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ON SETS OF VECTORS

PETER SCHERK

Given n non-void sets

$$A_1,\ldots,A_n \tag{1}$$

of vectors in a finite or infinite dimensional vector space V over an arbitrary field. The letters a_i, b_i, c_i denote elements of A_i, and $\dim A_i$ denotes the dimension of the subspace spanned by A_i, $i=1,\ldots,n$. Let

$$0 \leqq m \leqq n .$$

We consider the following two statements:

$\mathrm{P}_{n,m}$: *n vectors $a_1,\ldots,a_n$ always span a subspace of dimension* $\leqq m$.
$\mathrm{Q}_{n,m}$: *There exists an integer h, $0 \leqq h \leqq m$, and an h-space containing $h+(n-m)$ of the sets* (1).

Obviously, $\mathrm{Q}_{n,m}$ implies $\mathrm{P}_{n,m}$.

The purpose of this note is to show that conversely $\mathrm{P}_{n,m}$ *implies* $\mathrm{Q}_{n,m}$.

This assertion is trivial for $n=1$. We assume it has been proved up to $n-1$. From now on let $n>1$ be fixed. The case $m=0$ being trivial, we may assume $m>0$.

If some $n-1$ of the sets (1) satisfy $\mathrm{P}_{n-1,m-1}$, then they satisfy $\mathrm{Q}_{n-1,m-1}$ by induction assumption and the sets (1) themselves will satisfy $\mathrm{Q}_{n,m}$. Thus we may assume that no $n-1$ of the sets (1) satisfy $\mathrm{P}_{n-1,m-1}$, and hence, in particular, that $m<n$.

By the last assumption, there are $n-1$ vectors $b_1,\ldots,b_{n-1}$ spanning a subspace V_m of dimension $\geqq m$. By $\mathrm{P}_{n,m}$, we have $\dim V_m = m$ and every vector a_n lies in V_m. This yields $\dim A_m \leqq m$; more generally,

$$\dim A_i \leqq m , \qquad i=1,\ldots,n .$$

In particular, we may assume V to be finite dimensional.

Suppose $A_n=\{0\}$. Then $A_1,\ldots,A_{n-1}$ satisfy $\mathrm{P}_{n-1,m}$. If $m=n-1$, then $\mathrm{Q}_{n,m}$ is trivial with $h=0$. If $m<n-1$, then our induction assumption implies $\mathrm{Q}_{n-1,m}$ for the sets $A_1,\ldots,A_{n-1}$ and the sets (1) satisfy $\mathrm{Q}_{n,m}$. Thus we may assume

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(2) $$A_i \neq \{0\}, \qquad i=1,\ldots,n\,.$$

LEMMA. *Suppose there is an integer k, $1 \leqq k \leqq m$, and there is a k-space containing k of the sets* (1). *Then* $Q_{n,m}$ *holds.*

PROOF. Without loss of generality, we may assume that the sets

(3) $$A_1,\ldots,A_k$$

lie in a k-space V_k. We may also assume that k is minimal. Thus either $k=1$ or every h-space contains fewer than h of the sets (3) if $0<h<k$. Thus $Q_{k,k-1}$ and hence $P_{k,k-1}$ are false for the sets (3) if $k>1$. There are, therefore, k linearly independent vectors

$$c_1,\ldots,c_k\,.$$

They form a base of V_k. (If $k=1$, this remark follows directly from (2).)

We now project V parallel to V_k onto a subspace complementary to V_k. Dashes denote projections. For every choice of $a_{k+1},\ldots,a_n$, the vectors

$$c_1,\ldots,c_k,a_{k+1},\ldots,a_n$$

span a space of dimension $\leqq m$. Hence the projections

$$a'_{k+1},\ldots,a_n{}'$$

always span a subspace of dimension $\leqq m-k$ and the projections

(4) $$A'_{k+1},\ldots,A_n{}'$$

satisfy $P_{n-k,m-k}$ and hence $Q_{n-k,m-k}$. Thus there is an integer g, $0 \leqq g \leqq m-k$, and there are

$$g+(n-k)-(m-k) \;=\; g+(n-m)$$

distinct sets

$$A'_{i_1},\ldots,A'_{i_{g+n-m}}$$

among the sets (4) which lie in a g-space. The $k+g+n-m$ sets

$$A_1,\ldots,A_k,A_{i_1},\ldots,A_{i_{g+n-m}}$$

then lie in the $(g+k)$-space through V_k and that g-space. Thus our lemma is proved with $h=g+k$.

We now complete the induction proof of $Q_{n,m}$. The integer n was fixed. Put

$$f \;=\; \sum_1^n \dim A_i\,.$$

For $f<n$, our assertion is trivial; cf. (2). Suppose it has been proved up to $f-1$. On account of the Lemma, we may assume that

$$\dim A_n > 1 .$$

Let the set B_n consist of one single element $b_n \in A_n$, $b_n \neq 0$. Thus

$$\dim B_n = 1 < \dim A_n .$$

The sets

(5) $$A_1, \ldots, A_{n-1}, B_n$$

satisfy $P_{n,m}$. Also

$$\sum_1^{n-1} \dim A_i + \dim B_n < f .$$

Hence by our induction assumption for f and by (2), there is an integer k, $1 \leqq k \leqq m$, and a k-space containing $k+(n-m)$ of the sets (5). Thus it contains $k+(n-m-1) \geqq k$ of the sets (1). Our lemma now yields $Q_{n,m}$.

UNIVERSITY OF TORONTO, CANADA