $p = (n-1)k$.) Under the same assumption on S $\dim \operatorname{rad} m(S)^* = 1$ if and only if $S = S_2$. (Hence S_2 is the unique left amenable left cancellation semigroup for which $m(S)^*$ has a one dimensional radical.)

COROLLARY 3. Let S be an abelian semigroup with cancellation. Then $\dim \operatorname{rad} m(S)^*$ is either 0 or ∞ and $\dim \operatorname{rad} m(S)^* = 0$ if and only if S is an abelian finite group [4, p. 379].

In fact if S is a finite group then $m(S)^* = \ell_1(S)$ and $\ell_1(S)$ is semisimple. If $\dim \operatorname{rad} m(S)^* = N < \infty$ then S is finite and $S = G \times S_n$. But $n = 1$, since if $n \geqq 2$ then

$$(e, e_1)(e, e_2) = (e, e_2) \neq (e, e_1) = (e, e_2)(e, e_1),$$

(where e is the identity of G) i.e. S is not abelian. Hence $N = (n-1)k = 0k = 0$ and $S = G \times S_1$ which is isomorphic to G.

COROLLARY 4. Let S be a left amenable left cancellation semigroup. Then $m(S)^*$ is semisimple if and only if S is a finite group. (Use $N = (n-1)k = 0$.) It should be noted that in general, finite left amenable left cancellation semigroups need not be even embeddable in groups, as the example S_2 shows.

2. The main theorem.

If $S = G \times S_n$, where $G = \{g_1, \ldots, g_k\}$ is a finite group, we denote by $1_{g_i e_j}$ that element of $\ell_1(G \times S_n)$ which is one on $(g_i, e_j) \in G \times S_n$ and zero on any other element of $G \times S_n$.

LEMMA 1. *Let G be a finite group containing k elements and $S = G \times S_n$. Then* $\dim \operatorname{rad}[\ell_1(S)] = (n-1)k$ *and* $\operatorname{rad} \ell_1(S)$ *coincides with the left annihilator of* $\ell_1(S)$.

PROOF. Let $A_0 \subset \ell_1(S)$ be the linear span of $\{1_{ge_i} - 1_{ge_1} : i \geqq 2, g \in G\}$. This last set contains $(n-1)k$ elements and is linearly independent. In fact if $G = \{g_1, \ldots, g_k\}$ with $g_1 = e$ the identity, and if

$$\varphi = \sum_{i=1}^{k} \alpha_{i2}[1_{g_ie_2} - 1_{g_ie_1}] + \ldots + \sum_{i=1}^{k} \alpha_{in}[1_{g_ie_n} - 1_{g_ie_1}] = 0 ,$$

then

$$\left[\sum_{i=1}^{k} \alpha_{i2} i_{g_ie_2} + \ldots + \sum_{i=1}^{k} \alpha_{in} 1_{g_ie_n}\right] - \sum_{i=1}^{k} (\alpha_{i2} + \ldots + \alpha_{in}) 1_{g_ie_1} = 0 .$$

The carrier of the last term in this difference is included in $G \times e_1$ while the carrier of the first term in the above difference is included in $\bigcup_{i=2}^{n} G \times e_i$. Hence the first term in this difference has to be zero which implies immediately by the same argument that $\alpha_{ij} = 0$ for $1 \leqq i \leqq k$ and $2 \leqq j \leqq n$. This shows that $\dim A_0 = (n-1)k$. Let now

$$A = \{\varphi \in \ell_1(S) : \varphi \odot \psi = 0 \text{ for any } \psi \in \ell_1(S)\}$$

be the left annihilator of $\ell_1(S)$, where $\odot$ stands for the convolution multiplication in $\ell_1(S)$. Then $A_0 \subset A$ and A is a two sided ideal such that $A^2 = \{0\}$. Hence $A_0 \subset A \subset \operatorname{rad} \ell_1(S)$ (see N. Jacobson [6, p. 39, Corollary 1]) which shows that $\dim \operatorname{rad} \ell_1(S) \geqq (n-1)k$.

On the other hand the homomorphism P: $G \times S_n \to G$ defined by $P(g, e_i) = g$ induces an algebra homomorphism from $\ell_1(G \times S_n)$ onto $\ell_1(G)$. Since $\ell_1(G)$ is k dimensional, the kernel K of this homomorphism is $nk - k$ dimensional. But by [6, p. 10, Proposition 1], $\operatorname{rad} \ell_1(S) \subset K$ since $\ell_1(G)$ is semisimple, as known. Hence

$$(n-1)k = \dim A_0 \leqq \dim A \leqq \dim \operatorname{rad} \ell_1(S) \leqq \dim K = (n-1)k .$$

Hence $A_0 = A = \operatorname{rad} \ell_1(S)$ and $\dim \operatorname{rad} \ell_1(S) = (n-1)k$. [1]

Remark. The radical of arbitrary semigroup algebras $\ell_1(S)$, for finite S has been characterised by E. Hewitt and H. Zuckerman in [5, p. 108, Theorem 5.20]. However, the above lemma does not seem to follow in an obvious way from their characterisation and its proof is straightforward.

Theorem 1. *Let S be a left cancellation left amenable semigroup. Then* $\dim \operatorname{rad} m(S)^* = N$, $0 \leqq N < \infty$, *if and only if* $S = G \times S_n$ *where* G *is any finite group whose order* k *and* n *satisfy* $N = (n-1)k$.

Proof. Lemma 1 proves the "if" part of this theorem. The "only if" part is proved as follows:

By Civin and Yood [1, pp. 849–850],

$$J_1 = \{\varphi \in m(S)^* : \varphi(1) = 0, \ \varphi(f_s) = \varphi(f) \text{ for any } f \in m(S) \text{ and } s \in S\}$$

(where $1 \in m(S)$ is the constant 1 function) is a two sided ideal such that

[1] Thanks are due to S. Chase for a simplification of the original proof of this lemma.

$J_1^2 = \{0\}$ and hence is included in the radical of $m(S)^*$. If φ_0 is a fixed element of the set of left invariant means on S (denoted by $M\ell(S) \subset m(S)^*$, see [3, p. 33]) then for any φ in $M\ell(S)$,

$$\varphi = (\varphi - \varphi_0) + \varphi_0 \quad \text{with} \quad \varphi - \varphi_0 \in J_1$$

and hence $M\ell(S) \subset J_1 + \varphi_0$. The assumption that $\dim \operatorname{rad} m(S)^* = N < \infty$ implies that $\dim J_1 < \infty$ and so $\dim\{M\ell(S)\} < \infty$, where $\{M\ell(S)\}$ stands for the linear span of $M\ell(S)$. By [3, p. 56, Theorem E; see also footnote on p. 56] S has to be finite and $S = G \times S_n$ for some finite group G, whose order we denote by k. Hence $m(S)^* = \ell_1(G \times S_n)$ and

$$N = \dim \operatorname{rad} \ell_1(G \times S_n) = (n-1)k$$

by Lemma 1.

REMARKS. (1) One would like to replace in Theorem 1 the complicated algebra $m(S)^*$ by the more familiar one $\ell_1(S)$. However, this cannot be done even for groups G. In fact if G is an abelian countable (not finite) group then $\ell_1(G)$ is semisimple, that is, $\operatorname{rad} \ell_1(S) = \{0\}$ while according to Theorem 1 the radical of $m(G)^*$ is infinite dimensional.

(2) One cannot drop the assumption of left cancellation from Theorem 1 since there are finite abelian semigroups S for which $\ell_1(S) = m(S)^*$ is not semisimple (see Hewitt–Zuckerman [5, p. 108, Theorem 5.21]) and hence $\dim \operatorname{rad} m(S)^* = N$, $0 < N < \infty$, which contradicts Corollary 3.

(3) One would like to drop the assumption that S is left amenable, at least for groups, and prove that for *any* group G $\dim \operatorname{rad} m(G)^* = 0$ or ∞ and $\dim \operatorname{rad} m(G)^* = 0$ if and only if G is finite.

It is conjectured here that this statement holds true. The methods used in this paper do not seem to yield anything in this direction.

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