MATH. SCAND. 16 (1965), 25–28

ON THE BASES OF FREE ALGEBRAS

TAH-KAI HU

We consider a fixed class $\mathscr{C}$ of universal algebras of the same species. All the concepts we introduce will be relative to the class $\mathscr{C}$.

Let F be an algebra of the same species as the algebras in $\mathscr{C}$, and S a non-empty set of generators of F. We say that S is a *$\mathscr{C}$-free set of generators of F*, if any mapping of S into an algebra $A \in \mathscr{C}$ may be extended to a homomorphism of F into A. Assume that this is the case; then any mapping of S into $A \in \mathscr{C}$ is, in fact, uniquely extended to a homomorphism of F into A. The notations being as above, we say that F is a *$\mathscr{C}$-free algebra* and that S is a *$\mathscr{C}$-free base of F*, if S is a $\mathscr{C}$-free set of generators of F and $F \in \mathscr{C}$. It is easily seen that two $\mathscr{C}$-free algebras with equipotent $\mathscr{C}$-bases are isomorphic.

Let A be an algebra of the same species as the algebras in $\mathscr{C}$, and S a subset of A. We say that S is *$\mathscr{C}$-independent* if either $S=\emptyset$ or the subalgebra of A generated by S is $\mathscr{C}$-freely generated by S. S is said to be *$\mathscr{C}$-dependent* if it is not $\mathscr{C}$-independent. $\mathscr{C}$-independence in this sense obviously includes the notions of independence studied by E. Marczewski [4] and J. Schmidt [5].

Observe that any subset of an algebra A containing a set of generators of A is again a set of generators of A, and that any subset of a $\mathscr{C}$-independent set is $\mathscr{C}$-independent. Note that a $\mathscr{C}$-base is both a minimal non-empty set of generators and a maximal non-empty $\mathscr{C}$-independent set. There is thus a duality between "generator sets" and "$\mathscr{C}$-independent sets".

We consider the following properties of the class $\mathscr{C}$, letting F denote an *arbitrary* $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base containing n elements:

(Ia) *Any set of generators of F contains at least n elements.*

(Ib) *Any $\mathscr{C}$-independent subset of F contains at most n elements.*

(IIa) *Any set of generators of F containing precisely n elements is a $\mathscr{C}$-base of F.*

(IIb) *Any $\mathscr{C}$-independent subset of F containing precisely n elements is a $\mathscr{C}$-base of F.*

Received March 10, 1965

REMARK. If F has a $\mathscr{C}$-base consisting of exactly one element and F is generated by the empty set, then, strictly speaking, property (Ia) fails. But we shall admit this possible exception even when we state explicitly that $\mathscr{C}$ satisfies property (Ia).

Evidently, (IIa) $\Rightarrow$ (Ia) and (IIb) $\Rightarrow$ (Ib); and, moreover, either (Ia) or (Ib) implies that any two $\mathscr{C}$-bases of a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base have the same number of elements. Sufficient conditions for $\mathscr{C}$ to satisfy properties (Ia) and (IIb) were given by B. Jónsson and A. Tarski [2], and H. Jerome Keisler [3]. For the invariance property of the number of elements in a $\mathscr{C}$-base of a $\mathscr{C}$-free algebra, see [1] [6].

In the present note, we characterize properties (IIa) and (IIb) and derive sufficient conditions for properties (Ia) and (IIa), or properties (Ib) and (IIb).

LEMMA a. *Let F be a $\mathscr{C}$-free algebra with S as a $\mathscr{C}$-base, and A an algebra in $\mathscr{C}$ with T as a set of generators. Assume that there is a surjection $\varphi : S \to T$. Then φ is uniquely extended to an epimorphism $f : F \to A$.*

PROOF. Since $A \in \mathscr{C}$, φ is uniquely extended to a homomorphism $f : F \to A$. $f(F)$ is the subalgebra of A generated by $f(S) = \varphi(S)$. Since φ is surjective, $\varphi(S) = T$. But T generates A. Hence $f(F) = A$ and f is surjective.

LEMMA b. *Let F be a $\mathscr{C}$-free algebra with S as a $\mathscr{C}$-base, and A an algebra in $\mathscr{C}$ with T as a $\mathscr{C}$-independent subset. Assume that there is an injection $\varphi : S \to T$. Then φ is uniquely extended to a monomorphism $f : F \to A$.*

PROOF. Since $\varphi : S \to T$ is an injection, there is a surjection $\psi : T \to S$ such that $\psi \circ \varphi$ is the identity mapping of S, by the axiom of choice. Let B be the subalgebra of A generated by T. Then B is $\mathscr{C}$-freely generated by T. Since S is a $\mathscr{C}$-base of F and $A \in \mathscr{C}$, φ is uniquely extended to a homomorphism $f : F \to A$. Since $f(S) = \varphi(S) \subseteq T$ and B is generated by T, we have $f(F) \subseteq B$. Thus, we may regard f as a homomorphism of F into B. On the other hand, since T is a $\mathscr{C}$-independent subset of A and $F \in \mathscr{C}$, ψ is uniquely extended to a homomorphism $g : B \to F$. If $s \in S$, then

$$(g \circ f)(s) = g(\varphi(s)) = (\psi \circ \varphi)(s) = s .$$

Hence both $g \circ f$ and the identity mapping of F are homomorphisms extending $\psi \circ \varphi$. S being a $\mathscr{C}$-base of F, $g \circ f$ is the identity mapping of F. Therefore f is injective.

THEOREM a. *$\mathscr{C}$ satisfies property* (IIa) *if an only if any surjective endomorphism of a $\mathscr{C}$-free algebra with finite $\mathscr{C}$-base is an automorphism.*

PROOF. Let F be a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base containing n elements.

Assume that any surjective endomorphism of F is an automorphism. Let T be a set of generators of F containing precisely n elements. Then there is a surjection $\varphi : S \to T$, which is uniquely extended to a surjective endomorphism f of F, by Lemma a. By assumption, f is an automorphism of F. Therefore $T = \varphi(S) = f(S)$ is a $\mathscr{C}$-base of F.

Conversely, assume that $\mathscr{C}$ satisfies property (IIa). Then $\mathscr{C}$ also satisfies property (Ia). Let f be a surjective endomorphism of F. Then $f(S)$ contains at most n elements. On the other hand, $f(S)$ is a set of generators of $f(F) = F$ and, as such, must contain at least n elements, by property (Ia). Hence $f(S)$ contains precisely n elements and is consequently a $\mathscr{C}$-base of F, by property (IIa). But f maps S bijectively onto $f(S)$. In particular, this means that f is also injective, by Lemma b. Therefore f is an automorphism.

THEOREM b. *$\mathscr{C}$ satisfies property* (IIb) *if an only if any injective endomorphism of a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base is an automorphism.*

PROOF. Let F be a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base containing n elements.

Assume that any injective endomorphism of F is an automorphism. Let T be a $\mathscr{C}$-independent subset of F containing precisely n elements. Then there exists a bijection $\varphi : S \to T$, which is uniquely extended to an injective endomorphism of F, by Lemma b. By assumption, f is an automorphism of F. Therefore $T = \varphi(S) = f(S)$ is a $\mathscr{C}$-base of F.

Conversely, assume that $\mathscr{C}$ satisfies property (IIb). Let f be an injective endomorphism of F. Then $f(S)$ contains precisely n elements. Let A be the subalgebra of F generated by $f(S)$. Then f may be regarded as an isomorphism of F onto A, so that f^{-1} exists and is an isomorphism of A onto F. Let φ be any mapping of $f(S)$ into an algebra $C \in \mathscr{C}$. Denote by $f \mid S$ the restriction of f to S; then $f \mid S$ maps S into $f(S)$ so that $\varphi \circ (f \mid S)$ is defined and maps S into C. S being a $\mathscr{C}$-base of F, $\varphi \circ (f \mid S)$ is uniquely extended to a homomorphism $g : F \to C$. It follows that $g \circ f^{-1}$ is a homomorphism of A into C. Moreover, if $t \in f(S)$, then $t = f(s)$ for some $s \in S$, so that

$$\begin{aligned}(g \circ f^{-1})(t) &= (g \circ f^{-1})(f(s)) \\ &= ((g \circ f^{-1}) \circ f)(s) \\ &= (g \circ (f^{-1} \circ f))(s) = g(s) = (\varphi \circ (f \mid S))(s) \\ &\qquad\qquad = \varphi((f \mid S)(s)) = \varphi(f(s)) = \varphi(t);\end{aligned}$$

hence $g \circ f^{-1}$ extends φ. Thus, $f(S)$ generates A $\mathscr{C}$-freely and is consequently a $\mathscr{C}$-independent subset of F. Hence $f(S)$ is a $\mathscr{C}$-base of F, by property (IIb). But f maps S bijectively onto $f(S)$. In particular, this means that f is also surjective, by Lemma a. Therefore f is an automorphism.

COROLLARY. a. *Assume that the lattice of congruences of any $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base satisfies the ascending chain condition. Then $\mathscr{C}$ satisfies property* (IIa).

COROLLARY b. *Assume that the lattice of subalgebras of any $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base satisfies the descending chain condition. Then $\mathscr{C}$ satisfies property* (IIb).

There exist classes of algebras not satisfying any of the properties we considered (cf. [2] [6]). There exists a class $\mathscr{C}$ of algebras satisfying (Ia) but not (IIa) (cf. [2]). The class of groups satisfies (Ia) and (IIa), but not (Ib) and (IIb). The class of abelian groups satisfies (Ia), (IIa) and (Ib), but not (IIb).

The applications of our results to the class of distributive lattices, the class of Boolean algebras, and the class of left modules over a ring satisfying the ascending or descending chain condition for left ideals are immediate.

REFERENCES

1. T. Fujiwara, *Note on the isomorphism problem for free algebraic systems*, Proc. Japan Acad. 31 (1935), 135–136.
2. B. Jónsson and A. Tarski, *On two properties of free algebras*, Math. Scand. 9 (1961), 95–101.
3. H. Jerome Keisler, *On some results of Jónsson and Tarski concerning free algebras*, Math. Scand. 9 (1961), 102–106.
4. E. Marczewski, *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), 45–61.
5. J. Schmidt, *Algebraic operations and algebraic independence in algebras with infinitary operations*, Math. Japan. 6 (1961/62), 77–112.
6. S. Świerczkowski, *On isomorphic free algebras*, Fund. Math. 50 (1961/62), 35–44.

SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILL., U.S.A.