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SOME RESULTS CONCERNING A GENERAL SET THEORETICAL APPROACH TO LOGIC

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1. Introduction.

The meaning of compound propositions in a logical system is usually defined recursively by defining the truth or falsehood of propositions as dependent on the truth or falsehood of already meaningful propositions. The basis for the definition is a set of atomic propositions which are considered meaningful in advance. A relation of consequence is defined by stating that a proposition is a consequence of a set of propositions if it is true whenever all propositions of the set are true.

The aim of this paper is to study the general aspects of a similar approach to logic. The idea is also recursively to attach meanings to compound propositions and to define relations of consequence in accordance with these meanings.

The observation which leads to the theory is that the meaning of e.g. the conditional $b \Rightarrow c$ is given if $b \Rightarrow c$ is defined to be a proposition which together with b yields c. Of course many propositions may have this property, but no other one can be defined by this property alone.

So, in our concept of logic, a meaning is attached to the elements of a system of propositions by defining how each proposition together with already meaningful propositions implies other already meaningful propositions.

It is possible to give a definition of relations of consequence (derivations) which in a natural way expresses the above formulated idea. The main topics of the paper are the minimal derivations, which correspond to the intuitionistic logic, the weak classical derivations, which are the smallest derivations with the property that the double negation of a proposition always implies the proposition itself, and finally the relations of consequence derived by the above mentioned "truth-table" method. The technique of characterizing relations of consequence as minimal relations (derivations) having certain properties is due to L. Henkin [2].

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A general set-theoretical presentation is used in order to simplify notation and proofs. All propositions are considered elements of a set-theory, and the compound propositions are defined in the following way: If Δ_i, $i \in I$, are sets of propositions and y_i, $i \in I$, are propositions, then $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$ is the proposition the meaning of which is derived from the fact that the proposition, for all $k \in I$, together with Δ_k implies y_k.

The main result is that the minimal, the weak classical, and the strong classical derivations can be given general deductive characterizations, which resemble the formal deductive systems of Gentzen [1]. These deductive characterizations seem well suited for a kind of model theoretical treatment of formal logic, which is roughly sketched in the last section.

I wish to express my gratitude to professor L. Henkin for his valuable help in preparing this paper.

2. Set-theoretical preliminaries.

All objects of the theory which will be presented in the following sections are elements of a set-theory of the von Neumann–Bernays–Gödel-type with the axiom of regularity and possibly with individuals.

Below are listed some conventions which are not generally used or understandable:

V is the universal class.
$\emptyset$ is the empty set.
$S(A)$ is the class of all subsets of A.
$\langle a,b \rangle = \{\{a,b\},\{b\}\}$.
$\langle a,b,c \rangle = \langle \langle a,b \rangle, c \rangle$.
$\times\{a_\nu \mid \nu \in N\}$ = the class of functions from N to V such that $f(\nu) \in a_\nu$.

A very important convention is the following: An element $a \in V$ is called *basic with respect to a predicate* Φ if $\Phi(a)$ and $\Phi(x)$ does not hold for any x which is an element of the transitive closure $a \cup Ua \cup UUa \cup \ldots$ of a. It is a consequence of the normally used axioms of set theory that, if Φ is satisfied by some $b \in V$, then there exists an a which is basic with respect to Φ.

3. Fundamental definitions.

DEFINITION 3.1. An element x of the universe V is called *compound* if $x \in S(S(V) \times V)$, that is, if x is a set of the form $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$, where $\Delta_k \subseteq V$ for all $k \in I$. For any class K, let $C(K)$ be the class of all compound elements of K.

DEFINITION 3.2. An element x of the universe V is called *atomic* if $x \notin C(V)$. For any class K, let $A(K)$ be the class of all atomic elements of K.

DEFINITION 3.3. A *prelogic* is a set L with the *property* that

$$\{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$$

implies $\Delta_k \subseteq L$, $y_k \in L$ for all $k \in I$.

NOTATION. The letter L will always denote a prelogic. For the relations $\vdash$, $\Vdash$, etc., we shall use the normal notation, writing $\Gamma \vdash x$ instead of $\langle \Gamma, x \rangle \in \vdash$, $\Gamma, \Delta \vdash x$ instead of $\langle \Gamma \cup \Delta, x \rangle \in \vdash$, $\Gamma, y \vdash x$ instead of $\langle \Gamma \cup \{y\}, x \rangle \in \vdash$, etc.

DEFINITION 3.4. The relation $\vdash$ is a *derivation on* L, if $\vdash \subseteq S(L) \times L$ and the following four conditions are satisfied:

(0) If $\Gamma \vdash z$ for all $z \in \Delta$ and $\Delta \vdash x$, then $\Gamma \vdash x$.
(1) If $x \in \Gamma \subseteq L$, then $\Gamma \vdash x$.
(2) If $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$, then $x, \Delta_k \vdash y_k$ for all $k \in I$.
(3) If $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$ and $\Gamma, \Delta_k \vdash y_k$ for all $k \in I$, then $\Gamma \vdash x$.

Conditions (0) and (1) express fundamental properties of any relation of consequence. Condition (2) expresses that $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$ is to be considered as a proposition, the assumption of which enables one, for each $k \in I$, to draw the conclusion y_k provided all propositions of Δ_k are assumed. Condition (3) expresses that the meaning of $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$ is to be derived solely from this property; that is, if the set of propositions Γ permits one, for each $k \in I$, to draw the conclusion y_k provided all propositions of Δ_k are assumed, then $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$ is itself a consequence of Γ.

The following three definitions are listed at this point in order to facilitate the interpretation of the theory. Notice that the defined sets are not necessarily elements of the prelogic L.

DEFINITION 3.5. If $x \in V$, then $\neg_L x = \{\langle \{x\}, y \rangle \mid y \in A(L)\}$.

DEFINITION 3.6. If $x, y \in V$, then

$$x \Rightarrow y = \{\langle \{x\}, y \rangle\},$$
$$x \wedge y = \{\langle \emptyset, x \rangle, \langle \emptyset, y \rangle\},$$
$$x \vee_L y = \{\langle \{x \Rightarrow z, y \Rightarrow z\}, z \rangle \mid z \in A(L)\}.$$

DEFINITION 3.7. If $\Gamma \in S(V)$, then

$$\forall \Gamma = \{\langle \emptyset, x\rangle \mid x \in \Gamma\},$$
$$\exists_L \Gamma = \{\langle\{x \Rightarrow y \mid x \in \Gamma\}, y\rangle \mid y \in A(L)\}.$$

Below are listed without proof five simple consequences of the definitions. In the later sections these propositions will generally be used without any references.

PROPOSITION 3.1. *If* $\vdash$ *is a derivation on* L *and* $\Gamma' \subseteq \Gamma \subseteq L$, *then* $\Gamma' \vdash x$ *implies* $\Gamma \vdash x$.

PROPOSITION 3.2. *If* $\vdash$ *is a derivation on* L, $\langle \Delta, y\rangle \in \bigcup C(\Gamma)$, $\Gamma \vdash z$ *for all* $z \in \Delta$, *and* $\Gamma, y \vdash x$, *then* $\Gamma \vdash x$.

PROPOSITION 3.3. *If* $\vdash$ *is a derivation on* L *and*

$$\Gamma \vdash \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L),$$

then $\Gamma, \Delta_k \vdash y_k$ *for all* $k \in I$.

PROPOSITION 3.4. *The intersection of a non-empty set of derivations on* L *is a derivation on* L.

PROPOSITION 3.5. $S(L) \times L$ *is a derivation on* L.

4. The minimal derivations.

DEFINITION 4.1. $\Vdash_L$, *the minimal derivation on* L, is the intersection of all derivations on L.

DEFINITION 4.2. A *minimal deduction* is a triple $\langle \Gamma_0, x_0, \pi_0\rangle$ which is an element of a set P of triples $\langle \Gamma, x, \pi\rangle$ each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $x \in \Gamma$.

(2) $\pi = \{\langle \Gamma, z, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma \cup \{y\}, x, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma)$.

(3) $\pi = \{\langle \Gamma \cup \Delta_i, y_i, \pi_i\rangle \mid i \in I\}$, where $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$.

DEFINITION 4.3. $\Gamma \Vdash x$ means that there exists a minimal deduction $\langle \Gamma, x, \pi\rangle$.

THEOREM 4.1. *The relation* $\Vdash$ *satisfies the conditions*:

(0) *If* $\Gamma \Vdash z$ *for all* $z \in \Delta$ and $\Delta \Vdash x$, *then* $\Gamma \Vdash x$.

(1) *If* $x \in \Gamma \in V$, *then* $\Gamma \Vdash x$.

(2) *If* $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$, *then* $x, \Delta_k \Vdash y_k$ *for all* $k \in I$.

(3) *If* $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$ *and* $\Gamma, \Delta_k \Vdash y_k$ *for all* $k \in I$, *then* $\Gamma \Vdash x$.

PROOF. (1) If $x \in \Gamma \in V$, then $\langle \Gamma, x, \emptyset \rangle$ is a minimal deduction.

(2) If $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$ and $k \in I$, then

$$\left\langle \{x\} \cup \Delta_k, y_k, \left\{\langle \{x\} \cup \Delta_k, z, \emptyset \rangle \mid z \in \Delta_k\right\} \cup \left\{\langle \{x\} \cup \Delta_k \cup \{y_k\}, y_k, \emptyset \rangle\right\}\right\rangle$$

is a minimal deduction.

(3) If $\pi = \{\langle \Gamma \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$ is a set of minimal deductions and $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$, then $\langle \Gamma, x, \pi \rangle$ is a minimal deduction.

(0) is proved by lemmas 4.2, 4.3 and 4.4.

LEMMA 4.2. $\Gamma' \subseteq \Gamma \in V$ and $\Gamma' \Vdash x$ imply $\Gamma \Vdash x$.

PROOF. Assume $\delta_0 = \langle \Gamma_0', x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma', x, \pi \rangle$ is a minimal deduction and there exists a $\Gamma \in V$ such that $\Gamma' \subseteq \Gamma$ and $\Gamma \Vdash x$ is false. Assume $\Gamma_0 \in V$, $\Gamma_0' \subseteq \Gamma_0$ and $\Gamma_0 \Vdash x_0$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0'$. In this case $\langle \Gamma_0, x_0, \emptyset \rangle$ is a minimal deduction.

(2) $\pi_0 = \{\langle \Gamma_0', z, \pi_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0' \cup \{y\}, x_0, \pi' \rangle\right\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0')$. There exists a set $\bar{\pi}_0 = \{\langle \Gamma_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0 \cup \{y\}, x_0, \bar{\pi}' \rangle\right\}$ of minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

(3) $\pi_0 = \{\langle \Gamma_0' \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. There exists a set $\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i \rangle \mid i \in I\}$ of minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

LEMMA 4.3. *$\Gamma \Vdash \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$ implies $\Gamma, \Delta_k \Vdash y_k$ for all $k \in I$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, x, \pi \rangle$ is a minimal deduction, $x \in C(V)$, and $\Gamma, \Delta \Vdash y$ is false for some $\langle \Delta, y \rangle \in x$. Assume $\langle \Delta_0, y_0 \rangle \in x_0$ and $\Gamma_0, \Delta_0 \Vdash y_0$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. In this case

$$\langle \Gamma_0 \cup \Delta_0, y_0, \{\langle \Gamma_0 \cup \Delta_0, z, \emptyset \rangle \mid z \in \Delta_0\} \cup \left\{\Gamma_0 \cup \Delta_0 \cup \{y_0\}, y_0, \emptyset \rangle\right\}\rangle$$

is a minimal deduction.

(2) $\pi_0 = \{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0 \cup \{y\}, x_0, \pi' \rangle\right\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. Using lemma 4.2, we get the existence of a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0 \cup \Delta_0 \cup \{y\}, y_0, \bar{\pi}' \rangle\right\}$$

of minimal deductions, and $\langle \Gamma_0 \cup \Delta_0, y_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\}$. There is a $k \in I$ such that $\langle \Delta_0, y_0 \rangle = \langle \Delta_k, y_k \rangle$, and $\langle \Gamma_0 \cup \Delta_0, y_0, \pi_k \rangle$ is a minimal deduction.

LEMMA 4.4. *If $\Gamma \Vdash z$ for all $z \in \Delta$ and $\Gamma, \Delta \Vdash x$, then $\Gamma \Vdash x$.*

PROOF. Assume Δ_0 basic with respect to Φ: $\Phi(\Delta)$ iff there exist Γ and x such that $\Gamma \Vdash x$ is false although $\Gamma \Vdash z$ for all $z \in \Delta$ and $\Gamma, \Delta \Vdash x$.

Assume $\delta_0 = \langle \Gamma_0 \cup \Delta_0, x_0, \pi_0 \rangle$ basic with respect to Ψ: $\Psi(\delta)$ iff $\delta = \langle \Gamma \cup \Delta_0, x, \pi \rangle$ is a minimal deduction such that $\Gamma \Vdash z$ for all $z \in \Delta_0$ and $\Gamma \Vdash x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0 \cup \Delta_0$. If $x_0 \in \Gamma_0$, then $\langle \Gamma_0, x_0, \emptyset \rangle$ is a minimal deduction. If $x_0 \in \Delta_0$, then $\Gamma_0 \Vdash x_0$ is assumed.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \Delta_0 \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in C(\Gamma_0) \cup C(\Delta_0)$. Using that δ_0 is basic with respect to Ψ and lemma 4.2, we get the existence of a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, x_0, \bar{\pi}' \rangle\}$$

of minimal deductions. If $\langle \Delta, y \rangle \in C(\Gamma_0)$, then $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction. If $\langle \Delta, y \rangle \in C(\Delta_0)$, then lemma 4.3 implies $\Gamma_0, \Delta \Vdash y$. Since $\Gamma_0 \Vdash z$ for all $z \in \Delta$ and Δ_0 is basic with respect to Φ, we get $\Gamma_0 \Vdash y$. Finally, since $\Gamma_0, y \Vdash x_0$, we get $\Gamma_0 \Vdash x_0$, using on $\{y\}$ that Δ_0 is basic with respect to Φ.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. Using that δ_0 is basic with respect to Ψ and lemma 4.2, we get the existence of a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i \rangle \mid i \in I\}$$

of minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

Theorem 4.5. *$\Gamma \Vdash_L x$ iff $\Gamma \subseteq L$, $x \in L$ and $\Gamma \Vdash x$.*

Proof. Let $\Gamma \vdash' x$ mean that $\Gamma \subseteq L$, $x \in L$ and $\Gamma \Vdash x$. It is an immediate consequence of theorem 4.1 that $\vdash'$ is a derivation on L. In order to prove that $\vdash' \subseteq \Vdash_L$, assume $\delta_0 = \langle \Gamma_0, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, x, \pi \rangle$ is a minimal deduction, $\Gamma \subseteq L$, $x \in L$ and $\Gamma \Vdash_L x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. Then $\Gamma_0 \Vdash_L x_0$.

(2) $\pi_0 = \{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. We have $\Gamma_0 \Vdash_L z$ for all $z \in \Delta$ and $\Gamma_0, y \Vdash_L x_0$, and proposition 3.2 gives $\Gamma_0 \Vdash_L x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. We have $\Gamma_0, \Delta_k \Vdash_L y_k$ for all $k \in I$, but then $\Gamma_0 \Vdash_L x_0$.

The conditions in definition 4.2 of minimal deductions can be strengthened in various ways without decreasing the class of pairs $\langle \Gamma, x \rangle$ for which deductions $\langle \Gamma, x, \pi \rangle$ can be found. The following definition can be further strengthened, but is sufficient for our purpose.

Definition 4.4. A *regular minimal deduction* is a triple $\langle \Gamma_0, x_0, \pi_0 \rangle$, which is an element of a set P of triples $\langle \Gamma, x, \pi \rangle$, each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi=\emptyset$ and $x\in A(\Gamma)$.

(2) $\pi=\{\langle\Gamma,z,\pi_z\rangle \mid z\in\Delta\}\cup\{\langle\Gamma\cup\{y\},x,\pi'\rangle\}$, where $\langle\Delta,y\rangle\in\bigcup C(\Gamma)$, and $y=x$ if $y\in A(V)$.

(3) $\pi=\{\langle\Gamma\cup\Delta_i,y_i,\pi_i\rangle \mid i\in I\}$, where $x=\{\langle\Delta_i,y_i\rangle \mid i\in I\}\in C(V)$.

PROPOSITION 4.6. *$\Gamma\Vdash x$ implies the existence of a regular minimal deduction $\langle\Gamma,x,\pi\rangle$.*

PROOF. Assume x_0 basic with respect to Φ: $\Phi(x')$ iff there exists a minimal deduction $\delta'=\langle\Gamma',x',\pi'\rangle$ which is basic with respect to Ψ: $\Psi(\delta)$ iff $\delta=\langle\Gamma,x,\pi\rangle$ is a minimal deduction, and no regular minimal deduction $\langle\Gamma,x,\pi''\rangle$ exists. Assume $\langle\Gamma_0,x_0,\pi_0\rangle$ is basic with respect to Ψ.

(1) $\pi_0=\emptyset$ and $x_0\in\Gamma_0$. If $x_0\in A(V)$, then $\langle\Gamma_0,x_0,\emptyset\rangle$ is a regular minimal deduction. If $x_0=\{\langle\Delta_i,y_i\rangle \mid i\in I\}$, then the assumption that x_0 is basic with respect to Φ implies the existence of sets

$$\pi_i = \{\langle\Gamma_0\cup\Delta_i,z,\pi_{i,z}\rangle \mid z\in\Delta_i\}\cup\{\langle\Gamma\cup\Delta_i\cup\{y_i\},y_i,\pi_i'\rangle\},\quad i\in I,$$

of regular minimal deductions, and

$$\langle\Gamma_0,x_0,\{\langle\Gamma_0\cup\Delta_i,y_i,\pi_i\rangle \mid i\in I\}\rangle$$

is a regular minimal deduction.

(2) $\pi_0=\{\langle\Gamma_0,z,\pi_z\rangle \mid z\in\Delta\}\cup\{\langle\Gamma_0\cup\{y\},x_0,\pi'\rangle\}$, where $\langle\Delta,y\rangle\in\bigcup C(\Gamma_0)$. There exists a set

$$\bar{\pi}_0 = \{\langle\Gamma_0,z,\bar{\pi}_z\rangle \mid z\in\Delta\}\cup\{\langle\Gamma_0\cup\{y\},x_0,\bar{\pi}'\rangle\}$$

of regular minimal deductions. If $y\in C(V)$ or $y=x$, then $\langle\Gamma_0,x_0,\bar{\pi}_0\rangle$ is a regular minimal deduction. If $y\in A(V)$ and $y\neq x$, then lemma 4.7 below can be used.

(3) $\pi_0=\{\langle\Gamma_0\cup\Delta_i,y_i,\pi_i\rangle \mid i\in I\}$, where $x_0=\{\langle\Delta_i,y_i\rangle \mid i\in I\}\in C(V)$. There exists a set $\bar{\pi}_0=\{\langle\Gamma_0\cup\Delta_i,y_i,\bar{\pi}_i\rangle \mid i\in I\}$ of regular minimal deductions, and $\langle\Gamma_0,x_0,\bar{\pi}_0\rangle$ is a regular minimal deduction.

LEMMA 4.7. *If $\pi=\{\langle\Gamma,z,\pi_z\rangle \mid z\in\Delta\}\cup\{\langle\Gamma\cup\{y\},x,\pi'\rangle\}$, where $\langle\Delta,y\rangle\in\bigcup C(\Gamma)$ and $y\in A(V)$, is a set of regular minimal deductions, then there exists a regular minimal deduction $\langle\Gamma,x,\pi''\rangle$.*

PROOF. Assume $\langle\Gamma_0\cup\{y_0\},x_0,\pi_0'\rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta=\langle\Gamma\cup\{y\},x.\pi'\rangle$ is a regular minimal deduction, $\langle\Delta,y\rangle\in\bigcup C(\Gamma)$, $y\in A(V)$, and there exist regular minimal deductions $\langle\Gamma,z,\pi_z\rangle$ for all $z\in\Delta$, but no regular minimal deduction $\langle\Gamma,x,\pi\rangle$.

Assume that $\{\langle\Gamma_0,z,\pi_z\rangle \mid z\in\Delta\}$ is a set of regular minimal deductions, and that no minimal deduction $\langle\Gamma_0,x_0,\pi_0\rangle$ exists.

(1) $\pi_0'=\emptyset$. If $x_0 \in \Gamma_0$, then $\langle \Gamma_0, x_0, \emptyset\rangle$ is a regular minimal deduction. If $x_0=y_0$, then

$$\langle \Gamma_0, x_0, \{\langle \Gamma_0, z, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y_0\}, x_0, \pi_0'\rangle\}\rangle$$

is a regular minimal deduction.

(2) $\pi_0'=\{\langle \Gamma_0 \cup \{y_0\}, u, \pi_u'\rangle \mid u \in \Delta'\} \cup \{\langle \Gamma_0 \cup \{y_0\} \cup \{v\}, x_0, \pi''\rangle\}$, where $\langle \Delta', v\rangle \in \bigcup C(\Gamma_0)$. Using the obvious generalization of lemma 4.2 to regular minimal deductions, we get the existence of a set

$$\bar{\pi}_0' = \{\langle \Gamma_0, u, \bar{\pi}_u'\rangle \mid u \in \Delta'\} \cup \{\langle \Gamma_0 \cup \{v\}, x_0, \bar{\pi}''\rangle\}$$

of regular minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0'\rangle$ is a regular minimal deduction.

(3) $\pi_0'=\{\langle \Gamma_0 \cup \Delta_i \cup \{y_0\}, y_i, \pi_i\rangle \mid i \in I\}$, where $x_0=\{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$. Using again the generalization of lemma 4.2 to regular minimal deductions, we get a set

$$\bar{\pi}_0' = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i\rangle \mid i \in I\}$$

of regular minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0'\rangle$ is a regular minimal deduction.

5. *q*-minimal derivations.

The following proposition is easily proved by theorem 4.1:

PROPOSITION 5.1. *If $q \in V$, then the relation $\Vdash_L^q$ defined by*

$$\Gamma \Vdash_L^q x \quad \textit{iff} \quad \Gamma \in L,\ x \in L \text{ and } \Gamma, q \Vdash x\,,$$

is a derivation on L.

DEFINITION 5.1. $\Vdash_L^q$ is called the q-minimal derivation on L.

THEOREM 5.2. *If $q=\{\langle \Gamma_\nu, x_\nu\rangle \mid \nu \in N\} \subseteq S(L) \times L$, then $\Vdash_L^q$ is the intersection of all derivations $\vdash$ on L having the property that $\Gamma_\nu \vdash x_\nu$ for all $\nu \in N$, and $\Vdash_L^q$ has this property itself.*

PROOF. Theorem 4.1 implies that Γ_ν, $q \Vdash x_\nu$ for all $\nu \in N$; therefore, all we have to do is to prove, that $\Vdash_L^q \subseteq \vdash$ for all derivations $\vdash$ on L satisfying $\Gamma_\nu \vdash x_\nu$ for all $\nu \in N$. Let $\vdash$ be such a derivation.

Assume $\delta_0=\langle \Gamma_0 \cup \{q\}, x_0, \pi_0\rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \{q\}, x, \pi\rangle$ is a minimal deduction, $\Gamma \subseteq L$, $x \in L$, and $\Gamma \vdash' x$ is false.

(1) $\pi_0=\emptyset$ and $x_0 \in \Gamma_0$. But then $\Gamma_0 \vdash' x_0$.

(2) $\pi_0=\{\langle \Gamma_0 \cup \{q\}, z, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\} \cup \{q\}, x_0, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma_0) \cup q$. We have $\Gamma_0 \vdash' z$ for all $z \in \Delta$, and $\Gamma_0, y \vdash' x_0$. If $\langle \Delta, y\rangle \in$

$\bigcup C(\Gamma_0)$, then proposition 3.2 implies $\Gamma_0 \vdash' x_0$. If $\langle\Delta,y\rangle \in q$, then $\Delta \vdash' y$, and consequently $\Gamma_0 \vdash' x_0$.

(3) $\pi_0 = \{\langle\Gamma_0 \cup \{q\} \cup \Delta_i, y_i, \pi_i\rangle \mid i \in I\}$, where $x_0 = \{\langle\Delta_i, y_i\rangle \mid i \in I\} \in C(V)$. Here $\Gamma_0, \Delta_k \vdash' y_k$ for all $k \in I$, and consequently $\Gamma_0 \vdash' x_0$.

6. Properties of the negation $\neg_L$.

PROPOSITION 6.1. *$x, \neg_L x \Vdash y$ for all $x \in V$ and $y \in L$.*

PROOF. Assume y_0 basic with respect to Φ: $\Phi(y)$ iff $y \in L$, and $x, \neg_L x \Vdash y_0$ is false.

If $y_0 \in A(L)$, then $\langle\{x\}, y_0\rangle \in \neg_L x$, and $x, \neg_L x \Vdash y_0$. If

$$y_0 = \{\langle\Delta_i, y_i\rangle \mid i \in I\} \in C(L)\,,$$

then $x, \neg_L x \Vdash y_k$ for all $k \in I$.

Lemma 4.2 gives $x, \neg_L x, \Delta_k \Vdash y_k$ for all $k \in I$, and consequently $x, \neg_L x \Vdash y_0$.

PROPOSITION 6.2. *$\Gamma, x \Vdash y$ implies $\Gamma, \neg_L y \Vdash \neg_L x$.*

PROOF. $\Gamma, \neg_L y, x \Vdash y$ and $\Gamma, \neg_L y, x \Vdash \neg_L y$; consequently $\Gamma, \neg_L y, x \Vdash z$ for all $z \in A(L)$, and this implies $\Gamma, \neg_L y \Vdash \neg_L x$.

PROPOSITION 6.3. *$x \Vdash \neg_L \neg_L x$ for all $x \in V$.*

PROOF. $x, \neg_L x \Vdash z$ for all $z \in A(L)$, and consequently $x \Vdash \neg_L \neg_L x$.

PROPOSITION 6.4. *$\neg_L \neg_L \neg_L x \Vdash \neg_L x$ for all $x \in V$.*

PROOF. Proposition 6.2 applied to $x \Vdash \neg_L \neg_L x$ gives

$$\neg_L \neg_L \neg_L x \Vdash \neg_L x\,.$$

PROPOSITION 6.5. *If $A(L)$ contains more that one element, then $\neg_L \neg_L x \Vdash x$ is false for all $x \in A(L)$.*

PROOF. Assume $\langle\{\neg_L \neg_L x\}, x, \pi\rangle$ is a regular minimal deduction, and $x \in A(L)$. Then

$$\pi = \{\langle\{\neg_L \neg_L x\}, \neg_L x, \pi'\rangle, \langle\{\neg_L \neg_L x, x\}, x, \pi''\rangle\}\,.$$

This implies $\neg_L \neg_L x \Vdash \neg_L x$, and we have $x, \neg_L \neg_L x \Vdash y$ for all $y \in A(L)$. Since $x \Vdash \neg_L \neg_L x$, we have $x \Vdash y$ for some $y \in A(L) \setminus \{x\}$, which contradicts the obvious fact that no minimal deduction $\langle\{x\}, y, \pi\rangle$ exists, where $x, y \in A(V)$ and $x \neq y$.

7. The weak classical derivations.

DEFINITION 7.1. With $w_L = \{\langle\{\neg_L \neg_L x\}, x\rangle \mid x \in A(L)\}$, the relation $\Vdash_L^{w_L}$ is called the *weak classical derivation on L.*

LEMMA 7.1. *$\neg_L \neg_L x, w_L \Vdash x$ for all $x \in L$.*

PROOF. Obvious if $x \in A(L)$. Assume $x_0 = \{\langle \Delta_i, x_i\rangle \mid i \in I\}$ basic with respect to Φ: $\Phi(x)$ iff $x \in L$ and $\neg_L \neg_L x, w_L \Vdash x$ is false. We have that $\Delta_k, x_0 \Vdash y_k$ for all $k \in I$. Using proposition 6.2 twice, we obtain $\Delta_k, \neg_L \neg_L x_0 \Vdash \neg_L \neg_L y_k$ for all $k \in I$, and since $\neg_L \neg_L y_k, w_L \Vdash y_k$ for all $k \in I$, we have $\neg_L \neg_L x_0, \Delta_k, w_L \Vdash y_k$ for all $k \in I$, and consequently $\neg_L \neg_L x_0, w_L \Vdash x_0$.

THEOREM 7.2. *If $\neg_L x \in L$ for all $x \in L$, then $\Vdash_L^{w_L}$ is the intersection of all derivations $\vdash$ on L having the property that $\neg_L \neg_L x \vdash x$ for all $x \in L$, and $\Vdash_L^{w_L}$ has this property itself.*

PROOF. From theorem 5.2 it follows that $\Vdash_L^{w_L}$ is the smallest derivation $\vdash$ with the property that $\neg_L \neg_L x \vdash x$ for all $x \in A(L)$; and from lemma 7.1 it follows that $\neg_L \neg_L x \Vdash_L^{w_L} x$ for all $x \in L$.

DEFINITION 7.2. A weak classical deduction is a triple $\langle \Gamma_0, \theta_0, \pi_0\rangle$, which is an element of a set P of triples $\langle \Gamma, \theta, \pi\rangle$ each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $\Gamma \cap \theta \neq \emptyset$.
(2) $\pi = \{\langle \Gamma, \theta \cup \{z\}, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma \cup \{y\}, \theta, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma)$.
(3) $\pi = \{\langle \Gamma \cup \Delta_i, \theta \cup \{y_i\}, \pi_i\rangle \mid i \in I\}$, where $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(\theta)$.

DEFINITION 7.3. $\Gamma \Vdash^w \theta$ means that there exists a weak classical deduction $\langle \Gamma, \theta, \pi\rangle$.

NOTATION. We shall write $\Gamma \Vdash^w x$ instead of $\Gamma \Vdash^w \{x\}$, $\Gamma \Vdash^w x, \theta$ instead of $\Gamma \Vdash^w \{x\} \cup \theta$, etc.

THEOREM 7.3. $\Gamma \Vdash_L^{w_L} x$ iff $\Gamma \subseteq L$, $x \in L$ and $\Gamma \Vdash^w x$.

PROOF. The theorem is proved by a series of lemmas of which lemma 7.7 and lemma 7.8 directly imply the theorem.

LEMMA 7.4. *$\Gamma_0 \Vdash^w \theta_0$ implies $\Gamma_0, \Gamma_1 \Vdash^w \theta_0, \theta_1$ for all $\Gamma_1, \theta_1 \subseteq V$.*

PROOF. Analogous to the proof of lemma 4.2.

LEMMA 7.5. *$\Gamma, \neg_L x \Vdash^w \theta, x$ implies $\Gamma \Vdash^w \theta, x$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\}, \pi_0\rangle$ basic with respect to Φ:

$\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \{\neg_L x\}, \theta \cup \{x\}, \pi \rangle$ is a weak classical deduction, and $\Gamma \Vdash^w \theta, x$ is false.

(1) $\pi_0 = \emptyset$ and $(\Gamma_0 \cup \{\neg_L x_0\}) \cap (\theta_0 \cup \{x_0\}) \neq \emptyset$. If $\neg_L x_0 \in \theta_0$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \{\langle \Gamma_0 \cup \{x_0\}, \theta_0 \cup \{x_0\} \cup \{y\}, \emptyset \rangle \mid y \in A(L)\} \rangle$ is a weak classical deduction. If $\neg_L x \notin \theta_0$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \emptyset \rangle$ is a weak classical deduction.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\} \cup \{z\}, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{\neg_L x_0\} \cup \{y\}, \theta_0 \cup \{x_0\}, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in UC\ (\Gamma_0 \cup \{\neg_L x_0\})$. We have a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, \theta_0 \cup \{x_0\} \cup \{z\}, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{x_0\}, \bar{\pi}' \rangle\}$$

of weak classical deductions. If $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$ is a weak classical deduction. If $\langle \Delta, y \rangle \in \neg_L x_0$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_{x_0} \rangle$ is a weak classical deduction.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \{\neg_L x_0\} \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \pi_i \rangle \mid i \in I\}$, $\{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(\theta_0 \cup \{x_0\})$. There exists a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \bar{\pi}_i \rangle \mid i \in I\}$$

of weak classical deductions, and $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$ is a weak classical deduction.

LEMMA 7.6. *$\Gamma \Vdash^w \theta, \neg_L \neg_L x$ implies $\Gamma \Vdash^w \theta, x$ for all $x \in A(L)$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0, \theta_0 \cup \{\neg_L \neg_L x_0\}, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, \theta \cup \{\neg_L \neg_L x\}, \pi \rangle$ is a weak classical deduction, $x \in A(L)$, and $\Gamma \Vdash^w \theta, x$ is false.

(1) $\pi_0 = \emptyset$ and $\Gamma_0 \cap (\theta_0 \cup \{\neg_L \neg_L x_0\}) \neq \emptyset$. If $\Gamma_0 \cap \theta_0 \neq \emptyset$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \emptyset \rangle$ is a weak classical deduction. If $\neg_L \neg_L x_0 \in \Gamma_0$, then

$$\langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\}, \{\langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\} \cup \{\neg_L x_0\}, \emptyset \rangle, \langle \Gamma_0 \cup \{\neg_L x_0\} \cup \{x_0\}, \theta_0 \cup \{x_0\}, \emptyset \rangle\} \rangle$$

is a weak classical deduction. Therefore, $\Gamma_0, \neg_L x_0 \Vdash^w \theta_0, x_0$, and lemma 7.5 gives $\Gamma_0 \Vdash^w \theta_0, x_0$.

(2) $$\pi_0 = \{\langle \Gamma_0, \theta_0 \cup \{\neg_L \neg_L x_0\} \cup \{z\}, \pi_z \rangle \mid z \in \Delta\} \cup \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{\neg_L \neg_L x_0\}, \pi' \rangle\},$$

where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. There exists a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, \theta_0 \cup \{x_0\} \cup \{z\}, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{x_0\}, \bar{\pi}' \rangle\}$$

of weak classical deductions, and $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$ is a weak classical deduction.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{\neg_L \neg_L x_0\} \cup \{y_i\}, \pi_i \rangle \mid i \in I\}$, where

$$\{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(\theta_0 \cup \{\neg_L \neg_L x_0\}) .$$

There exists a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \bar{\pi}_i \rangle \mid i \in I\}$$

of weak classical deductions. If $\{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(\theta_0)$, then

$$\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$$

is a weak classical deduction. If $\{\langle \Delta_i, y_i \rangle \mid i \in I\} = \neg_L \neg_L x_0$, then $x_0 = y_k$ for some $k \in I$, and $\langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\}, \bar{\pi}_k \rangle$ is a weak classical deduction. This proves that $\Gamma_0, \neg_L x_0 \Vdash^w \theta_0, x_0$, and lemma 7.5 gives $\Gamma_0 \Vdash^w \theta_0, x_0$.

Lemma 7.7. *$\Gamma, \{\neg_L y \mid y \in \theta\}, w_L \Vdash x$ implies $\Gamma \Vdash^w \theta, x$.*

Proof. Let $\bar{\theta} = \{\neg_L y \mid y \in \theta\}$ and assume $\delta_0 = \langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\}, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \bar{\theta} \cup \{w_L\}, x, \pi \rangle$ is a *regular* minimal deduction and $\Gamma \Vdash^w \theta, x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. Then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \emptyset \rangle$ is a weak classical deduction.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\}, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\} \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\})$. Using lemma 7.4 we get a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, \theta_0 \cup \{x_0\} \cup \{z\}, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{x_0\}, \bar{\pi}' \rangle\}$$

of weak classical deductions. If $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$ is a weak classical deduction. If $\langle \Delta, y \rangle \in \bigcup \bar{\theta}_0$, then $\Delta = \{z_0\}$, where $z_0 \in \theta_0$, and $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$ is a weak classical deduction. If $\langle \Delta, y \rangle \in w_L$, then $\langle \Delta, y \rangle = \langle \{\neg_L \neg_L x_0\}, x_0 \rangle$ and $x_0 \in A(L)$. Consequently $\Gamma_0 \Vdash^w \theta_0, x_0, \neg_L \neg_L x_0$, and lemma 7.6 gives $\Gamma_0 \Vdash^w \theta_0, x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\} \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in \} \in C(V)$. Using lemma 7.4. we get a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \bar{\pi}_i \rangle\}$$

of weak classical deductions, and $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0 \rangle$ is a weak classical deduction.

Lemma 7.8. *$x \in L$ and $\Gamma \Vdash^w \theta, x$ imply $\Gamma, \{\neg_L y \mid y \in \theta\}, w_L \Vdash x$.*

Proof. Let $\bar{\theta} = \{\neg_L y \mid y \in \theta\}$, and assume $\delta_0 = \langle \Gamma_0, \theta_0 \cup \{x_0\}, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, \theta \cup \{x\}, \pi \rangle$ is a weak classical deduction, $x \in L$ and $\Gamma, \bar{\theta}, w_L \Vdash x$ is false.

(1) $\pi_0 = \emptyset$ and $\Gamma_0 \cap [\theta_0 \cup \{x_0\}] \neq \emptyset$. If $x_0 \in \Gamma_0$, then $\Gamma_0, \bar{\theta}_0, w_L \Vdash x_0$. If $\Gamma_0 \cap \theta_0 \neq \emptyset$, then proposition 6.1 implies $\Gamma_0, \bar{\theta}_0, w_L \Vdash x_0$.

(2) $\pi_0 = \{\langle \Gamma_0, \theta_0 \cup \{x_0\} \cup \{z\}, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{x_0\}, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. We have $\Gamma_0, \bar\theta_0, \neg_L x_0, w \Vdash z$ for all $z \in \Delta$ and consequently $\Gamma_0, \bar\theta_0, \neg_L x_0, w_L \Vdash y$. Furthermore, $\Gamma_0, \bar\theta_0, y, w_L \Vdash x_0$, and proposition 6.2 gives $\Gamma_0, \bar\theta_0, \neg_L x_0, w_L \Vdash \neg_L y$. Now proposition 6.1 gives $\Gamma_0, \bar\theta_0, w_L, \neg_L x \Vdash z$ for all $z \in A(L)$, and this implies $\Gamma_0, \bar\theta_0, w_L \Vdash \neg_L \neg_L x_0$, and finally lemma 7.1 gives $\Gamma_0, \bar\theta_0, w_L \Vdash x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \pi_i \rangle \mid i \in I\}$, where

$$\{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(\theta_0 \cup \{x_0\}) .$$

We have $\Gamma_0, \bar\theta_0, \neg_L x_0, w_L, \Delta_k \Vdash y_k$ for all $k \in I$, and consequently $\Gamma_0, \bar\theta_0, \neg_L x_0, w_L \Vdash \{\langle \Delta_i, y_i \rangle \mid i \in I\}$. Since $\neg_L \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in \bar\theta_0 \cup \{\neg_L x_0\}$, we have $\Gamma_0, \bar\theta_0, \neg_L x_0, w_L \Vdash z$ for all $z \in A(L)$, and consequently $\Gamma_0, \bar\theta_0, w_L \Vdash \neg_L \neg_L x_0$. Now lemma 7.1 gives $\Gamma_0, \bar\theta_0, w_L \Vdash x_0$.

8. The strong classical derivations.

DEFINITION 8.1. Using the notations

$$\tilde\Lambda_L = \Lambda \cup \{\neg_L y \mid y \in A(L) \setminus \Lambda\} \quad \text{for all } \Lambda \subseteq A(L) ,$$
$$s_L = \{\langle \{\langle \tilde\Lambda_L, y \rangle \mid \Lambda \subseteq A(L)\}, y \rangle \mid y \in A(L)\} ,$$

$\Vdash_L^{s_L}$ is called the *strong classical derivation on L.*

Intuitively s_L states that one of the sets $\tilde\Lambda_L$, $\Lambda \subseteq A(L)$, contains nothing but true sentences. (Compare with the definitions of $x \vee_L y$ and $\exists_L \Gamma$ in section 3.)

PROPOSITION 8.1. *If $\Lambda \subseteq A(L)$, then $\tilde\Lambda_L \Vdash s_L$.*

PROPOSITION 8.2. *If $\Lambda \subseteq A(L)$ and $x \in L$, then $\tilde\Lambda_L \Vdash x$ or $\tilde\Lambda_L \Vdash \neg_L x$.*

PROOF. Obvious if $x \in A(L)$. Assume $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$ basic with respect to Φ: $\Phi(x)$ iff $\tilde\Lambda_L \Vdash x$ and $\tilde\Lambda \Vdash \neg_L x$ are both false.
Since $\tilde\Lambda_L \Vdash x_0$ is false, we have $\Delta_k, \tilde\Lambda_L \Vdash y_k$ is false for some $k \in I$. Consequently $\tilde\Lambda_L \Vdash y_k$ is false, and $\tilde\Lambda_L \Vdash \neg_L y_k$ is true. Since $\tilde\Lambda_L \Vdash \neg_L z$ for some $z \in \Delta_k$ would imply $\Delta_k, \tilde\Lambda_L \Vdash y_k$, we have $\tilde\Lambda_L \Vdash z$ for all $z \in \Delta_k$, and from this $\tilde\Lambda_L, x_0 \Vdash y_k$ follows. But $\tilde\Lambda_L, x_0 \Vdash y_k$ and $\tilde\Lambda_L, x_0 \Vdash \neg_L y_k$ imply $\tilde\Lambda_L, x_0 \Vdash z$ for all $z \in A(L)$, and consequently $\tilde\Lambda_L \Vdash \neg_L x_0$.

DEFINITION 8.2. A *truth-assignment on L* is a set $M \subseteq L$ with the property that

$$\{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(M)$$

iff no $k \in I$ exists such that $\Delta_k \subseteq M$ and $y_k \notin M$.

Proposition 8.3. *If* $\Lambda \subseteq A(L)$, *then* $M = \{x \mid x \in L \text{ and } \tilde{\Lambda}_L \Vdash x\}$ *is the only truth-assignment on* L *with the property that* $A(M) = \Lambda$.

Proof. $A(M) \supseteq \Lambda$ is obvious. To prove $A(M) \subseteq \Lambda$, assume $\delta_0 = \langle \tilde{\Lambda}_L, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \tilde{\Lambda}_L, x, \pi \rangle$ is a *regular* minimal deduction and $\neg_L x \in \tilde{\Lambda}_L$. Then

$$\pi_0 = \{\langle \tilde{\Lambda}_L, y_0, \pi_0' \rangle, \langle \tilde{\Lambda}_L \cup \{x_0\}, x_0, \pi_0'' \rangle\},$$

where $\neg_L y_0 \in \tilde{\Lambda}_L$, and this contradicts that δ_0 is basic with respect to Φ.

To prove that M is a truth-assignment on L, assume

$$x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L).$$

If $\tilde{\Lambda}_L \Vdash x$ and $\tilde{\Lambda}_L \Vdash z$ for all $z \in \Delta_k$, where $k \in I$, then $\tilde{\Lambda}_L \Vdash y_k$; and if no $k \in I$ exists such that $\tilde{\Lambda}_L \Vdash z$ for all $z \in \Delta_k$ and $\tilde{\Lambda}_L \Vdash y_k$ is false, then $\Delta_k, \tilde{\Lambda}_L \Vdash y_k$ for all $k \in I$ because of proposition 8.2, and consequently $\tilde{\Lambda}_L \Vdash x$.

Finally, assume that M' is a truth-assignment on L, and that $A(M') = A(M) = \Lambda$. Assume that $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$ is basic with respect to Ψ: $\Psi(x)$ iff either $x \in M$ and $x \notin M'$ or $x \notin M$ and $x \in M'$. Then $x_0 \in C(M)$ iff no $k \in I$ exists such that $\Delta_k \subseteq M$ and $y_k \notin M$ iff no $k \in I$ exists such that $\Delta_k \subseteq M'$ and $y_k \notin M'$ iff $x_0 \in C(M')$.

Definition 8.3. A *strong classical deduction* is a triple $\langle \Gamma_0, \theta_0, \pi_0 \rangle$ which is an element of a set P of triples $\langle \Gamma, \theta, \pi \rangle$ each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $\Gamma \cap \theta \neq \emptyset$.

(2) $\pi = \{\langle \Gamma \cup \Gamma_f, \theta \cup \theta_f, \pi_f \rangle \mid f \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N\}\}$, where $\Gamma_f = \{y_\nu \mid \nu \in N \text{ and } f(\nu) = \Delta_\nu\}$, $\theta_f = \{f(\nu) \mid \nu \in N \text{ and } f(\nu) \in \Delta_\nu\}$ and $\langle \Delta_\nu, y_\nu \rangle \in \bigcup C(\Gamma)$ for all $\nu \in N$.

(3) $\pi = \{\langle \Gamma \cup \Gamma_g, \theta \cup \theta_g, \pi_g \rangle \mid g \in \times\{x_\nu \mid \nu \in N\}$, where $\Gamma_g = \bigcup\{\Delta_{g,\nu} \mid \nu \in N\}$, $\theta_g = \{y_{g,\nu} \mid \nu \in N\}$, $g(\nu) = \langle \Delta_{g,\nu}, y_{g,\nu} \rangle$ and $x_\nu \in C(\theta)$ for all $\nu \in N$.

Notice that the weak classical deductions can be considered as strong classical deductions, where the index sets N are always singletons. Intuitively each step in a strong classical deduction is the simultaneous execution of a number of weak classical steps; one step for each element of the index set N. It is not hard to prove that any strong classical deduction with a finite index set N can be replaced by a weak classical deduction. As we shall see in the next section, this does not hold for strong classical deductions in general.

Theorem 8.4. *The following four conditions are equivalent*:

(a) $\Gamma \Vdash_L^{sL} x$.

(b) *$\Gamma \subseteq L$, $x \in L$, and $\tilde{\Lambda}_L, \Gamma \Vdash x$ for all $\Lambda \subseteq A(L)$.*
(c) *$\Gamma \subseteq L$, $x \in L$, and $\Gamma \subseteq M$ implies $x \in M$ for all truth-assignments M on L.*
(d) *$\Gamma \subseteq L$, $x \in L$, and there exists a strong classical deduction $\langle \Gamma, \{x\}, \pi \rangle$.*

PROOF. The theorem is an immediate consequence of the following five lemmas, which prove (a) $\to$ (c), (c) $\to$ (b), (b) $\to$ (a), (d) $\to$ (c) and (c) $\to$ (d).

LEMMA 8.5. *If $\Gamma \Vdash_L^{sL} x$, then no truth-assignment M on L exists such that $\Gamma \subseteq M$ and $x \notin M$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0 \cup \{s_L\}, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \{s_L\}, x, \pi \rangle$ is a regular minimal deduction, and there exists a truth-assignment M on L such that $\Gamma \subseteq M$ and $x \notin M$. Assume $M_0 = \{x \mid x \in L$ and $\tilde{\Lambda}_L \Vdash x\}$, where $\Lambda \subseteq A(L)$, $\Gamma_0 \subseteq M_0$ and $x_0 \notin M_0$.

(1) $\pi_0 = \emptyset$ and $x_0 \in A(\Gamma_0)$. Since $x_0 \in \Gamma_0$, $\Gamma_0 \subseteq M_0$ and $x_0 \notin M_0$ cannot both be true.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \{s_L\}, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{s_L\} \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0 \cup \{s_L\})$, and $x_0 = y$ if $y \in A(L)$. We have $z \in M_0$ for all $z \in \Delta$ because δ_0 is basic with respect to Φ. If $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$, then $y \in M_0$; but $\Gamma_0 \cup \{y\} \subseteq M_0$ implies $x_0 \in M_0$ since δ_0 is basic with respect to Φ. If $\langle \Delta, y \rangle \in s_L$, then $y = x_0$ and $\langle \tilde{\Lambda}_L \cup \Gamma_0 \cup \{s_L\}, x_0, \pi'' \rangle \in \pi_z$ for some $z \in \Delta$, and since $\tilde{\Lambda}_L \Vdash u$ for all $u \in \Gamma_0 \cup \{s_L\}$, we have $\tilde{\Lambda}_L \Vdash x_0$, which implies $x_0 \in M_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i \cup \{s_L\}, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$. Since $x_0 \notin M_0$, a $k \in I$ exists such that $\Delta_k \subseteq M_0$ and $y_k \notin M_0$. But since $\Gamma_0 \cup \Delta_k \subseteq M_0$ and δ_0 is basic with respect to Φ, we have $y_k \notin M_0$, which is a contradiction.

LEMMA 8.6. *If $\Lambda \subseteq A(L)$, $\Gamma \subseteq L$, $x \in L$, and $\tilde{\Lambda}_L, \Gamma \Vdash x$ does not hold, then $\Gamma \subseteq M$ and $x \notin M$, where $M = \{y \mid y \in L$* and *$\tilde{\Lambda}_L \Vdash y\}$.*

PROOF. We have $\tilde{\Lambda}_L \Vdash z$ for all $z \in \Gamma$, since otherwise $\tilde{\Lambda}_L \Vdash \neg_L z$ and $\tilde{\Lambda}_L, \Gamma \Vdash x$; consequently $\Gamma \subseteq M$. Since $\tilde{\Lambda}_L, \Gamma \Vdash x$ is false, $\tilde{\Lambda}_L \Vdash x$ is false, and $x \notin M$.

LEMMA 8.7. *If $\Gamma \subseteq L$, $x \in L$, and $\tilde{\Lambda}_L, \Gamma \Vdash x$ for all $\Lambda \subseteq A(L)$, then $\Gamma, s_L \Vdash x$.*

PROOF. Assume x_0 basic with respect to Φ: $\Phi(x)$ iff $x \in L$, there exists a $\Gamma \subseteq L$ such that $\tilde{\Lambda}_L, \Gamma \Vdash x$ for all $\Lambda \subseteq A(L)$, but $\Gamma, s_L \Vdash x$ does not hold. Assume $\Gamma_0 \subseteq L$ and $\tilde{\Lambda}_L, \Gamma_0 \Vdash x_0$ for all $\Lambda \subseteq A(L)$, but $\Gamma_0, s_L \Vdash x_0$ is false.

If $x_0 \in A(L)$, then $\Gamma_0 \Vdash \{\langle \tilde{\Delta}_L, x_0 \rangle \mid \Delta \subseteq A(L)\}$ and thus $\Gamma_0, s_L \Vdash x_0$. If $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$, then $\tilde{\Delta}_L, \Delta_k, \Gamma_0 \Vdash y_k$ for all $\Delta \subseteq A(L)$ and all $k \in I$; consequently $\Delta_k, \Gamma_0, s_L \Vdash y_k$ for all $k \in I$, and $\Gamma_0, s_L \Vdash x_0$.

Lemma 8.8. *If $\Gamma \subseteq L$, $\theta \subseteq L$, and there exists a strong classical deduction $\langle \Gamma, \theta, \pi \rangle$, then no truth-assignment M on L exists, for which $\Gamma \subseteq M$ and $\theta \subseteq L \setminus M$.*

Proof. Assume $\delta_0 = \langle \Gamma_0, \theta_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, \theta, \pi \rangle$ is a strong classical deduction, $\Gamma \subseteq L$, $\theta \subseteq L$, and there exists a truth-assignment M on L such that $\Gamma \subseteq M$ and $\theta \subseteq L \setminus M$. Assume M_0 is a truth-assignment on L, $\Gamma_0 \subseteq M_0$ and $\Theta_0 \subseteq L \setminus M_0$.

(1) $\pi_0 = \emptyset$ and $\Gamma_0 \cap \theta_0 \neq \emptyset$. Impossible since $\Gamma_0 \subseteq M_0$ and $\theta_0 \cap M_0 = \emptyset$.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \Gamma_f, \theta_0 \cup \theta_f, \pi_f \rangle \mid f \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N\}\}$, where

$$\begin{aligned} \Gamma_f &= \{y_\nu \mid \nu \in N \text{ and } f(\nu) = \Delta_\nu\}, \\ \theta_f &= \{f(\nu) \mid \nu \in N \text{ and } f(\nu) \in \Delta_\nu\}, \\ \langle \Delta_\nu, y_\nu \rangle &\in \bigcup C(\Gamma_0) \qquad \text{for all } \nu \in N. \end{aligned}$$

At least one $f_0 \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N\}$ has the property that $y_\nu \in M_0$ if $f_0(\nu) = \Delta_\nu$, and $f_0(\nu) \in L \setminus M$ if $f_0(\nu) \in \Delta_\nu$. But then $\Gamma_0 \cup \Gamma_{f_0} \subseteq M_0$ and $\theta_0 \cup \theta_{f_0} \subseteq L \setminus M_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Gamma_g, \theta \cup \theta_g, \pi_g \rangle \mid g \in \times\{x_\nu \mid \nu \in N\}\}$, where $\Gamma_g = \bigcup\{\Delta_{g,\nu} \mid \nu \in N\}$, $\theta_g = \{y_{g,\nu} \mid \nu \in N\}$, $g(v) = \langle \Delta_{g,\nu}, y_{g,\nu} \rangle$, and $x_\nu \in C(\theta_0)$ for all $\nu \in N$. At least one $g_0 \in \times\{x_\nu \mid \nu \in N\}$ has the property that $\Delta_{g_0,\nu} \subseteq M_0$, $y_{g_0,\nu} \notin M_0$ for all $\nu \in N$. But then $\Gamma_0 \cup \Gamma_{g_0} \subseteq M_0$, $\theta_0 \cup \theta_{g_0} \subseteq L \setminus M_0$.

Lemma 8.9. *If $\Gamma \subseteq L$, $\theta \subseteq L$, and no strong classical deduction $\langle \Gamma, \theta, \pi \rangle$ exists, then there is a truth-assignment M on L such that $\Gamma \subseteq M$ and $\theta \subseteq L \setminus M$.*

Proof. If no classical deduction $\langle \Gamma, \theta, \pi \rangle$ exists, then there exists a sequence $\{\langle \Gamma_n, \theta_n \rangle \mid n = 0, 1, 2, \ldots\}$, where $\Gamma_0 = \Gamma$, $\theta_0 = \theta$, $\Gamma_{n+1} = \Gamma_n \cup \Gamma_{h_n}$, $\theta_{n+1} = \theta_n \cup \theta_{h_n}$ for $n = 0, 1, 2, \ldots$, and such that for all $k = 0, 1, 2, \ldots$:

(1) $h_{2k} \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N_{2k}\}$,

$$\begin{aligned} \Gamma_{h_{2k}} &= \{y_\nu \mid \nu \in N_{2k} \text{ and } h_{2k}(\nu) = \Delta_\nu\}, \\ \theta_{h_{2k}} &= \{h_{2k}(\nu) \mid \nu \in N_{2k} \text{ and } h_{2k}(\nu) \in \Delta_\nu\}, \\ N_{2k} &= \bigcup C(\Gamma_{2k}), \\ \langle \Delta_\nu, y_\nu \rangle &= \nu \qquad \text{for all } \nu \in N_{2k}, \end{aligned}$$

and no strong classical deduction $\langle \Gamma_{2k} \cup \Gamma_{h_{2k}}, \theta_{2k} \cup \theta_{h_{2k}}, \pi \rangle$ exists.

(2) $h_{2k+1} \in \times\{x_\nu \mid \nu \in N_{2k+1}\}$,
$\Gamma_{h_{2k+1}} = \bigcup\{\Delta_\nu \mid \nu \in N_{2k+1}\}$,
$\theta_{h_{2k+1}} = \{y_\nu \mid \nu \in N_{2k+1}\}$,
$N_{2k+1} = C(\theta_{2k+1})$,
$\langle \Delta_\nu, y_\nu \rangle = h_{2k+1}(\nu)$ for all $\nu \in N_{2k+1}$,

and no strong classical deduction $\langle \Gamma_{2k+1} \cup \Gamma_{h_{2k+1}}, \theta_{2k+1} \cup \theta_{h_{2k+1}}, \pi \rangle$ exists.

Let $\Gamma_\omega = \bigcup\{\Gamma_n \mid n = 0,1,2,\ldots\}$ and $\theta_\omega = \bigcup\{\theta_n \mid n = 1,2,\ldots\}$. Now we claim that $\Gamma_\omega \subseteq M$ and $\theta_\omega \subseteq L \setminus M$, where $M = \{x \mid x \in L \text{ and } \tilde{\Lambda}_L \Vdash x\}$ and $\Lambda = A(\Gamma_\omega)$.

Assume that x_0 is basic with respect to Φ: $\Phi(x)$ iff either $x \in \Gamma_\omega$ and $x \notin M$, or $x \in \theta_\omega$ and $x \in M$.

$x_0 \in A(L)$ is impossible: If $x_0 \in A(\Gamma_\omega)$, then $x_0 \in M$ by the definition of M, and if $x_0 \in A(\theta_\omega) \cap M$, then there exists an n such that $x_0 \in A(\Gamma_n \cap \theta_n)$, and consequently $\langle \Gamma_n, \theta_n, \emptyset \rangle$ is a strong classical deduction.

Assume that $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$. Chose k such that $x_0 \in \Gamma_{2k} \cup \theta_{2k}$. If $x_0 \in \Gamma_{2k}$, then $\Delta_j \subseteq M$ implies $h_{2k}(\langle \Delta_j, y_j \rangle) = \Delta_j$ and consequently $y_j \in \Gamma_{h_{2k}}$ and $y_j \in M$, for all $j \in I$, and we have $x_0 \in M$. If $x_0 \in \theta_{2k}$, then $\Delta \subseteq M$ and $y \notin M$, where $\langle \Delta, y \rangle = h_{2k+1}(x_0)$, and consequently $x_0 \notin M$.

9. Relations between the weak and the strong classical derivations.

THEOREM 9.1. $s_L \Vdash w_L$ and $\Vdash_L^{sL} \supseteq \Vdash_L^{wL}$.

PROOF. The theorem follows from the fact that the weak classical deductions are special cases of strong classical deductions.

PROPOSITION 9.2. *$w_L \Vdash s_L$ iff $A(L)$ is finite.*

PROOF. That $w_L \Vdash s_L$ if $A(L)$ is finite, is a consequence of the next theorem 9.3.

Assume that $A(L)$ is infinite and $w_L \Vdash s_L$. Define

$$\bar{x} = \{\langle \tilde{\Lambda}_L, x \rangle \mid \Lambda \subseteq A(L)\} \quad \text{for all } x \in A(L)\,.$$

Then we have $\bar{x}, w_L \Vdash x$ for all $x \in A(L)$, and consequently there exists a weak classical deduction $\langle \{\bar{x}_0\}, \{x_0\}, \bar{\pi} \rangle$, where $x_0 \in A(L)$. Assume $\delta_0 = \langle \{\bar{x}_0\} \cup \Gamma_0, \theta_0' \cup \theta_0'', \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff

$$\delta = \langle \{\bar{x}_0\} \cup \Gamma, \theta' \cup \theta'', \pi \rangle$$

is a weak classical deduction, Γ, θ', θ'' are finite sets,

$$\Gamma \subseteq A(L), \quad \theta' \subseteq A(L), \quad \theta'' = \{\neg_L y \mid y \in \Gamma'\}$$

for some $\Gamma' \subseteq \Gamma$, and $\Gamma \cap \theta' = \emptyset$. Notice that we have assumed

$$\Phi(\langle\{\bar{x}_0\},\{x_0\},\bar{\pi}\rangle)\ .$$

(1) $\pi_0=\emptyset$ and $[\{\bar{x}_0\}\cup\Gamma_0]\cap[\theta_0'\cup\theta_0'']\neq\emptyset$. This contradicts that $\Phi(\delta_0)$.

(2) $\pi_0 = \{\langle\{\bar{x}_0\}\cup\Gamma_0,\theta_0'\cup\theta_0''\cup\{z\},\pi_z\rangle \mid z\in\Delta\}\ \cup$
$\cup\{\langle\{\bar{x}_0\}\cup\Gamma_0\cup\{x_0\},\theta_0'\cup\theta_0'',\pi'\rangle\}$,

where $\Delta=\tilde{\Lambda}_L$ for some $\Lambda\subseteq A(L)$. Since Γ_0, θ_0' and θ_0'' are finite sets and $A(L)$ is infinite, there exists a $z_0\in\Delta$ such that

$$\Phi(\langle\{\bar{x}_0\}\cup\Gamma_0,\theta_0'\cup\theta_0''\cup\{z_0\},\pi_{z_0}\rangle)\ ,$$

which contradicts that δ_0 is basic with respect to Φ.

(3) $\pi_0=\{\langle\{\bar{x}_0\}\cup\Gamma_0\cup\{y_0\},\theta_0'\cup\theta_0''\cup\{z\},\pi_z\rangle \mid z\in A(L)\}$, where $\neg_L y_0\in\theta_0''$. Since θ_0' and θ_0'' are finite and $A(L)$ is infinite, there exists a $z_0\in A(L)$ such that $\Phi(\langle\{\bar{x}_0\}\cup\Gamma_0\cup\{y_0\},\theta_0'\cup\theta_0''\cup\{z_0\},\pi_{z_0}\rangle)$, which contradicts that δ_0 is basic with respect to Φ.

THEOREM 9.3. *If* $\bigcup C(L)$ *is finite or denumerable, then* $\Vdash_L^{wL} = \Vdash_L^{sL}$.

PROOF. Let $\langle\Delta_n,y_n\rangle$, $n=0,1,2,\ldots$, be an enumeration of the elements of $\bigcup C(L)$, which repeats each element infinitely many times.

Since $\Vdash_L^{wL}\subseteq\Vdash_L^{sL}$ is generally true, we only have to prove that $\Vdash_L^{wL}\supseteq\Vdash_L^{sL}$. Assume that $\Gamma\Vdash_L^{wL}x$ does not holds. Then there exists a sequence $\langle\Gamma_n,\theta_n,u_n\rangle$, $n=0,1,2,\ldots$, such that weak classical deductions $\langle\Gamma_n,\theta_n,\pi\rangle$ do not exist for any $n=0,1,2,\ldots$, $\langle\Gamma_0,\theta_0,n_0\rangle=\langle\Gamma,\{x\},x\rangle$, and for all $n=0,1,2,\ldots$ the following conditions hold:

$\langle\Gamma_{2n+1},\theta_{2n+1},u_{2n+1}\rangle = \langle\Gamma_{2n},\theta_{2n},u_{2n}\rangle$, if $u_n\in A(L)$;
$= \langle\Gamma_{2n}\cup\Delta,\theta_{2n}\cup\{y\},y\rangle$, where $\langle\Delta,y\rangle\in u_n$, if $u_n\in C(L)$.

$\langle\Gamma_{2n+2},\theta_{2n+2},u_{2n+2}\rangle = \langle\Gamma_{2n+1},\theta_{2n+1},u_{2n+1}\rangle$, if $\langle\Delta_n,y_n\rangle\notin\bigcup C(\Gamma_{2n+1})$;
$= \langle\Gamma_{2n+1}\cup\{y_n\},\theta_{2n+1},u_{2n+1}\rangle$, if $\langle\Delta_n,y_n\rangle\in\bigcup C(\Gamma_{2n+1})$ and no weak classical deduction $\langle\Gamma_{2n+1}\cup\{y_n\},\theta_{2n+1},u_{2n+1}\rangle$ exists;
$= \langle\Gamma_{2n+1},\theta_{2n+1}\cup\{z\},z\rangle$, where $z\in\Delta_n$, if $\langle\Delta_n,y_n\rangle\in\bigcup C(\Gamma_{2n+1})$ and there exists a weak classical deduction $\langle\Gamma_{2n+1}\cup\{y_n\},\theta_{2n+1},u_{2n+1}\rangle$.

Let $\Gamma_\omega=\bigcup\{\Gamma_n \mid n=0,1,2,\ldots\}$ and $\theta_\omega=\bigcup\{\theta_n \mid n=0,1,2,\ldots\}$. We claim that $\Gamma_\omega\subseteq M$ and $\theta_\omega\subseteq L\setminus M$, where $M=\{y \mid y\in L$ and $\tilde{\Lambda}_L\Vdash y\}$ and $\Lambda=A(\Gamma_\omega)$.

Assume x_0 basic with respect to Φ: $\Phi(x)$ iff $x\in\Gamma_\omega$ and $x\notin M$, or $x\in\theta_\omega$ and $x\in M$.

$x_0\in A(L)$ is impossible: If $x_0\in A(\Gamma_\omega)$, then $x_0\in M$ by the definition of M, if $x_0\in A(\theta_\omega)\cap M$, then there exists an n such that $x_0\in A(\Gamma_n\cap\theta_n)$, and consequently $\langle\Gamma_n,\theta_n,\emptyset\rangle$ is a weak classical deduction.

If $x_0 \in C(\theta_\omega)$, then $x_0 = u_n$ for some n, and $\Delta \subseteq \Gamma_{2n+1}$, $y \in \theta_{2n+1}$ for some $\langle \Delta, y \rangle \in x_0$. But then $\Delta \subseteq M$, $y \in L \setminus M$, and consequently $x_0 \notin M$.

If $x_0 \in C(\Gamma_\omega)$ and $\langle \Delta, y \rangle \in x_0$, then there exists an n such that $\langle \Delta, y \rangle = \langle \Delta_n, y_n \rangle$, and $x_0 \in \Gamma_{2n+1}$. We have either $y \in \Gamma_{2n+2}$ or $z \in \theta_{2n+2}$ for some $z \in \Delta$, and since $\langle \Delta, y \rangle$ was an arbitrary element of x_0 this proves that $x_0 \in M$.

THEOREM 9.4. *If* $A(L)$ *is finite, then* $\Vdash_L^{w_L} = \Vdash_L^{s_L}$.

PROOF. Follows from the fact that $w_L \Vdash s_L$ (proposition 9.2).

10. Application to first order logic.

The purpose of this final section is briefly to indicate how the theory of the previous sections may be applied to formal logic, and to show how some of the theorems are related to well known results. In order to simplify the exposition, only first order logic with a single binary relation R and variables $x_1, x_2, \ldots$, but with no constants or functions, is considered. N denotes the set of positive natural numbers.

DEFINITION 10.1. $\mathscr{P}$ is the smallest set of formulas which satisfies all of the conditions:

(1) $R(x_i, x_j)$ belongs to $\mathscr{P}$ for all $i,j \in \mathsf{N}$.

(2) If A belongs to $\mathscr{P}$ then $\neg A$ belongs to $\mathscr{P}$.

(3) If A and B belong to $\mathscr{P}$ then $(A \wedge B)$, $(A \vee B)$ and $(A \Rightarrow B)$ belong to $\mathscr{P}$.

(4) If A belongs to $\mathscr{P}$ then $\forall x_i A$ and $\exists x_i A$ belong to $\mathscr{P}$ for all $i \in \mathsf{N}$.

PROPOSITION 10.1. *Let* M *be a set such that* $M \times M$ *is a prelogic containing only atomic elements, and let* $\mathscr{F}$ *be the set of functions* $\mathsf{N} \to M$. *Then there exist exactly one function* μ *and one prelogic* L *such that (definitions* 3.5, 3.6 *and* 3.7 *are used):*

(1) $A(L) = M \times M$.

(2) μ *maps* $\mathscr{F} \times \mathscr{P}$ *onto* L.

(3) $\mu\langle f, R(x_i, x_j)\rangle = \langle f(i), f(j)\rangle$ *for all* $f \in \mathscr{F}$, $i,j \in \mathsf{N}$.

(4) $\mu\langle f, \neg A\rangle = \neg_L \mu\langle f, A\rangle$ *for all* $f \in \mathscr{F}$, $A \in \mathscr{P}$.

(5) $\mu\langle f, (A \wedge B)\rangle = \mu\langle f, A\rangle \wedge \mu\langle f, B\rangle$ *for all* $f \in \mathscr{F}$, $A, B \in \mathscr{P}$.

(6) $\mu\langle f, (A \vee B)\rangle = \mu\langle f, A\rangle \vee_L \mu\langle f, B\rangle$ *for all* $f \in \mathscr{F}$, $A, B \in \mathscr{P}$.

(7) $\mu\langle f, (A \Rightarrow B)\rangle = \mu\langle f, A\rangle \Rightarrow \mu\langle f, B\rangle$ *for all* $f \in \mathscr{F}$, $A, B \in \mathscr{P}$.

(8) $\mu\langle f, \forall x_i A\rangle = \forall \{\mu\langle g, A\rangle \mid g \in \mathscr{F}$ and $g(j) = f(j)$ for all $j \in \mathsf{N} \setminus \{i\}\}$ *for all* $f \in \mathscr{F}$, $i \in \mathsf{N}$, $A \in \mathscr{P}$.

(9) $\mu\langle f, \exists x_i A\rangle = \exists_L \{\mu\langle g, A\rangle \mid g \in \mathscr{F}$ and $g(j) = f(j)$ for all $j \in \mathsf{N} \setminus \{i\}\}$ *for all* $f \in \mathscr{F}$, $i \in \mathsf{N}$, $A \in \mathscr{P}$.

DEFINITION 10.2. An *interpretation* of $\mathscr{P}$ is a quadruple $\langle M,f,\varphi,L'\rangle$, where M satisfies the conditions of proposition 10.1, and (using the terminology introduced in proposition 10.1) $f \in \mathscr{F}$; φ is the function $\mathscr{P} \to L$ such that $\varphi[A] = \mu\langle f,A\rangle$ for all $A \in P$, and L' is the smallest prelogic containing the image of $\mathscr{P}$ by φ.

DEFINITION 10.3. Let $\Gamma \subseteq \mathscr{P}$ and $A \in \mathscr{P}$. The formula A is a *minimal consequence* of Γ (written $\Gamma \Vdash A$), if $\varphi[\Gamma] \Vdash \varphi[A]$ for all interpretations $\langle M,f,\varphi,L\rangle$. The formula A is a *weak classical consequence* of Γ (written $\Gamma \Vdash^{w} A$) if $\varphi[\Gamma] \Vdash_{L}^{wL} \varphi[A]$ for all interpretations $\langle M,f,\varphi,L\rangle$. The formula A is a *strong classical consequence* of Γ (written $\Gamma \Vdash^{s} A$) if $\Gamma \Vdash_{L}^{sL} A$ for all interpretations $\langle M,f,\varphi,L\rangle$.

THEOREM 10.2. *Let $\Gamma \subseteq \mathscr{P}$, $A \in \mathscr{P}$. If there exitists an intuitionistic Gentzen type proof of the sequent $\Gamma \to A$ (see* [1], *or* [3, *chapter XV*]), *then $\Gamma \Vdash A$. If there exists a classical Gentzen type proof of the sequent $\Gamma \to A$, then $\Gamma \Vdash^{w} A$.*

THEOREM 10.3. *Let $\langle M,f,\varphi,L\rangle$ be any interpretation of $\mathscr{P}$ such that f is univalent and M contains a denumerable set of elements which are not images of N by f, let $\Gamma \subseteq \mathscr{P}$ and $A \in \mathscr{P}$. If $\varphi[\Gamma] \Vdash \varphi[A]$, then there exists an intuitionistic Gentzen type proof without cut of the sequent $\Gamma \to A$.* If $\varphi[\Gamma] \Vdash_{L}^{wL} \varphi[A]$, *then there exists a classical Gentzen type proof without cut of the sequent $\Gamma \to A$.*

Theorem 10.3 may be proved using strong variants of theorem 4.1 and theorem 7.3. (Proposition 4.6 is an example of a strong variant of theorem 4.1.)

THEOREM 10.4. *The weak and the strong classical relations of consequence are identical on $\mathscr{P}$.*

Theorem 10.4 is proved by applying theorem 9.3 to an interpretation $\langle M,f,\varphi,L\rangle$ which satisfies the conditions of theorem 10.3 with the extra condition that M is denumerable. Theorem 10.4 is a variant of the Gödel completeness theorem, and it is natural to consider theorem 9.3 as a generalisation.

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