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A QUASI-SPECTRAL OPERATOR

E. R. DEAL

In [1] we defined a quasi-spectral operator and stated sufficient conditions to insure that an operator be quasi-spectral. The operator $Tf(x) = xf(x)$ on $C[0,1]$ was used as a guiding example, but it is clear that for any continuous monotone function g on $[0,1]$, such that $g(0)=0$, $g(1)=1$, $Tf(x)=g(x)f(x)$ satisfies the sufficient conditions of [1] if we pick $\langle f,\Phi(\lambda)\rangle = f\big(g^{-1}(\lambda)\big)$. This choice is necessary in order to make $\overline{(\lambda I-T)X}$ the nullspace of $\Phi(\lambda)$. Therefore $Tf(x)=g(x)f(x)$ on $C[0,1]$ is a quasi-spectral operator.

The restrictions of monotonicity and that $g(0)=0$, $g(1)=1$ seem to be rather strong, so it is natural to attempt to remove them. The restriction that $g(0)=0$, $g(1)=1$ can easily be removed by a change of scale. To study monotonicity, let us consider an example. Let $g(x)$ be a continuous function on $[0,1]$ with a single relative maximum. In order to fix our ideas, let us say that for some z, $0<z<1$, $g(z)=1$, $g(0)=0$, $g(1)=0$, g is monotone increasing from 0 to z, and monotone decreasing from z to 1. We now define projections $E(a,b)$ for intervals (a,b) or $(a,b]$ or $[a,b)$ or $[a,b]$ on the range of g, which is the spectrum of T. As before, we use $E(0,b)$ whether 0 is in the spectrum of T or not. Let

$$\delta(a,b) \;=\; g^{-1}(a,b) \;=\; \{x \mid a<g(x)<b\} .$$

Then

$$\delta(a,b) \;=\; (a',b') \cup (a'',b'') .$$

Define, for $a \neq 0$

$$\begin{aligned} E(a,b)f(x) &= 0 && x \leqq a' \\ &= f(x)-f(a') && a' \leqq x \leqq b' \\ &= f(b')-f(a') && b' \leqq x \leqq a'' \\ &= f(x)-f(a'')+f(b')-f(a') && a'' \leqq x \leqq b'' \\ &= f(b'')-f(a'')+f(b')-f(a') && b'' \leqq x . \end{aligned}$$

For $a=0$,

$$\begin{aligned} E(0,b)f(x) &= f(x) && x \leqq b' \\ &= f(b') && b' \leqq x \leqq a'' \\ &= f(x)-f(a'')+f(b') && a'' \leqq x . \end{aligned}$$

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Since g is monotone on the intervals $[0,z]$ and $[z,1]$, it determines two invertable functions $g_1(x)$ on $[0,z]$ and $g_2(x)$ on $[z,1]$. Given an x in the range of g, let $x'=g_1^{-1}(x)$ and $x''=g_2^{-1}(x)$.

Theorem 1. *Let* $0=x_0<x_1<\ldots<x_n=1$ *be a partition of* $[0,1]$. *Let* $\delta_i=(x_{i-1},x_i)$. *Then* $\sum_{i=1}^{n}E(\delta_i)=I$.

Proof. Let f and x be arbitrary, and pick k so that $x_{k-1}\leqq g(x)<x_k$. Suppose first that $x<z$. Then

$$\begin{aligned}\sum_{i=1}^{n}E(\delta_i)f(x) &= f(x_1')+\sum_{i=2}^{k-1}[f(x_i')-f(x_{i-1}')]+f(x)-f(x_{k-1}')+0\\ &= f(x)\,.\end{aligned}$$

Next suppose $x>z$. Then

$$\begin{aligned}\sum_{i=1}^{n}E(\delta_i)f(x) &= f(x_1')+\sum_{i=2}^{k-1}[f(x_i')-f(x_{i-1}')]+f(x)-f(x_k'')+\\ &\quad +f(x_k')-f(x_{k-1}')+\sum_{i=k+1}^{n}[f(x_{i-1}'')-f(x_i'')+f(x_i')-f(x_{i-1}')]\\ &= f(x)+f(x_n')-f(x_n'')\\ &= f(x)+f(z)-f(z)\\ &= f(x)\,.\end{aligned}$$

Since $\sum_{i=1}^{n}E(\delta_i)f(x)=f(x)$ for all x in $[0,1]$ and for all f in $C[0,1]$, $\sum_{i=1}^{n}E(\delta_i)=I$.

Theorem 2. *Let* δ_i *be as in Theorem* 1. *Then*

$$\sum_{i=1}^{n}\lambda_i E(\delta_i) = T-N$$

where $\lambda_i\in\delta_i$, *and where*

$$\begin{aligned}Nf(x) &= \int_0^x f(t)\,dg(t) && x\leqq z\,,\\ &= \int_0^z f(t)\,dg(t)-\int_z^x f(t)\,dg(t) && z\leqq x\,.\end{aligned}$$

Proof. Let f and x be arbitrary, and pick k so that $x_{k-1}\leqq g(x)<x_k$. Suppose first that $x<z$. Then

$$\sum_{i=1}^{n} \lambda_i E(\delta_i) f(x)$$
$$= \lambda_1 f(x_1') + \sum_{i=2}^{k-1} \lambda_i [f(x_i') - f(x'_{i-1})] + \lambda_k [f(x) - f(x'_{k-1})]$$
$$= g(\lambda_1') f(x_1') + \sum_{i=2}^{k-1} g(\lambda_i')[f(x_i') - f(x'_{i+1})] + g(\lambda_k')[f(x) - f(x'_{k-1})] .$$

Rearrangement of this sum by partial summation gives

$$\sum_{i=1}^{k-1} f(x_i')[g(\lambda_i') - g(\lambda'_{i-1})] + g(\lambda_k') f(x) .$$

Passage to the limit gives

$$\int_{\sigma(T)} \lambda E(d\lambda) f(x) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \lambda_i E(\delta_i) f(x)$$
$$= -\int_0^x f(t)\, dg(t) + g(x) f(x) = Tf(x) - Nf(x) .$$

Next suppose $x > z$. Then

$$\sum_{i=1}^{n} \lambda_i E(\delta_i) f(x) = \lambda_1 f(x_1') + \sum_{i=2}^{k-1} \lambda_i [f(x_i') - f(x'_{i-1})] +$$
$$+ \lambda_k [f(x) - f(x_k'') + f(x_k') - f(x'_{k-1})] +$$
$$+ \sum_{i=k+1}^{n} \lambda_i [f(x''_{i-1}) - f(x_i'') + f(x_i') - f(x'_{i-1})] .$$

Rearrangement of this sum by partial summation gives

$$\sum_{i=1}^{n-1} f(x_i')[\lambda_i - \lambda_{i+1}] + \lambda_n f(x_n') + \sum_{i=k}^{n-1} f(x_i'')[\lambda_{i+1} - \lambda_i] - \lambda_n f(x_n'') + \lambda_k f(x)$$
$$= \sum_{i=1}^{n-1} f(x_i')[g(\lambda_i') - g(\lambda'_{i+1})] + \sum_{i=k}^{n-1} f(x_i'')[g(\lambda''_{i+1}) - g(\lambda_i'')] + g(\lambda_k'') f(x) .$$

Passage to the limit gives

$$\int_{\sigma(T)} \lambda E(d\lambda) f(x) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \lambda_i E(\delta_i) f(x)$$
$$= g(x) f(x) - \int_0^z f(t)\, dg(t) + \int_z^x f(t)\, dg(t) .$$

Thus in either case

$$\int_{\sigma(T)} \lambda E(d\lambda) f(x) = Tf(x) - Nf(x) ,$$

and since f and x are arbitrary,

$$\int_{\sigma(T)} \lambda E(d\lambda) = T - N .$$

Thus T is a quasi-spectral operator, where $Tf(x) = g(x)f(x)$ for $g \in C[0,1]$ such that $g(0)=0$, $g(1)=0$, and such that g has a single relative maximum at z, $g(z)=1$. Again by a change of scale we can remove the restriction on the values of $g(0)$, $g(z)$, and $g(1)$. Also, it is clear that only notational difficulties would be introduced by allowing g to have a finite number of extreme points. Thus we have the following theorem.

THEOREM 3. *Let $Tf(x)=g(x)f(x)$ where g is continuous and has a finite number of extreme points. Then T is a quasi-spectral operator on $C[0,1]$.*

BIBLIOGRAPHY

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COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO, U. S. A.