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ON CERTAIN INTERPOLATION SPACES RELATED TO GENERALIZED SEMI-GROUPS

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Introduction.

Let $D(A)$ be the domain of the infinitesimal generator A of a strongly continuous semi-group $G(\sigma)$ of bounded linear operators in a Banach-space E (see [2], [3]). In [4] Lions has described interpolation spaces between E and $D(A)$. Later these results were extended to interpolation spaces between E and $D(A^m)$ $(m \geqq 1)$, (see [6], [7]). The description was made by means of the first order difference $G(\sigma)-1$ and the m-th order difference

$$\big(G(\sigma)-1\big)^m ,$$

respectively.

The purpose of this paper is to describe interpolation spaces between $D(A^{m-1})$ and $D(A^m)$ $(m \geqq 1)$, where now A is the infinitesimal generator of a generalized semi-group of a certain class (see [8]; see also [5], [9]). Since differences are not bounded operators they cannot be used any longer. We use instead the remainder of order m in Taylor's formula, i.e.

$$\frac{1}{(m-1)!}\int_0^s (s-\sigma)^{m-1}\,G(\sigma)\,A^m\,d\sigma .$$

The precise result will be found in section 3.

The first two sections are devoted to a brief presentation of definitions and results in the theory of interpolation spaces and generalized semi-groups, based on [6] and [7] and [8] (cf. also [5], [9]), respectively. In section 4 the result obtained in section 3 is used to extend theorems of Butzer and Tillmann [1] to generalized semi-groups. Finally, section 5 contains a further generalization of the results of the preceding sections.

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1. Preliminaries; interpolation spaces.

Let X_0 and X_1 be Banach spaces, continuously embedded in a Banach space E. The corresponding norms are denoted by

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$$a \to \|a\|_{X_0}$$

and

$$a \to \|a\|_{X_1},$$

respectively. In X_0+X_1 we consider the family of norms

$$K(s,a) = \inf_{a=a_0+a_1} (\|a_0\|_{X_0}+s\|a_1\|_{X_1}), \qquad 0<s<\infty .$$

Further, let Φ be a function norm, i.e. a positive functional defined on the set of positive functions on $(0,\infty)$ which are measurable with respect to the measure dt/t such that:

(1.1) $$\Phi(\varphi) = 0 \quad \text{if and only if} \quad \varphi = 0 \text{ a.e.}$$

(1.2) $$\Phi(\varphi) < \infty \quad \text{implies} \quad \varphi < \infty \text{ a.e.}$$

(1.3) $$\Phi(\alpha\varphi) = \alpha\Phi(\varphi) \quad \text{if} \quad \alpha \geqq 0 .$$

(1.4) $$\varphi \leqq \sum_{n=1}^{\infty} \varphi_n \text{ a.e.} \quad \text{implies} \quad \Phi(\varphi) \leqq \sum_{n=1}^{\infty} \Phi(\varphi_n) .$$

Then

$$\mathsf{K}(\Phi) = \mathsf{K}(\Phi; X_0, X_1) = \{a \mid a \in X_0+X_1,\ \Phi(K(s,a))<\infty\}$$

is a Banach space with the norm

$$a \to \Phi(K(s,a)) .$$

It can be shown that $\mathsf{K}(\Phi)$ is an interpolation space (see [6], [7]).

If

(1.5) $$\Phi(\min(1,s)) < \infty ,$$

then

$$X_0 \cap X_1 \subset \mathsf{K}(\Phi) \subset X_0+X_1 ,$$

(see [7]). On the other hand, if (1.5) does not hold, then $\mathsf{K}(\Phi)=\{0\}$. In the sequel we will therefore assume (1.5).

In the special case that we will consider, we have

(1.6) $$\|a\|_{X_0} \leqq \|a\|_{X_1} ,$$

so that $X_0 \supset X_1$.

From (1.6) it follows that

(1.7) $$K(s,a) = \|a\|_{X_0} \quad \text{if } s \geqq 1 .$$

Indeed, obviously $K(s,a) \leqq \|a\|_{X_0}$ for all $s>0$, and

$$\|a\|_{X_0} \leqq \|a_0\|_{X_0} + \|a_1\|_{X_1} \leqq \|a_0\|_{X_0} + s\|a_1\|_{X_1} ,$$

if $a=a_0+a_1$, $s \geqq 1$. From the first inequality we also get

$$s\,\|a\|_{X_0} \leqq K(s,a) \quad \text{if } s<1\ ,$$

and thus

(1.8)
$$K(s,a) \geqq \min(1,s)\,\|a\|_{X_0}\ .$$

REMARK 1.1. This formula also shows that $\mathsf{K}(\Phi)=\{0\}$ if (1.5) does not hold.

2. Preliminaries; generalized semi-groups.

Let $\mathscr{D}_+$ be the space of infinitely differentiable complex-valued functions with compact supports contained in $\mathsf{R}_+=\{t \mid t>0\}$. Moreover, let $\mathscr{L}(E)$ be the algebra of bounded linear operators in E. A generalized semi-group on E is a linear mapping G from $\mathscr{D}_+$ into $\mathscr{L}(E)$ such that:

$$G(\varphi*\psi) = G(\varphi)G(\psi), \qquad \varphi,\psi \in \mathscr{D}_+\ ,$$

$$G(\varphi) \to G(\varphi_0) \text{ uniformly} \quad \text{when} \quad \varphi \to \varphi_0 \text{ in } \mathscr{D}_+$$

(see [5], [8], [9]).

For example, let $G(t)$ be a strongly continuous semi-group, that is, a mapping from R_+ into $\mathscr{L}(E)$ such that:

$$G(s+t) = G(s)G(t), \qquad G(s) \to G(s_0) \text{ strongly} \quad \text{when} \quad s \to s_0\ .$$

Then, if we define G on $\mathscr{D}_+$ by the formula

$$G(\varphi)a = \int_0^\infty \varphi(t)G(t)a\,dt\ ,$$

G is a generalized semi-group.

Now it is possible to extend the domain of G to the space $\overline{\mathscr{E}}_+'$ of distributions with compact supports in $\overline{\mathsf{R}}_+=\{t \mid t\geqq 0\}$, so that:

$$G(T*S)a = G(T)G(S)a, \qquad S,T \in \overline{\mathscr{E}}_+'\ ,$$

$$G(T)a \to G(T_0)a, \qquad T \to T_0 \text{ in } \overline{\mathscr{E}}_+'\ ,$$

where

$$a \in R = \{a \mid a=\textstyle\sum G(\varphi_n)a_n,\ \varphi_n \in \mathscr{D}_+,\ a_n \in E\}\ ,$$

which is assumed dense in E (see [8], [9]). Thus $G(T)$ is, in general, an unbounded operator with domain R.

Put

$$G(\delta_\sigma)a = G(\sigma)a, \qquad a \in R,\ \sigma>0\ ,$$

where δ_σ is the Dirac measure in the point σ. If $T \in \overline{\mathscr{E}}_+'$ then, as can be shown,

$$G(T)a = \langle T(\sigma); G(\sigma)a\rangle = \int_0^\infty T(\sigma)G(\sigma)a\, d\sigma , \tag{2.1}$$

where $a \in R$. Write $\overline{G(T)}$ for the closure of $G(T)$. Then, by definition,

$$A = -\overline{G(\delta_0')}$$

is the infinitesimal generator and

$$A^m = (-1)^m\overline{G(\delta_0^{(m)})}, \qquad m \geqq 1 . \tag{2.2}$$

Now we say that the semi-group G is of class $\sigma(k)$, if for every $\varphi \in \mathscr{D}_+$ we have that

$$\|G(\varphi)\| \leqq C \int_0^\infty t^k|\varphi^{(k)}(t)|\, dt . \tag{2.3}$$

Then the same inequality holds for $\varphi \in \overline{\mathscr{D}}_+{}^k$, where $\overline{\mathscr{D}}_+{}^k$ is the space of k-times continuously differentiable functions with compact supports in R_+, and even for φ in the space M_k of measures T with compact supports in R_+ such that $|T|\{0\}=0$ and such that the restriction of $T^{(k)}$ to R_+ is a measure with

$$\int_{+0}^\infty t^k|T^{(k)}(t)|\, dt < \infty$$

(see [8]). After a change of variable (2.3) becomes

$$\|G(\varphi_s)\| \leqq C \int_{+0}^\infty t^k|\varphi^{(k)}(t)|\, dt, \qquad \varphi \in M_k , \tag{2.4}$$

where

$$\varphi_s(t) = s^{-1}\varphi(ts^{-1}) .$$

From (2.2) it follows that

$$\frac{d^n}{ds^n} G(s)a = G(s)\, A^n a \tag{2.5}$$

(cf. section 5). Hence

$$G(s)a - a = \int_0^s G(\sigma)\, Aa\, d\sigma .$$

Iteration of this leads to Taylor's formula

$$G(s)a = \sum_{n=0}^{m-1} \frac{s^n}{n!} A^n a + R_m(s)a , \tag{2.6}$$

where

$$(2.7) \qquad R_m(s)\,a = \int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}\, G(\sigma)\, A^m a\, d\sigma, \qquad a \in D(A^m)\,.$$

3. Interpolation between $D(A^{m-1})$ and $D(A^m)$.

In this section we put

$$X_0 = D(A^{m-1}), \qquad X_1 = D(A^m)\,.$$

The norms in $D(A^{m-1})$ and $D(A^m)$ will be denoted

$$\|a\|_{m-1} \quad \text{and} \quad \|a\|_m\,,$$

respectively, and are defined by

$$\|a\|_{m-1} = \|a\| + \|A^{m-1}a\|, \qquad \|a\|_0 = \|a\|\,,$$
$$\|a\|_m = \|a\| + \|A^m a\|\,,$$

where $\|\cdot\|$ is the norm in E. Then we may suppose

$$(3.1) \qquad \|a\|_{m-1} \leqq \|a\|_m\,.$$

(We have $\|a\|_{m-1} \leqq C\|a\|_m$. By replacing $\|a\|_m$ with the equivalent norm $C\|a\|_m$ we get (3.1)). We also put (cf. (2.7))

$$\varrho_m(s,a) = s^{1-m} \sup_{0<\sigma\leqq s} \|R_m(\sigma)a\|\,.$$

THEOREM 3.1. *If G is of class $\sigma(k)$ and if $m>k$, then for $s>0$,*

$$C^{-1}K(s,a) \leqq \varrho_m(s,a) + \min(1,s)\,\|a\|_{m-1} \leqq CK(s,a)\,,$$

where C is independent of a and s.

PROOF. (i) The second inequality:

Let $a=a_0+a_1$, $a_0 \in D(A^{m-1})$, $a_1 \in D(A^m)$, be a given decomposition of a. Then

$$R_m(\sigma)a_1 = \frac{\sigma^m}{(m-1)!}\frac{1}{\sigma}\int_0^\sigma (1-\tau\sigma^{-1})^{m-1}\, G(\tau)\, A^m a_1\, d\tau\,.$$

But

$$\frac{1}{\sigma}\int_0^\sigma (1-\tau\sigma^{-1})^{m-1}\, G(\tau)\, b\, d\tau = G(\varphi_\sigma)b\,,$$

if

$$\varphi(t) = \begin{cases} (1-t)^{m-1}, & 0 \leqq t < 1\,, \\ 0, & t \geqq 1\,. \end{cases}$$

(cf. (2.1)). According to (2.4) we obtain

$$\|R_m(\sigma)a_1\| \leqq \frac{\sigma^m}{(m-1)!}\|G(\varphi_\sigma)A^m a_1\| \leqq C\sigma^m\|A^m a_1\|,$$

and thus

$$\varrho_m(s,a_1) \leqq Cs\|A^m a_1\| \leqq Cs\|a_1\|_m .$$

Now we have

$$R_m(\sigma)a_0 = R_{m-1}(\sigma)\,a_0 - \frac{\sigma^{m-1}}{(m-1)!}A^{m-1}a_0 , \tag{3.2}$$

(we put $R_0(\sigma)b = G(\sigma)b$). Since

$$R_{m-1}(\sigma)a_0 = \frac{\sigma^{m-1}}{(m-2)!}\frac{1}{\sigma}\int_0^\sigma (1-\tau\sigma^{-1})^{m-2}\,G(\tau)\,A^{m-1}a_0\,d\tau ,$$

we again obtain from (2.4) that

$$\|R_{m-1}(\sigma)a_0\| \leqq C\sigma^{m-1}\|A^{m-1}a_0\| .$$

Hence

$$\|R_m(\sigma)a_0\| \leqq C\sigma^{m-1}\|A^{m-1}a_0\| ,$$

and thus

$$\varrho_m(s,a_0) \leqq C\|A^{m-1}a_0\| \leqq C\|a_0\|_{m-1} .$$

Finally we obtain

$$\varrho_m(s,a) \leqq \varrho_m(s,a_0)+\varrho_m(s,a_1) \leqq C(\|a_0\|_{m-1}+s\|a_1\|_m) ,$$

from which follows

$$\varrho_m(s,a) \leqq CK(s,a) .$$

From (1.8) we get

$$\min(1,s)\,\|a\|_{m-1} \leqq K(s,a) ,$$

and hence the second inequality of the theorem is proved.

(ii) The first inequality: Put

$$a_1 = ms^{-m}\int_0^s (s-\sigma)^{m-1}\,G(\sigma)a\,d\sigma = ms^{-1}\int_0^s (1-\sigma s^{-1})^{m-1}\,G(\sigma)a\,d\sigma$$

and

$$a_0 = a - a_1 .$$

Then, according to (2.4),

$$\|a_1\| \leqq C\|a\| \leqq C\|a\|_{m-1} . \tag{3.3}$$

Moreover we have

$$a_0 = -ms^{-m}\int_0^s (s-\sigma)^{m-1}\big(G(\sigma)-1\big)a\,d\sigma$$

$$= -ms^{-m}\int_0^s (s-\sigma)^{m-1}\Big(\int_0^\sigma G(\tau)\,Aa\,d\tau\Big)\,d\sigma$$

$$= -ms^{-m}\int_0^s \Big(\int_\tau^s (s-\sigma)^{m-1}\,d\sigma\Big)\,G(\tau)\,Aa\,d\tau$$

$$= -s^{-m}\int_0^s (s-\tau)^m\,G(\tau)\,Aa\,d\tau = -\int_0^s (1-\tau s^{-1})^m\,G(\tau)\,Aa\,d\tau$$

if $m>1$. Thus

(3.4) $$\|a_0\| \leqq Cs\,\|Aa\| \leqq Cs\,\|a\|_{m-1}, \qquad m>1\,.$$

If $m=1$ we easily get

(3.4′) $$\|a_0\| = s^{-1}\Big\|\int_0^s (G(\sigma)-1)a\,d\sigma\Big\| \leqq \varrho_1(s,a)\,.$$

Now

(3.5) $$A^m a_1 = m!\,s^{-m}\int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}\,G(\sigma)A^m a\,d\sigma = m!\,s^{-m}R_m(s)a\,,$$

and thus

(3.6) $$s\|A^m a_1\| \leqq C\varrho_m(s,a)\,.$$

Moreover we have

$$A^{m-1}a_1 = m!\,s^{-m}\int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}\,G(\sigma)A^{m-1}a\,d\sigma$$

$$= m!\,s^{-m}\int_0^s\int_0^\sigma \frac{(\sigma-\tau)^{m-2}}{(m-2)!}\,G(\tau)A^{m-1}a\,d\tau d\sigma = m!\,s^{-m}\int_0^s R_{m-1}(\sigma)a\,d\sigma\,.$$

It follows that

$$A^{m-1}a_0 = A^{m-1}a - A^{m-1}a_1 = m!\,s^{-m}\int_0^s \Big[\frac{\sigma^{m-1}}{(m-1)!}A^{m-1} - R_{m-1}(\sigma)\Big]\,a\,d\sigma$$

$$= -m!\,s^{-m}\int_0^s R_m(\sigma)a\,d\sigma\,,$$

which gives

(3.7) $$\|A^{m-1}a_0\| \leqq C\varrho_m(s,a)\,.$$

From (3.3) and (3.6) we obtain

$$(3.8) \qquad s\|a\|_m \leqq C\big(\varrho_m(s,a)+s\|a\|_{m-1}\big)\,,$$

and from (3.4) (or (3.4′)) and (3.7)

$$(3.9) \qquad \|a_0\|_{m-1} \leqq C\big(\varrho_m(s,a)+s\|a\|_{m-1}\big)\,.$$

Finally

$$K(s,a) \leqq \|a_0\|_{m-1}+s\|a_1\|_m \leqq C\big(\varrho_m(s,a)+s\|a\|_{m-1}\big)\,,$$

which is the first inequality for $0<s<1$. But if $s \geqq 1$ it follows from (1.7) that

$$K(s,a) = \|a\|_{m-1} \leqq \varrho_m(s,a)+\|a\|_{m-1}\,.$$

Hence the first inequality is proved.

Now we put

$$W_\Phi{}^m = \{a \mid a \in D(A^{m-1}),\ \Phi\big(\varrho_m(s,a)\big)<\infty\}\,,$$

where Φ is a function norm such that (1.5) holds. Then $W_\Phi{}^m$ is a Banach space with the norm

$$a \to \Phi\big(\varrho_m(s,a)\big)+\|a\|_{m-1}\,.$$

From theorem 3.1 we immediately get

COROLLARY 3.1. *If G is of class $\sigma(k)$ and if $m>k$ then*

$$W_\Phi{}^m = \mathsf{K}\big(\Phi;\, D(A^{m-1}), D(A^m)\big)\,,$$

except for an equivalence of norms.

REMARK 3.1. For $k=0$, $m=1$ we get that the space

$$W_\Phi{}^1 = \{a \mid a \in E,\ \Phi\big(\sup_{0<\sigma\leqq s}\|G(\sigma)a-a\|\big)<\infty\}$$

is identical with the interpolation space $\mathsf{K}\big(\Phi;\, E, D(A)\big)$, except for an equivalence of norms. This is the description made in [7, pp. 9–10] if

$$\Phi(\varphi) = \Phi_{\theta,p}(\varphi) = \left(\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p}, \qquad 0<\theta<1,\ 1\leqq p\leqq\infty\,,$$

(cf. also [4] and [6, pp. 55–57]).

4. Extension of theorems of Butzer and Tillmann.

If G is of class $\sigma(k)$ and if $m>k$, then

$$m s^{-m}\int_0^s (s-\sigma)^{m-1}\, G(\sigma)b\, d\sigma \to b, \qquad s \to +0\,,$$

(see [8]). This implies that if $a \in D(A^m)$, then

$$m!\, s^{-m} R_m(s)a = m!\, s^{-m} \int_0^s (s-\sigma)^{m-1}\, G(\sigma)\, A^m a \, d\sigma \to A^m a\,.$$

We have the following converse:

THEOREM 4.1. *If G is of class $\sigma(k)$, if $m > k$ and if there is an element $b \in E$ such that*

$$\lim_{s \to +0} \|m!\, s^{-m} R_m(s)a - b\| = 0, \qquad a \in D(A^{m-1})\,, \tag{4.1}$$

then

$$a \in D(A^m) \qquad \text{and} \qquad b = A^m a\,.$$

REMARK 4.1. This and the following theorem were proved by Butzer and Tillmann [1] for ordinary semi-groups. After a slight modification their proofs can be carried over to generalized semi-groups. Here we give alternative proofs, along the lines of theorem 3.1.

PROOF OF THEOREM 4.1. In the proof of theorem 3.1 we constructed a decomposition $a = a_0 + a_1$, such that

$$A^m a_1 = m!\, s^{-m} R_m(s)\, a \tag{4.2}$$

(formula (3.5)), and such that ((3.8) and (3.9))

$$\|a_0\|_{m-1} + s\|a_1\|_m \leqq C(\varrho_m(s,a) + s\,\|a\|_{m-1})\,.$$

But according to the assumptions we have

$$\varrho_m(s,a) = s \sup_{0<\sigma \leqq s} \|s^{-m} R_m(\sigma)a\| = O(s), \qquad s \to +0\,.$$

Hence

$$\|a_0\|_{m-1} + s\|a_1\|_m = O(s), \qquad s \to +0\,,$$

so that

$$a_0 \to 0 \quad \text{in } D(A^{m-1})\,,$$

which implies

$$a_1 \to a \quad \text{in } D(A^{m-1})\,.$$

But, by (4.1) and (4.2),

$$A^m a_1 \to b\,.$$

Since A^m is a closed operator the conclusion follows.

THEOREM 4.2. *If E is reflexive, if G is of class $\sigma(k)$ and if $m > k$, then*

$$\sup_{0<s \leqq 1} \|m!\, s^{-m} R_m(s)\, a\| < \infty, \qquad a \in D(A^{m-1})\,,$$

implies

$$a \in D(A^m)\,.$$

PROOF. Since $\varrho_m(s,a)=O(s)$ we get from theorem 3.1 that there is a decomposition $a=a_{0s}+a_{1s}$ such that

$$\|a_{0s}\|_{m-1}+s\|a_{1s}\|_m \leqq CK(s,a) = O(s), \qquad s \to 0 .$$

Hence $a_{0s} \to 0$ in $D(A^{m-1})$ and

$$\sup_{0<s<1} \|a_{1s}\|_m < \infty .$$

The space $D(A^m)$ with the graph-norm $\|a\|+\|A^m a\|$ can be identified with the graph of A^m. It is then a closed sub-space of $E \times E$, which is reflexive. This implies that $D(A^m)$ is reflexive. Then the set $\{a_{1s}, 0<s<1\}$, being bounded, is weakly sequentially compact. Therefore the sequence $a_{1,1/n}$ has a weakly convergent subsequence b_ν. Then, if $b_\nu \to b$ weakly in $D(A^m)$, we easily obtain $a=b$ and $a \in D(A^m)$. (Concerning these general results on reflexive spaces see e.g. [2, pp. 65–69].)

5. Generalization of the preceding results.

Let $T \in M_k$ and G be of class $\sigma(k)$. We recall that

$$T_s(x) = s^{-1}T(xs^{-1}) .$$

Now we have

$$\frac{d}{ds}T_s(x) = -s^{-2}T(xs^{-1})-s^{-1}\frac{d}{dx}T(xs^{-1}) = -\frac{d}{dx}(xT)_s = -\delta_0'*(xT)_s ,$$

and in general

$$\frac{d^n}{ds^n}T_s = (-1)^n\delta_0^{(n)}*(x^nT)_s .$$

From (2.2) it follows that

$$\frac{d^n}{ds^n}G(T_s)a = G\left(\frac{d^n}{ds^n}T_s\right)a = G\big((x^nT)_s\big)A^na ,$$

and from this we obtain Taylor's formula:

$$G(T_s)a = \sum_{n=0}^{m-1}\frac{C_n(T)s^n}{n!}A^na+R_m(T_s), \tag{5.1}$$

where

$$R_m(T_s) = \int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}G\big((x^mT)_\sigma\big)A^ma\,d\sigma$$

and

$$C_n(T) = \int_0^\infty x^n T(x)\, dx$$

(cf. (2.6)). We put

$$\varrho_m(T; s,a) = s^{1-m} \sup_{0<\sigma \leqq s} \|R_m(T_\sigma)a\| .$$

The proof of theorem 3.1 depends essentially on Taylor's formula (5.1) (for $T = \delta_1$) and on (2.4). We can thus prove the following generalization.

Theorem 5.1. *If G is of class $\sigma(k)$, if $T \in M_k$ and if $m > k$, $C_m(T) \neq 0$, then*

$$C^{-1}k(s,a) \leqq \varrho_m(T; s,a) + \varrho_m(xT; s,a) + \min(1; s)\, \|a\|_{m-1} \leqq Ck(s,a) .$$

Remark 5.1. The special case $m = 1$ can be found in [8].

Proof. (i) The second inequality:
Let $a = a_0 + a_1$, $a_0 \in D(A^{m-1})$, $a_1 \in D(A^m)$ be a decomposition of a. Since

$$R_m(T_\sigma)a_1 = \frac{\sigma^m}{(m-1)!} \frac{1}{\sigma} \int_0^\sigma (1 - \tau\sigma^{-1})^{m-1} G\big((x^m T)_\tau\big) A^m a_1\, d\tau ,$$

we have, according to (2.5),

$$\|R_m(T_\sigma)a_1\| \leqq C\sigma^{m-1} \|A^m a_1\| .$$

Moreover

$$R_m(T_\sigma)a_0 = R_{m-1}(T_\sigma)a_0 - \frac{C_{m-1}(T)\sigma^{m-1}}{(m-1)!} A^{m-1} a_0 ,$$

which by (2.4) gives

$$\|R_m(T_\sigma a_0\| \leqq C\sigma^{m-1} \|A^{m-1} a_0\| .$$

Hence we easily get

$$\varrho_m(T; s,a) \leqq CK(s,a) .$$

By substituting xT for T we get

$$\varrho_m(xT; s,a) \leqq CK(s,a) .$$

Now the second inequality follows by applying (1.8).
(ii) The first inequality: Put

$$a_1 = m s^{-m} \int_0^s (s-\sigma)^{m-1}\, G\big((x^m T)_\sigma\big)\, a\, d\sigma, \qquad a_0 = a - a_1 ,$$

and suppose that $C_m(T) = 1$. From (2.5) we obtain

(5.2) $$\|a_1\| \leqq C\|a\|_{m-1}\,.$$

A simple calculation, analogous to the corresponding one in the proof of theorem 3.1 gives

$$\begin{aligned}-a_0 &= ms^{-m}\int_0^s (s-\sigma)^{m-1}\big(G((x^mT)_\sigma)-1\big)a\,d\sigma \\ &= ms^{-m}\int_0^s (s-\sigma)^{m-1}\left(\int_0^\sigma G((x^{m+1}T)_\tau)Aa\,d\tau\right)d\sigma \\ &= \int_0^s (1-\tau s^{-1})^m G((x^{m+1}T)_\tau)Aa\,d\tau, \qquad m>1\,,\end{aligned}$$

and thus

(5.3) $$\|a_0\| \leqq Cs\|a\|_{m-1}, \qquad m>1\,.$$

If $m=1$ we easily get

(5.3′) $$\|a_0\| \leqq C\varrho_1(xT,s,a)\,.$$

Since

$$A^m a_1 = m!\,s^{-m}R_m(T_s)a\,,$$

we obtain

(5.4) $$\|A^m a_1\| \leqq C\varrho_m(T;s,a)\,.$$

We also have

$$\begin{aligned}A^m a_0 &= m!\,s^{-m}\int_0^s \left[\frac{\sigma^{m-1}}{(m-1)!}A^{m-1}-R_{m-1}((xT)_\sigma)\right]a\,d\sigma \\ &= -m!\,s^{-m}\int_0^s R_m((xT)_\sigma)a\,d\sigma\,.\end{aligned}$$

Hence

(5.5) $$\|A^m a_0\| \leqq C\varrho_m(xT;s,a)\,.$$

As in the proof of theorem 3.1 it follows that

$$K(s,a) \leqq C\big(\varrho_m(T;s,a)+\varrho_m(xT;s,a)+s\|a\|_{m-1}\big)\,.$$

From (1.7) we obtain

$$K(s,a) = \|a\|_{m-1} \quad \text{if } s\geqq 1\,,$$

and hence the first inequality follows.

Now, let $W^m_{\Phi,T}$ be the Banach space of all $a\in E$ such that the norm

$$\|a\|_{W^m_{\Phi,T}} = \Phi(\varrho_m(T;s,a)) + \Phi(\varrho_m(xT;s,a)) + \|a\|_{m-1}\,,$$

is finite. If (1.5) holds, theorem 5.1 implies that

$$C^{-1}\Phi\big(K(s,a)\big) \leqq \|a\|_{W^m_{\Phi,T}} \leqq C\Phi\big(K(s,a)\big)\,.$$

We have proved

COROLLARY 5.1. *Except for an equivalence of norms we have*

$$W^m_{\Phi,T} = \mathsf{K}\big(\Phi;\, D(A^{m-1}),\, D(A^m)\big) = W_\Phi{}^m\,.$$

Obviously we also have

COROLLARY 5.2. $a \in \mathsf{K}\big(\Phi;\, D(A^{m-1}),\, D(A^m)\big)$ *if and only if*

$$a \in D(A^{m-1}) \quad \textit{and} \quad \Phi\big(\varrho_m(T;\, s,a)\big) < \infty\,,$$

for all $T \in M_k$ (cf. [8, p. 96–98]).

From theorem 5.1 also follows

THEOREM 5.2. *Under the assumptions of theorem* 5.1 *and if there is a* $b \in E$ *such that*

$$\lim_{s \to +0} \|m!\, s^{-m} R_m(T_s)a - b\| = 0, \qquad a \in D(A^{m-1})\,, \tag{5.6}$$

then $a \in D(A^m)$, $b = A^m a$. *Conversely, if* $a \in D(A^m)$, *then* (5.6) *holds with* $b = A^m a$.

THEOREM 5.3. *If* E *is reflexive and if*

$$\sup_{0<s\leqq 1} \|m!\, s^{-m} R_m(T_s)a\| < \infty, \qquad a \in D(A^{m-1})\,,$$

then the assumptions of theorem 5.1 *imply that* $a \in D(A^m)$.

The proofs of these theorems are analogous to those of theorems 4.1 and 4.2 and will be omitted.

REFERENCES

1. P. L. Butzer and H. G. Tillmann, *Approximation theorems for semi-groups of bounded linear transformations*, Math. Ann. 140 (1960), 256–262.
2. N. Dunford and J. T. Schwartz, *Linear operators* I, New York, 1958.
3. E. Hille and R. S. Phillips, *Functional analysis and semi-groups* (Revised edition), New York, 1957.
4. J. L. Lions, *Théorèmes de trace et d'interpolation* (I), Ann. Scoula Norm. Sup. Pisa 13 (1959), 389–403.
5. J. L. Lions, *Les semi-groupes distributions*, Portugal. Math. 19 (1960), 141–164.
6. J. L. Lions et J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1963), 5–68.

7. J. Peetre, *Espaces d'interpolation; generalisations, applications*, Rend. Sem. Mat. Fis. Milano 34 (1964), 2–34.
8. J. Peetre, *Sur la théorie des semi-groupes distributions*, Collège de France, Séminaire sur les équations aux dérivées partielles II, novembre 1963 – mai 1964, 76–99.
9. K. Yoshinaga, *Ultra-distributions and semi-groups distributions*, Bull. Kyushu Inst. Techn. 10 (1963), 1–24.

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