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ON WEAK SEPARATION OF CONVEX SETS AND ON α-UBIQUITOUS SETS

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1. Introduction.

We say that a linear functional f on a real vector space X *weakly separates* two subsets A and B of X iff for some real number k, the functional f is $\geqq k$ on one of the subsets and $\leqq k$ on the other but is not identically $=k$ on $A \cup B$. We say that a subset A of X is *radial* at a point x of X or that x is an *inner point* of A iff for each $y \in X$ there exists a positive number t so that $x+[0,t)y \subset A$.

The most fundamental result concerning weak separation of convex sets in real vector spaces is the following

THEOREM A. *If A and B are disjoint convex subsets of a real vector space X and if one of these subsets has an inner point, then there exists a linear functional weakly separating A and B.*

This theorem is a direct consequence of the Hahn–Banach Theorem which is in turn implied by this separation theorem. Any attempt to weaken the hypotheses of this separation theorem may therefore be regarded as an attempt to generalize the Hahn–Banach Theorem.

A number of years ago A. P. Morse proposed the following

PROBLEM. *If A and B are disjoint convex subsets of a real vector space X, does there exist a linear functional weakly separating A and B provided that some member of A is an inner point of $X \setminus B$?*

The author has shown in a part of his dissertation [9] as yet unpublished that the separation cannot in general be effected under these hypotheses in spaces of infinite dimension. More precisely it was shown that

THEOREM B. *If X is a real vector space of linear dimension $\aleph_0$, then there exists a convex subset C of X such that the origin θ, is an inner point of*

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$X \setminus C$ and such that any linear functional which is non-negative on C is identically zero on X.

The *linear dimension* referred to in this theorem is the cardinal number of a Hamel basis for X.

In light of subsequent remarks in this introduction it will be seen that this theorem is considerably generalized in Theorem 3.2 of this paper.

The separation problems under discussion can be rephrased in terms of the *lin* operation which is defined for any subset A of a real vector space X by

$$\operatorname{lin} A = A \cup \{x : \exists y \neq x,\ x + (0,1]y \subset A\}.$$

(Here $(0,1]$ denotes a half-open interval on the real line.) That is to say $x \in \operatorname{lin} A$ iff either $x \in A$ or x is the endpoint of a segment contained in A. One further defines

$$\operatorname{lin}^{0} A = A$$

and for each ordinal $\alpha > 0$

$$\operatorname{lin}^{\alpha} A = \begin{cases} \operatorname{lin} \operatorname{lin}^{\alpha-1} A & \text{if } \alpha \text{ is a successor ordinal}, \\ \bigcup_{\beta<\alpha} \operatorname{lin}^{\beta} A & \text{if } \alpha \text{ is a limit ordinal}. \end{cases}$$

Clearly $\operatorname{lin}^{\beta} A \subset \operatorname{lin}^{\alpha} A$ if $\beta < \alpha$. One finally defines the *order* of a subset A of a real vector space X by

$$\operatorname{order} A = \text{the smallest ordinal } \alpha \text{ for which } \operatorname{lin}^{\alpha+1} A = \operatorname{lin}^{\alpha} A.$$

The following results have been obtained by O. M. Nikodym [7] [8] and subsequently demonstrated somewhat more simply by V. L. Klee [4]. In each of the statements enumerated below, X is a real vector space and C a convex subset of X.

1) Order $C \leqq \Omega$ where, Ω denotes the first non-countable ordinal.
2) If $\dim X = \aleph_0$, then order $C < \Omega$.
3) If $\dim X = \aleph_0$, then for each ordinal $\alpha < \Omega$ there exists a convex set C with order $C = \alpha$.
4) If $\dim X < \aleph_0$, then $\operatorname{lin} C = \bar{C}$ where $\bar{C}$ is the closure of C in the Euclidean topology for X, so that order $C = 0$ or 1 according as whether C is closed or not.
5) If C is convex, then for each ordinal α, $\operatorname{lin}^{\alpha} C$ is convex.

It is emphasized that 4) does not hold in general in the absence of the hypothesis that C is convex.

In order to see how the lin operation relates to the separation problem, refer to Theorem B and note that, in view of the convexity of C, the condition that θ is an inner point of $X \setminus C$ is equivalent to the condition that $\theta \notin \text{lin}\, C$. Further note that for any set C, convex or not, if a linear functional f is $\geqq k$ on C then f is $\geqq k$ on $\text{lin}^\alpha C$ for each ordinal α. Thus it is seen that Theorem B will follow if we can demonstrate:

THEOREM C. *If X is a real vector space of dimension $\aleph_0$, then there exists a convex subset C of X such that $\theta \notin \text{lin}\, C$ while $\text{lin}^2 C = X$.*

It was by proving essentially this theorem that the author obtained Theorem B in [9].

In this paper a more general theorem than Theorem C will be obtained. Klee [2] defines a set, A, to be *ubiquitous* iff $\text{lin}\, A = X$ and shows that a space of dimension $\aleph_0$, can be represented as the union of two (or in fact infinitely many) disjoint ubiquitous convex sets. [It is worthy of note that this result affords one way of demonstrating the impossibility in spaces of infinite dimension of weakly separating two convex sets under the sole hypothesis that they be disjoint. That weak separation can always be effected under this hypothesis in spaces of finite dimension has long been known.] In [4] Klee proposes the problem: for what (if any) ordinals α other than 0 and 1 do there exist convex sets C of order α such that $\text{lin}^\alpha C = X$.

The answer to this question is provided in the principal result of this paper, Theorem 3.2. If X is a real vector space with $\dim X = \aleph_0$ and α is an ordinal with $\alpha < \Omega$, then there exists a convex set C of order α such that $\text{lin}^\alpha C = X$.

It is clear that this theorem solves the cited problem of Klee and provides a negative answer to the separation problem of Morse implying as it does Theorems C and B.

We will define a subset A of a vector space X to be *α-ubiquitous* iff order $A = \alpha$ and $\text{lin}^\alpha A = X$.

Since the submission of this paper, some of the results have been obtained by V. L. Klee using different methods.

2. Preliminary definitions and lemmas.

The proofs of the main theorems of this paper depend on finding general conditions under which for a family Γ of convex sets we may conclude that

$$\text{lin}^\alpha \langle \bigcup \{C : C \in \Gamma\} \rangle = \langle \bigcup \{\text{lin}^\alpha C : C \in \Gamma\} \rangle, \tag{2.1}$$

where $\langle S \rangle$ denotes the *convex hull* of S, that is $\bigcap \{C : S \subset C \wedge C \text{ convex}\}$. This relation does not hold in general even in finite dimensional spaces when $\alpha = 1$. We give two examples in E_2.

EXAMPLE 2.1. For $n=1,2,3,\ldots$, let C_n be the segment of the line through the origin with argument n lying in the closed unit disc. Here $\text{lin}\langle\bigcup\{C_n: n=1,2,\ldots\}\rangle$ is closed while $\langle\bigcup\{\text{lin}\, C_n: n=1,2,\ldots\}\rangle$ is not.

EXAMPLE 2.2. Let $C_1=\{\theta\}$ and $C_2=\{(x,y): x\geqq 0, y=1\}$. Here

$$\text{lin}\langle\bigcup\{C_n: n=1,2\}\rangle = \{(x,y): x\geqq 0, 0\leqq y\leqq 1\}$$

while

$$\langle\bigcup\{\text{lin}\, C_n: n=1,2\}\rangle = \{\theta\}\cup\{(x,y): x\geqq 0, 0<y\leqq 1\}.$$

In the first example the conclusion (2.1) fails because the set Γ is infinite; in the second example because the members of Γ are not linearly bounded, where a convex set is said to be *linearly bounded* iff it contains no ray. If neither of these exceptions occurs then the conclusion holds with $\alpha=1$ in spaces of finite dimension. That is

REMARK 2.1. If $C_1, C_2, \ldots, C_n$ are linearly bounded convex subsets of a finite dimensional vector space, then

$$\text{lin}\langle\bigcup\{C_i: i=1,2,\ldots,n\}\rangle = \langle\bigcup\{\text{lin}\, C_i: i=1,2,\ldots,n\}\rangle.$$

This is easily proved using compactness, see for example [4]. This remark inspires the following Definition 2.1. Before stating this fundamental definition we introduce some notation which will be used throughout the remainder of this section and the next.

NOTATION: We let X denote a real vector space with $\dim X=\aleph_0$. We let I denote the set of non-negative integers, I' the set of positive integers, R the set of real numbers, P the interval $[1,\infty)$, Q the interval $(0,1]$. For use in discussing convex combinations we let K be the set of functions t satisfying:

(1) range $t\subset[0,1]$,
(2) $\{i: t(i)>0\}$ is finite,
(3) $\sum\{t(i): i\in \text{domain}\, t\}=1$.

We further agree that if b is a real number, A a set of real numbers, y a vector and C a set of vectors, then

$$AC = \{ax: a\in A \wedge x\in C\}, \qquad bC = \{bx: x\in C\}, \qquad Ay = \{ay: a\in A\}.$$

For each set S of subsets of X we define $\sigma S=\bigcup\{H: H\in S\}$. We use $S\subset\subset T$ to mean $\forall A\in S\, \exists B\in T\, (A\subset B)$. The *natural cone* of a basis $\mathscr{B}$ for X is defined to be the set of vectors which when expressed as linear combinations of members of $\mathscr{B}$ involve only non-negative coefficients.

DEFINITION 2.1. We say that a convex set C is *enveloped* (respectively, *delineated*) *by* Γ iff

(1) Γ is a family of linearly bounded convex sets (respectively, segments) with $C=\langle\sigma\Gamma\rangle$;
(2) for each finite dimensional subspace, Y, of X there is a finite dimensional subspace, Z, of X and a finite subset, Δ, of Γ such that

$$Y\cap C \subset \langle \bigcup\{Z\cap G : G\in\Delta\}\rangle .$$

We say that C is *enveloped* (respectively, *delineated*) iff for some Γ, C is enveloped (respectively, delineated) by Γ. We say Γ is an *enveloping* (respectively, *delineating*) *family* iff $\langle\sigma\Gamma\rangle$ is enveloped (respectively, delineated) by Γ. (It should be emphasized that it is *not* required of an enveloping family Γ that each finite dimensional subspace of X should intersect only finitely many members of Γ.)

It is clear that if C is delineated by Γ then C is enveloped by Γ.

Partial motivation for this definition is furnished by the following lemma.

LEMMA 2.1. *If Γ is an enveloping family, then*

$$\operatorname{lin}\langle\bigcup\{G : G\in\Gamma\}\rangle = \langle\bigcup\{\operatorname{lin} G : G\in\Gamma\}\rangle .$$

It is noted that the family in Example 2.1 above fails to satisfy condition (2) of the above definition while that in Example 2.2 fails to satisfy condition (1).

PROOF. Let $C=\langle\bigcup\{G : G\in\Gamma\}\rangle$ and let $x\in\operatorname{lin} C$ with $x+Qy\subset C$. Let Y be the (at most two-dimensional) subspace of X spanned by x and y. Let Z be a finite dimensional subspace of X and Δ a finite subset of Γ so that

$$Y\cap C \subset \langle\bigcup\{Z\cap G : G\in\Delta\}\rangle .$$

The family $\{Z\cap G : G\in\Delta\}$ is a finite family of linearly bounded convex sets contained in the finite dimensional space Z so that

$$\operatorname{lin}\langle\bigcup\{Z\cap G : G\in\Delta\}\rangle = \langle\bigcup\{\operatorname{lin}(Z\cap G) : G\in\Delta\}\rangle$$

by Remark 2.1. Now

$$x+Qy \subset Y\cap C \subset \langle\bigcup\{Z\cap G : G\in\Delta\}\rangle$$

so that

$$\begin{aligned} x \in \operatorname{lin}\langle\bigcup\{Z\cap G : G\in\Delta\}\rangle &= \langle\bigcup\{\operatorname{lin}(Z\cap G) : G\in\Delta\}\rangle \\ &\subset \langle\bigcup\{\operatorname{lin} G : G\in\Gamma\}\rangle . \end{aligned}$$

This shows that

$$\operatorname{lin}\langle \bigcup \{G : G \in \Gamma\}\rangle \subset \langle \bigcup \{\operatorname{lin} G : G \in \Gamma\}\rangle .$$

The inclusion the other way follows immediately from the observation that for each $G \in \Gamma$,

$$\operatorname{lin} G \subset \operatorname{lin}\langle \bigcup \{G : G \in \Gamma\}\rangle$$

and from the fact that $\operatorname{lin} A$ is convex whenever A is convex. The proof is therefore complete.

DEFINITION 2.2: C is *α-enveloped* (respectively, *α-delineated*) *by* F iff:

(1) F is a function on the set of ordinals β with $\beta \leqq \alpha$, whose values are families of subsets of X;
(2) for each ordinal $\beta \leqq \alpha$, $\operatorname{lin}^\beta C$ is enveloped (respectively, delineated) by $F(\beta)$;
(3) for $\gamma < \beta \leqq \alpha$, $F(\gamma) \subset \subset F(\beta)$.

We say that C is *α-enveloped* (respectively, *α-delineated*) iff for some function F, C is α-enveloped (respectively, α-delineated) by F. We say that F is an *α-enveloping* (respectively, *α-delineating*) function iff $\langle \sigma F(0)\rangle$ is α-enveloped (respectively, α-delineated) by F. If conditions (1), (2) and (3) hold with $\beta \leqq \alpha$ replaced by $\beta < \alpha$ in each of these conditions then we say that C is *α-subenveloped* (respectively, *α-subdelineated*) by F, etc.

The following remarks are self-evident consequences of the above definition.

REMARKS 2.1. 1. If C is α-enveloped (respectively, α-delineated) by F and $\beta < \alpha$ then C is β-enveloped (respectively, β-delineated) by $F \mid \{\gamma : \gamma \leqq \beta\}$.

2. If C is α-enveloped (respectively, α-delineated) by F and order $C \leqq \alpha$ then for each ordinal $\beta > \alpha$, C is β-enveloped (respectively, β-delineated) by G where $G(\gamma) = F(\gamma)$ for $\gamma \leqq \alpha$ and $G(\gamma) = F(\alpha)$ for $\alpha < \gamma \leqq \beta$.

3. If α is a successor ordinal then C is α-subenveloped (respectively, α-subdelineated) by F iff C is $(\alpha-1)$-enveloped (respectively $(\alpha-1)$-delineated) by F.

4. If C is α-subenveloped (respectively, α-subdelineated) by F then for each ordinal $\beta < \alpha$, C is β-enveloped (respectively, β-delineated) by $F \mid \{\gamma : \gamma \leqq \beta\}$.

5. If C is α-enveloped (respectively, α-delineated) by F then C is α-subenveloped (respectively, α-subdelineated) by $F \mid \{\beta : \beta < \alpha\}$.

6. If C is α-subenveloped (respectively, α-subdelineated) by F and $\operatorname{lin}^\alpha C$ is enveloped (respectively, delineated) by Γ and for each ordinal

$\beta<\alpha$, $F(\beta)\subset\subset\Gamma$, then C is α-enveloped (respectively, α-delineated) by G where $G(\beta)=F(\beta)$ for $\beta<\alpha$ and $G(\alpha)=\Gamma$.

7. If C is α-delineated by F then C is α-enveloped by F.

We next present some lemmas which give conditions under which

$$P \operatorname{lin}^{\alpha} C = \operatorname{lin}^{\alpha}(PC) .$$

The following example suffices to show that this is not in general true even when $\alpha=1$ and X is two-dimensional.

EXAMPLE 2.3. Let $C=\{(x,y) : (x-1)^2+y^2\leqq 1\}$. Then

$$\operatorname{lin}(PC) = \{(x,y) : x\geqq 0\}$$

while

$$P \operatorname{lin} C = \{\theta\} \cup \{(x,y) : x>0\} .$$

However,

LEMMA 2.2. *If C is a convex set in two dimensional Euclidean space, Y, and either $\theta\notin\operatorname{lin} C$ or $\operatorname{lin} C$ is a polygonal region, then*

$$P \operatorname{lin} C = \operatorname{lin}(PC) .$$

Before proving this lemma we state and prove

LEMMA 2.3. *For all $C\subset X$ and for all $\alpha<\Omega$*

$$P \operatorname{lin}^{\alpha} C \subset P \operatorname{lin}^{\alpha}(PC) = \operatorname{lin}^{\alpha}(PC) .$$

PROOF. The inclusions $P \operatorname{lin}^{\alpha} C\subset P \operatorname{lin}^{\alpha}(PC)$ and $\operatorname{lin}^{\alpha}(PC)\subset P \operatorname{lin}^{\alpha}(PC)$ are obvious. It only remains to show that $P \operatorname{lin}^{\alpha}(PC)\subset\operatorname{lin}^{\alpha}(PC)$ which we do first for $\alpha=1$. Let $x\in P \operatorname{lin}(PC)$. Then for some $k\in Q$, $kx\in \operatorname{lin}(PC)$ so that for some y, $kx+Qy\subset PC$. Thus since $1/k\in P$ we have

$$x+Q\frac{1}{k}y = \frac{1}{k}(kx+Qy) \subset PC ,$$

whence $x\in\operatorname{lin}(PC)$ which yields the desired inclusion for $\alpha=1$. Suppose now that the lemma has been demonstrated for all ordinals less than α where $\alpha>1$. Then if α is a successor ordinal,

$$\begin{aligned} P \operatorname{lin}^{\alpha}(PC) = P \operatorname{lin}\operatorname{lin}^{\alpha-1}(PC) &\subset \operatorname{lin} P \operatorname{lin}^{\alpha-1}(PC) \\ &\subset \operatorname{lin}\operatorname{lin}^{\alpha-1}(PC) = \operatorname{lin}^{\alpha}(PC) . \end{aligned}$$

If α is a limit ordinal then,

$$\begin{aligned} P \operatorname{lin}^{\alpha}(PC) = P \textstyle\bigcup_{\beta<\alpha}\operatorname{lin}^{\beta}(PC) &= \textstyle\bigcup_{\beta<\alpha} P \operatorname{lin}^{\beta}(PC) \\ &\subset \textstyle\bigcup_{\beta<\alpha}\operatorname{lin}^{\beta}(PC) = \operatorname{lin}^{\alpha}(PC) . \end{aligned}$$

This completes the proof.

PROOF OF LEMMA 2.2. In light of Lemma 2.3 it need only be shown that $\text{lin}(PC) \subset P \,\text{lin}\, C$.

Assume first that $\theta \notin \text{lin}\, C$. Let x and y be such that $x \in \text{lin}(PC)$ and $x+Qy \subset PC$. Suppose that $x \notin P \,\text{lin}\, C$. Then $[0,1]x \subset Y \setminus \text{lin}\, C$. Since $[0,1]x$ is compact and $\text{lin}\, C$ is closed in the Euclidean topology for Y, there is an open set U with $[0,1]x \subset U \subset Y \setminus \text{lin}\, C$ and a $z \in x+Qy$ such that $[0,1]z \subset U$. Hence $Qz \cap \text{lin}\, C = \emptyset$ or equivalently $z \notin P \,\text{lin}\, C$ whence a fortiori $z \notin PC$. This contradiction assures us that $x \in P \,\text{lin}\, C$ which completes the proof in the case that $\theta \notin \text{lin}\, C$.

Next assume that $\text{lin}\, C$ is a polygonal region. In view of the part of the proof already given we need only consider the case that $\theta \in \text{lin}\, C$. If θ is an interior point of $\text{lin}\, C$ then $P \,\text{lin}\, C = Y$ which gives the desired result. Finally consider the case that θ is a boundary point of $\text{lin}\, C$. Let S be the smallest sector (convex cone) with vertex at the origin containing $\text{lin}\, C$. Since $\text{lin}\, C$ is a polygonal region there are segments contained in $\text{lin}\, C$ on each of the bounding rays of S. Thus $P \,\text{lin}\, C = S$. Since $PC \subset S$ we have $\text{lin}(PC) \subset \text{lin} S = S$. This completes the proof.

Lemmas 2.2 and 2.3 facilitate the proofs of the important lemmas 2.4 and 2.6.

LEMMA 2.4. *If C is convex, $\alpha < \Omega$ and $\theta \notin \text{lin}^\alpha C$, then* $P \,\text{lin}^\alpha C = \text{lin}^\alpha (PC)$.

PROOF (by transfinite induction). From Lemma 2.3 we know that $P \,\text{lin}^\alpha C \subset \text{lin}^\alpha (PC)$. Let x and y be such that $x \in \text{lin}(PC)$ and $x + Qy \subset PC$. Let Y be the (at most two-dimensional) subspace of X spanned by x and y and let $C' = Y \cap C$. Now C' is a convex subset of Y with $\theta \notin \text{lin}\, C' \subset \text{lin}\, C$. Since $x + Qy \subset Y \cap (PC) = PC'$ we see that $x \in \text{lin}(PC')$. Invoking Lemma 2.2 we obtain $x \in P \,\text{lin}\, C' \subset P \,\text{lin}\, C$. This proves the lemma in the case that $\alpha = 1$. Assuming the lemma to have been proved for all ordinals less than α, we obtain the result for α by means of the following familiar computations. If α is a successor ordinal, then

$$\text{(2.2)} \qquad \begin{aligned} \text{lin}^\alpha (PC) = \text{lin}\,\text{lin}^{\alpha-1}(PC) &= \text{lin}\, P \,\text{lin}^{\alpha-1} C \\ &= P \,\text{lin}\,\text{lin}^{\alpha-1} C = P \,\text{lin}^\alpha C , \end{aligned}$$

while if α is a limit ordinal

$$\text{(2.3)} \qquad \begin{aligned} \text{lin}^\alpha (PC) = \textstyle\bigcup_{\beta<\alpha} \text{lin}^\beta (PC) &= \textstyle\bigcup_{\beta<\alpha} P \,\text{lin}^\beta C \\ &= P \textstyle\bigcup_{\beta<\alpha} \text{lin}^\beta C = P \,\text{lin}^\alpha C . \end{aligned}$$

This completes the proof.

LEMMA 2.5. *If C is delineated, then* $\text{lin}(PC) = P \,\text{lin}\, C$.

PROOF. Again, by Lemma 2.3, we need only show that $\mathrm{lin}(PC) \subset P\,\mathrm{lin}\,C$. Let Γ be a family which delineates C. Let x and y be such that $x \in \mathrm{lin}(PC)$ and $x + Qy \subset PC$. Let Y be the space (at most two-dimensional) spanned by x and y and let Z be such a finite dimensional subspace of X and Δ such a finite subset of Γ that

$$(2.4) \qquad Y \cap C \subset \bigcup \langle \{Z \cap G : G \in \Delta\} \rangle .$$

Letting $C' = Y \cap C$, (2.4) yields

$$(2.5) \qquad C' = Y \cap C \subset Y \cap \langle \bigcup \{Z \cap G : G \in \Delta\} \rangle \subset Y \cap C = C' .$$

Since the family $\{Z \cap G : G \in \Delta\}$ is a finite collection of points and segments, the set $\mathrm{lin}\langle \bigcup \{Z \cap G : G \in \Delta\} \rangle$ is a polytope so that $C' = Y \cap C$ is by (2.5) a polygonal region in Y. Now $x + Qy \subset Y \cap (PC) = PC'$. Thus $x \in \mathrm{lin}(PC')$ whence, by Lemma 2.2, $x \in P\,\mathrm{lin}\,C'$. This completes the proof.

LEMMA 2.6. *If C is α-subdelineated, then* $\mathrm{lin}^\alpha(PC) = P\,\mathrm{lin}^\alpha C$.

PROOF (by transfinite induction). The case $\alpha = 1$ is covered by the preceding lemma. Assume that the lemma holds for all ordinals less than α where $\alpha > 1$. A repetition of the arguments (2.2) and (2.3) completes the proof.

The following lemma which will be employed in the final stages of the proof of the main theorem of Section 3 is so obvious as to require no proof.

LEMMA 2.7. *If C is contained in the natural cone of a basis $\mathscr{B}$ for X, then for each ordinal α,* $\mathrm{lin}^\alpha C$ *is contained in this cone.*

A few words of orientation may be helpful to the reader at this point. In the proof of the main Theorem 3.2 in the case that α is a limit ordinal there will be constructed a convex set C of order α such that θ is an inner point of $\mathrm{lin}^\alpha C$. The steps in the argument require that C be linearly bounded. We next pass to the set PC which is not linearly bounded. It is clear from Lemma 2.3 that $X = P\,\mathrm{lin}^\alpha C \subset \mathrm{lin}^\alpha(PC)$. It will next be necessary to show that order (PC) is not less than α. This will be achieved by use of Lemma 2.6 applied to C. The hypotheses of the said lemma require that C have a certain "polygonal" character namely that C be α-subdelineated. The notions of delineated, α-delineated, and α-subdelineated sets are required solely for the proof of Theorem 3.2 in the case that α is a limit ordinal as are Lemmas 2.5, 2.6, 2.7 and the subsequent Lemma 3.5 and the parts of Lemmas 3.2 and 3.4 which deal with delineated sets. The proof of Theorem 3.2 in the case that α is a limit ordinal also involves Theorem 3.1 in the case that α is a successor ordinal which in turn employs all the lemmas not mentioned above. It is seen then that the proof of Theorem 3.2 in the case that α is a limit ordinal requires a good deal more machinery than in the case that α is a successor ordinal.

3. Main results.

NOTATION. We denote by $Y \oplus Z$ the direct sum of the subspaces Y and Z and by $\sum^d \mathscr{X}$ the direct sum of the family, $\mathscr{X}$, of subspaces of X. If $M \subset \sum^d \mathscr{X}$ then for each $Z \in \mathscr{X}$ we define

$$\operatorname{proj} M \text{ on } Z(\mathscr{X}) = \{x : x \in Z \wedge \exists y \in \textstyle\sum^d (\mathscr{X} \setminus \{Z\}) (x+y \in M)\} .$$

We write simply $\operatorname{proj} M$ on Z when the family $\mathscr{X}$ is clearly understood. We denote by $[M]$ the space spanned by M.

LEMMA 3.1. *Suppose that* $X = \sum^d \{X_i : i \in I\}$. *Suppose that for each* $i \in I'$, C_i *is a linearly bounded convex subset of* X_i *and that* C_0 *is a linearly bounded, finite dimensional convex set. Let* $u_0 = \theta$ *and for* $k \in I'$ *let*

$$u_k \in \textstyle\sum^d \{X_i : i = 0, 1, \ldots, k-1\} .$$

Further suppose that $\{u_k : \theta \in C_k\} \subset C_0$. *Then the set*

$$C = \langle \bigcup \{u_i + C_i : i \in I\} \rangle$$

is enveloped by the family $\{u_i + C_i : i \in I\}$ *so that*

$$\operatorname{lin} C = \langle \bigcup \{u_i + \operatorname{lin} C_i : i \in I\} \rangle .$$

PROOF. The first of the conditions of Definition 2.1 is clearly satisfied. In order to establish condition (2) of this definition let Y be a finite dimensional subspace of X. Let m be the smallest integer such that

$$C_0 \cup Y \subset \textstyle\sum^d \{X_i : i = 0, 1, \ldots, m\} .$$

Let $W = [Y, C_0, u_1, u_2, \ldots, u_m]$. Note that W is finite dimensional and that

$$W \subset \textstyle\sum^d \{X_i : i = 0, 1, \ldots, m\} .$$

Now let

$$Z = \textstyle\sum^d \{\operatorname{proj} W \text{ on } X_i : i = 0, 1, \ldots, m\} .$$

Again Z is finite dimensional and

$$W \subset Z \subset \textstyle\sum^d \{X_i : i = 0, 1, \ldots, m\} .$$

Next let $x \in Y \cap C$. Then there exists $t \in K$ with domain $t = I$ so that

$$x = \sum \{t_i(u_i + x_i) \ : i \in I\}$$

where $x_i \in C_i$ when $t_i \neq 0$ and $x_i = \theta$ when $t_i = 0$. Let

$$I_0 = \{i : \ x_i = \theta \wedge t_i \neq 0\}, \qquad I_1 = \{i : \ x_i \neq \theta \wedge t_i \neq 0\}.$$

Now I_0 and I_1 are disjoint finite sets of non-negative integers and recall that for each $i \in I_0$,

$$u_i \in C_0 \subset W \subset \textstyle\sum^d \{X_i : \ i = 0, 1, \ldots, m\}.$$

Furthermore

$$x = \sum \{t_i u_i : i \in I_0\} + \sum \{t_i(u_i + x_i) \ : i \in I_1\}. \tag{3.1}$$

Observe that $u_i \in C_0$ for $i \in I_0$ and use the convexity of C_0 to write

$$\sum (t_i u_i : i \in I_0\} = t_0' x_0'$$

where

$$x_0' \in C_0, \qquad t_0' = \sum \{t_i : i \in I_0\}.$$

Now we have

$$x = t_0' x_0' + \sum \{t_i(u_i + x_i) \ : i \in I_1\}. \tag{3.2}$$

Let $k = \max I_1$, and using (3.2) write

$$t_k x_k = x - t_0' x_0' - t_k u_k - \sum \{t_i(u_i + x_i) \ : \ i \in I_1 \wedge i < k\}. \tag{3.3}$$

It is readily seen that each term on the righthand side of (3.3) belongs either to $\sum^d \{X_i : i \leqq m\}$ or to $\sum^d \{X_i : i < k\}$. Since $t_k \neq 0$ and $x_k \in X_k \setminus \{\theta\}$ we see that $k \leqq m$. It therefore follows that for $i \in I_1$ we have $u_i \in W$ and since $x \in Y \subset W$ and $x_0' \in C_0 \subset W$ we have from (3.2)

$$\sum \{t_i x_i : i \in I_1\} = x - t_0' x_0' - \sum \{t_i u_i : i \in I_1\} \in W. \tag{3.4}$$

The definition of Z together with (3.4) now assures us that $x_i \in Z$ for $i \in I_1$. And since as we have already noted, $u_i \in W \subset Z$ for $i \in I_1$ we find that

$$u_i + x_i \in Z \cap (u_i + C_i) \qquad \text{for} \quad i \in I_1$$

And now, since $t_0' + \sum \{t_i : i \in I_1\} = 1$, $x_0' \in C_0$, and $\max I_1 \leqq m$ we have

$$\begin{aligned} x &= t_0' x_0' + \sum \{t_i(u_i + x_i) \ : i \in I_1\} \\ &\in \langle \bigcup \{Z \cap (u_i + C_i) \ : \ i \in I_1 \vee i = 0\} \rangle \\ &\subset \langle \bigcup \{Z \cap (u_i + C_i) \ : \ i = 0, 1, 2, \ldots, m\} \rangle. \end{aligned}$$

Thus we see that

$$Y \cap C \subset \langle \bigcup \{Z \cap (u_i + C_i) \ : \ i = 0, 1, 2, \ldots, m\} \rangle.$$

Thus C is enveloped by the family $\{u_i+C_i : i \in I\}$. That $\operatorname{lin} C = \langle \bigcup \{u_i + \operatorname{lin} C_i : i \in I\} \rangle$ follows from Lemma 2.1. This completes the proof.

LEMMA 3.2. *If all the hypotheses of Lemma* 3.1 *hold and in addition, for each $i \in I$, C_i is enveloped* (*respectively, delineated*) *by a family Γ_i, then C is enveloped* (*respectively, delineated*) *by the family $\{u_i + G : G \in \Gamma_i \wedge i \in I\}$.*

PROOF. It is clear that we only need verify condition (2) of Definition 2.1. To this end let Y be a finite dimensional subspace of X and employ Lemma 3.1 to find a finite dimensional subspace Z' of X and an integer m so that

$$Y \cap C \subset \langle \bigcup \{Z' \cap (u_i + C_i) \;:\; i = 0,1,2,\ldots,m\} \rangle \,. \tag{3.5}$$

For each $i = 0,1,2,\ldots,m$ we have

$$Z' \cap (u_i + C_i) \;=\; u_i + (Z' - u_i) \cap C_i \subset u_i + [Z', u_i] \cap C_i \,. \tag{3.6}$$

And, since C_i is enveloped (respectively, delineated) by Γ_i, there exists a finite dimensional subspace, Z_i, of X_i and a finite subset, Δ_i, of Γ_i so that

$$[Z', u_i] \cap C_i \subset \langle \bigcup \{Z_i \cap G : G \in \Delta_i\} \rangle \,. \tag{3.7}$$

Now (3.6) and (3.7) yield for $i = 0,1,2,\ldots,m$

$$\begin{aligned} Z' \cap (u_i + C_i) &\subset u_i + [Z', u_i] \cap C_i \\ &\subset u_i + \langle \bigcup \{Z_i \cap G : G \in \Delta_i\} \rangle \\ &\subset \langle \bigcup \{[Z_i, u_i] \cap (u_i + G) \;:\; G \in \Delta_i\} \rangle \,. \end{aligned} \tag{3.8}$$

Let $Z = [Z_0, Z_1, \ldots, Z_m, u_1, u_2, \ldots, u_m]$ and note that Z is finite dimensional. Now combining (3.5) and (3.8) we have

$$\begin{aligned} Y \cap C &\subset \langle \bigcup \{Z' \cap (u_i + C_i) \;:\; i = 0,1,2,\ldots,m\} \rangle \\ &\subset \langle \bigcup \{[Z_i, u_i] \cap (u_i + G) \;:\; G \in \Delta_i \wedge i = 0,1,2,\ldots,m\} \rangle \\ &\subset \langle \bigcup \{Z \cap (u_i + G) \;:\; G \in \Delta_i \wedge i = 0,1,2,\ldots,m\} \rangle \,. \end{aligned}$$

The finiteness of each of the Δ_i, $i = 0,1,2,\ldots,m$, yields the desired result.

LEMMA 3.3. *Let X_i, C_i, u_i for $i \in I$ and C satisfy the hypotheses of Lemma* 3.1. *Let α be an ordinal > 0 and define by induction for each ordinal $\beta \leqq \alpha$*

$$C_0{}^\beta = \begin{cases} C_0 & \text{if } \beta = 0 \,, \\ \langle \operatorname{lin} C_0{}^{\beta-1} \cup \{u_i : \; \theta \in \operatorname{lin}^\beta C_i\} \rangle & \text{if } \beta \text{ is a successor ordinal} \,, \\ \bigcup_{\gamma < \beta} C_0{}^\gamma & \text{if } \beta \text{ is a limit ordinal} \,. \end{cases}$$

Suppose that for each $\beta < \alpha$ the sets $C_0{}^\beta$ and $\operatorname{lin}^\beta C_i$, $i \in I$, are linearly bounded and that $C_0{}^\beta$ is finite dimensional. Then C is α-subenveloped by F where

$$F(\beta) = \{C_0{}^\beta\} \cup \{u_i + \mathrm{lin}^\beta C_i \ : i \in I'\} \quad \text{for} \quad \beta < \alpha .$$

Furthermore

$$\mathrm{lin}^\alpha C = \langle C_0{}^\alpha \cup \bigcup \{u_i + \mathrm{lin}^\alpha C_i \ : i \in I'\}\rangle . \tag{3.9}$$

PROOF. It is readily checked that Lemma 3.1 is just the case $\alpha = 1$ of this lemma. It is also easily checked that for $\gamma < \beta < \alpha$, $F(\gamma) \subset\subset F(\beta)$. Assume that the lemma has been demonstrated for all ordinals $\delta < \alpha$ where α is an ordinal greater than 1. It therefore follows that for $\delta < \alpha$, C is δ-subenveloped by $F \mid \{\beta : \beta < \delta\}$ and that

$$\mathrm{lin}^\delta C = \langle C_0{}^\delta \cup \bigcup \{u_i + \mathrm{lin}^\delta C_i \ : i \in I'\}\rangle .$$

Moreover, since $\delta < \alpha$, the hypotheses assure us that $C_0{}^\delta$ is finite dimensional and linearly bounded and that each of the sets $\mathrm{lin}^\delta C_i$, $i \in I$, is linearly bounded. Reference to the definition of $C_0{}^\delta$ assures us that the hypotheses of Lemma 3.1 are satisfied with C_0 replaced by $C_0{}^\delta$ and C_i replaced by $\mathrm{lin}^\delta C_i$. Therefore $\mathrm{lin}^\delta C$ is enveloped by

$$\{C_0{}^\delta\} \cup \bigcup \{u_i + \mathrm{lin}^\delta C_i \ : i \in I'\} .$$

Therefore (see for example Remark 2.1.6) C is δ-enveloped by $F \mid \{\beta : \beta \leqq \delta\}$. This shows that C is α-subenveloped by F. In order to establish (3.9) first consider the case that α is a successor ordinal. Now

$$\mathrm{lin}^{\alpha-1} C = \langle C_0{}^{\alpha-1} \cup \bigcup \{u_i + \mathrm{lin}^{\alpha-1} C_i \ : i \in I'\}\rangle .$$

Since the hypotheses of Lemma 3.1 are satisfied with C_0 replaced by $C_0{}^{\alpha-1}$, C_i replaced by $\mathrm{lin}^{\alpha-1} C_i$ for $i \in I'$ and C replaced by $\mathrm{lin}^{\alpha-1} C$ we have by that lemma

$$\begin{aligned}\mathrm{lin}^\alpha C = \mathrm{lin}\, \mathrm{lin}^{\alpha-1} C &= \langle \mathrm{lin}\, C_0{}^{\alpha-1} \cup \bigcup \{u_i + \mathrm{lin}^\alpha C_i \ : i \in I'\}\rangle \\ &= \langle C_0{}^\alpha \cup \bigcup \{u_i + \mathrm{lin}^\alpha C_i \ : i \in I'\}\rangle ,\end{aligned}$$

where the last equality is obtained by reference to the definition of $C_0{}^\alpha$. Thus (3.9) holds for α a successor ordinal.

In the case that α is a limit ordinal,

$$\begin{aligned}\mathrm{lin}^\alpha C = \bigcup_{\beta<\alpha} \mathrm{lin}^\beta C &= \bigcup_{\beta<\alpha} \langle C_0{}^\beta \cup \bigcup \{u_i + \mathrm{lin}^\beta C_i \ : i \in I'\}\rangle \\ &= \langle \bigcup_{\beta<\alpha} C_0{}^\beta \cup \bigcup \{u_i + \mathrm{lin}^\alpha C_i \ : i \in I'\}\rangle . \\ &= \langle C_0{}^\alpha \cup \bigcup \{u_i + \mathrm{lin}^\alpha C_i \ : i \in I'\}\rangle .\end{aligned}$$

This establishes (3.9) when α is a limit ordinal and completes the proof of the lemma.

LEMMA 3.4. *Suppose that the hypotheses of Lemma 3.3 hold and that in addition for each $i \in I'$ there is a function H_i so that C_i is α-subenveloped (respectively, α-subdelineated) by H_i. Further suppose that there is a function H_0 on the set of ordinals less than α so that for each $\beta < \alpha$, $H_0(\beta)$ is a finite family of linearly bounded convex sets (respectively, segments) with $C_0{}^\beta = \langle \sigma H_0(\beta) \rangle$ and that for $\gamma < \beta < \alpha$, $H_0(\gamma) \subset\subset H_0(\beta)$. Then C is α-subenveloped (respectively, α-subdelineated) by H where*

$$H(\beta) = \{u_i + G \ : \ G \in H_i(\beta) \wedge i \in I\}.$$

PROOF. Conditions (1) and (3) of Definition 2.2 are clearly satisfied. Moreover for $\beta < \alpha$

$$\begin{aligned} \operatorname{lin}^\beta C &= \langle C_0{}^\beta \cup \bigcup \{u_i + \operatorname{lin}^\beta C_i \ : i \in I'\} \rangle \\ &= \langle u_0 + \langle \sigma H_0(\beta) \rangle \cup \bigcup \{u_i + \langle \sigma H_i(\beta) \rangle \ : i \in I'\} \rangle \\ &= \langle \bigcup \{u_i + \sigma H_i(\beta) \ : i \in I\} \rangle = \langle \sigma H(\beta) \rangle . \end{aligned}$$

Note that for each $\beta < \alpha$ the hypotheses of Lemma 3.2 are satisfied with C replaced by $\operatorname{lin}^\beta C$, C_0 replaced by $C_0{}^\beta$ and with C_i replaced by $\operatorname{lin}^\beta C_i$ for $i \in I'$, Γ_i replaced by $H_i(\beta)$ for $i \in I$ and with Γ replaced by $H(\beta)$. Applying Lemma 3.2 we find therefore that $\operatorname{lin}^\beta C$ is enveloped (respectively, delineated) by

$$\{u_i + G : \ G \in H_i(\beta), i \in I\}.$$

This completes the proof.

LEMMA 3.5. *If $X = \sum^d \{X_i : i \in I'\}$ and C_i is a convex subset of X_i for $i \in I'$ with $\theta \notin \operatorname{lin}^\Omega C_i$ and $\operatorname{lin}^\Omega C_i$ linearly bounded and $C = \langle \bigcup \{C_i : i \in I'\} \rangle$, then*

$$\operatorname{order} C = \sup \{\operatorname{order} C_i : i \in I'\}.$$

PROOF. Let $\beta = \operatorname{order} C$ and $\alpha = \sup\{\operatorname{order} C_i : i \in I'\}$. Apply Lemma 3.3 with $C_0 = \emptyset$ and $u_0 = u_1 = u_2 = \ldots = \theta$ to find for each ordinal $\gamma < \Omega$ that

$$\operatorname{lin}^\gamma C = \langle \bigcup \{\operatorname{lin}^\gamma C_i : i \in I'\} \rangle .$$

Thus by the definition of α,

$$\begin{aligned} \operatorname{lin}^{\alpha+1} C &= \langle \bigcup \{\operatorname{lin}^{\alpha+1} C_i : i \in I'\} \rangle \\ &= \langle \bigcup \{\operatorname{lin}^{\alpha} C_i : i \in I'\} \rangle = \operatorname{lin}^\alpha C . \end{aligned}$$

It therefore follows that $\beta \leqq \alpha$. It will suffice now to show that for $\gamma < \Omega$

$$(\operatorname{lin}^\gamma C) \cap X_n = \operatorname{lin}^\gamma C_n \qquad \text{for each } n \in I' .$$

Accordingly let $x \in (\operatorname{lin}^\gamma C) \cap X_n$. Then

$$x = \sum\{t_i x_i : i \in I'\},$$

where $x_i \in \mathrm{lin}^\gamma C_i \subset X_i$, $t \in K$ and domain $t = I'$. Since $\theta \notin \mathrm{lin}^\gamma C_i$, we can only have $x \in X_n$ if $t_i = 0$ for $i \neq n$, whence $t_n = 1$ and $x = x_n \in \mathrm{lin}^\gamma C_n$. Therefore for $n \in I'$ and $\gamma < \Omega$,

$$\mathrm{lin}^\gamma C_n \subset (\mathrm{lin}^\gamma C) \cap X_n \subset (\mathrm{lin}^\beta C) \cap X_n \subset \mathrm{lin}^\beta C_n$$

so that for each $n \in I'$, order $C_n \leqq \beta$, whence $\alpha \leqq \beta$. As we have already shown that $\beta \leqq \alpha$ the proof is complete.

THEOREM 3.1. *If* $\dim X = \aleph_0$, *then for each* $\alpha < \Omega$ *there is an* α*-delineated convex set* C *of order* α *lying in the natural cone of some basis* $\mathscr{B}$ *for* X. *Moreover in the case that* α *is a successor ordinal we may further require either that* C *satisfy*

(3.10) $$\theta \notin \mathrm{lin}^\alpha C$$

or

(3.11) $$\mathrm{lin}^\alpha C = \langle\{\theta\} \cup \mathrm{lin}^{\alpha-1} C\rangle .$$

PROOF (by transfinite induction). For $\alpha = 1$ let $u \in X$ with $u \neq \theta$. Then $(1,2]u$ and $(0,1]u$ are 1-delineated sets of order 1 satisfying respectively (3.10) and (3.11). Furthermore if u is chosen as a basis element, these sets are in the natural cone regardless of how the other basis elements are chosen. Suppose next that the theorem has been proved for all ordinals $< \alpha$ where $\alpha > 1$. Let $X = \sum^d\{X_i : i \in I\}$ where $X_0 = Ru$, $u \neq \theta$ and for $i \in I'$, $\dim X_i = \aleph_0$. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a sequence of successor ordinals satisfying:

$\alpha_i = \alpha - 1$ if α and $\alpha - 1$ are successor ordinals;

otherwise

$\alpha_1, \alpha_2, \alpha_3, \ldots$ is a strictly increasing sequence with limit α or $\alpha - 1$ whichever is a limit ordinal.

For $i \in I'$ let C_i be an α_i-delineated convex subset of X_i which lies in the natural cone with respect to some basis $\mathscr{B}_i$ of X_i and such that order $C_i = \alpha_i$. Let $\mathscr{B}$ be the basis for X defined by $\mathscr{B} = \{u\} \cup \bigcup\{\mathscr{B}_i : i \in I'\}$. We now divide the remainder of the proof into two cases.

CASE I. *α is a successor ordinal.* Require that

$$\mathrm{lin}^{\alpha_i} C_i = \langle\{\theta\} \cup \mathrm{lin}^{\alpha_i - 1} C_i\rangle .$$

Let $a_1, a_2, a_3, \ldots$ be a strictly decreasing sequence of positive numbers and let $a_0 = \lim a_n$. Let

$$C = \langle \bigcup\{a_i u + C_i : i \in I'\}\rangle .$$

It is clear that C lies in the natural cone of $\mathscr{B}$. We note that by Remark 2.1.2 each of the C_i, $i \in I'$, is $(\alpha+1)$-delineated. Check that the hypotheses of Lemma 3.4 are satisfied with α replaced by $\alpha+1$, with u_i replaced by $a_i u$, and with

$$C_0{}^\beta = \begin{cases} \langle\{a_i u : \alpha_i \leqq \beta\}\rangle & \text{for } \beta < \alpha ; \\ [a_0, a_1]u & \text{for } \beta = \alpha, \alpha+1 . \end{cases} \tag{3.12}$$

It follows from Lemma 3.4 that C is $\alpha+1$-subdelineated and by Remark 2.1.4 that C is α-delineated. The hypotheses of Lemma 3.3 are satisfied with u_i replaced by $a_i u$ and with $C_0{}^\beta$ defined for $\beta \leqq \alpha$ by (3.12) so that the conclusion (3.9) of that lemma yields

$$\begin{aligned} \operatorname{lin}^\alpha C &= \langle C_0{}^\alpha \cup \bigcup \{a_i u + \operatorname{lin}^\alpha C_i \ : i \in I'\}\rangle \\ &= \langle [a_0, a_1]u \cup \bigcup \{a_i u + \operatorname{lin}^{\alpha_i} C_i \ : i \in I'\}\rangle . \end{aligned}$$

The hypotheses of Lemma 3.1 are seen to hold with C_0 replaced by $[a_0, a_1]u$ with u_i replaced by $a_i u$ and with C_i replaced by $\operatorname{lin}^{\alpha_i} C_i$, $i \in I'$. Therefore

$$\begin{aligned} \operatorname{lin}^{\alpha+1} C = \operatorname{lin} \operatorname{lin}^\alpha C &= \langle \operatorname{lin}([a_0, a_1]u) \cup \bigcup \{a_i u + \operatorname{lin} \operatorname{lin}^{\alpha_i} C_i \ : i \in I'\}\rangle \\ &= \langle [a_0, a_1]u \cup \bigcup \{a_i u + \operatorname{lin}^{\alpha_i} C_i \ : i \in I'\}\rangle = \operatorname{lin}^\alpha C . \end{aligned}$$

whence $\operatorname{order} C \leqq \alpha$. Application of Lemma 3.3 with α replaced by β where $\beta < \alpha$ yields

$$\operatorname{lin}^\beta C = \langle C_0{}^\beta \cup \bigcup \{a_i u + \operatorname{lin}^\beta C_i\}\rangle \qquad \text{for } \beta < \alpha .$$

It is now easily verified that $a_0 u \notin \operatorname{lin}^\beta C$ for $\beta < \alpha$ and since $a_0 u \in \operatorname{lin}^\alpha C$ we find that $\operatorname{order} C \geqq \alpha$. Thus $\operatorname{order} C = \alpha$. The condition (3.10) or (3.11) is satisfied according to whether $a_0 > 0$ or $a_0 = 0$. This establishes the result in the case that α is a successor ordinal.

CASE II. *α is a limit ordinal.* Require that $\theta \notin \operatorname{lin}^{\alpha_i} C_i$ for $i \in I'$. Let

$$C = \langle \bigcup \{C_i : i \in I'\}\rangle .$$

Lemma 3.5 assures us that $\operatorname{order} C = \alpha$ and it is obvious that C is contained in the natural cone of $\mathscr{B}$.

Since each of the sets C_i is an α_i-delineated set of order α_i, it follows from Remark 2.1.2 that for each $i \in I'$, C_i is $(\alpha+1)$-delineated. Applying Lemma 3.4 we find that C is $\alpha+1$-subdelineated and hence by Remark 2.1.4, C is α-delineated. This completes the proof.

It might shed some light on the procedure employed in this paper to observe that in the proof of case I of the above theorem we used the fact that

$$\mathrm{lin}^{\beta} C = \langle C_0^{\beta} \cup \bigcup \{a_i u + \mathrm{lin}^{\beta} C_i \ : i \in I'\}\rangle$$

whence it is easily checked that

$$\mathrm{lin}^{\beta} C = \langle \bigcup \{a_i u + \mathrm{lin}^{\beta} C_i \ : i \in I'\}\rangle .$$

It might be thought therefore that the sets C_0^{β} could be dispensed with; this is not the case however since $\mathrm{lin}^{\beta} C$ is *not* for all $\beta \leqq \alpha$ enveloped by the family $\{a_i u + \mathrm{lin}^{\beta} C_i \ : i \in I'\}$ as required for the induction. A similar remark applies to Theorem 3.2.

THEOREM 3.2. *If* $\dim X = \aleph_0$, *then for each ordinal* α *with* $\alpha < \Omega$ *there is a convex* C' *of order* α *such that*

$$\mathrm{lin}^{\alpha} C' = X .$$

PROOF. CASE I. *α is a successor ordinal.* Let

$$X = X_0 \oplus \textstyle\sum^d \{X_n{}^i : \ i = 1,2;\ n \in I'\} .$$

For each $i = 1,2$ and each $n \in I'$, let $C_n{}^i$ be a linearly bounded convex set of order α with

(3.13) $$C_n{}^i \subset X_n{}^i \quad \text{and} \quad \mathrm{lin}^{\alpha} C_n{}^i = \langle \mathrm{lin}^{\alpha-1} C_n{}^i \cup \{\theta\}\rangle .$$

Let $e_1, e_2, e_3, \ldots$ be a basis for X with

$$e_n \in X_0 \cup \bigcup \{X_k{}^i : \ i = 1,2 \wedge k = 1,2,\ldots,n-1\} .$$

Let

$$C = \langle \bigcup \{(-1)^i e_n + C_n{}^i \ : \ i = 1,2 \wedge n \in I'\}\rangle .$$

By Lemma 3.3, for $\beta < \alpha$ we have

$$\mathrm{lin}^{\beta} C = \langle \bigcup \{(-1)^i e_n + \mathrm{lin}^{\beta} C_n{}^i \ : \ i = 1,2 \wedge n \in I'\}\rangle$$

and by the same lemma

(3.14) $$\mathrm{lin}^{\alpha} C = \langle \bigcup \{(-1)^i e_n \ : \ i = 1,2 \wedge n \in I'\} \cup \bigcup \{(-1)^i e_n + \mathrm{lin}^{\alpha} C_n{}^i \ : \ i = 1,2 \wedge n \in I'\}\rangle .$$

Thus $\mathrm{lin}^{\alpha} C$ contains each basis element e_n and its negative, hence also θ. It may be seen however that $\theta \notin \mathrm{lin}^{\beta} C$ for $\beta < \alpha$ as follows. Suppose on the contrary that for some $\beta < \alpha$ we have $\theta \in \mathrm{lin}^{\beta} C$. Then θ can be expressed in the form

$$\theta = \sum \{t_{ni} x_n{}^i : \ i = 1,2 \wedge n \in I'\} ,$$

where $x_n{}^i \in \left((-1)^i e_n + C_n{}^i\right)$ and $t \in K$ with domain $t = I' \times \{1,2\}$. Let m

be the largest integer n for which $t_{n1}+t_{n2}>0$ and let j be the largest integer i for which $t_{mi}>0$ and let $j'=3-j$. Set

$$x_m^j = (-1)^j e_m + y_m^j \quad \text{with} \quad y_m^j \in \operatorname{lin}^\beta C_m^j \subset X_m^j$$

and note that y_m^j cannot be θ by (3.13). Now

$$\begin{aligned} t_{mj} y_m^j &= -t_{mj}(-1)^j e_m - t_{mj'} x_m^{j'} - \sum \{t_{ni} x_n^i :\ i=1,2 \wedge n=1,2,\ldots,m-1\} \\ &\in X_m^{j'} \oplus X_0 \oplus \textstyle\sum^d \{X_n^i :\ i=1,2 \wedge n=1,2,\ldots,m-1\} \end{aligned}$$

which is impossible since $t_{mj} \neq 0$ and $\theta \neq y_m^j \in C_m^j \subset X_m^j$.

Since $\theta \notin \operatorname{lin}^\beta C$ for $\beta<\alpha$ we see by Lemma 2.4 that

$$\theta \notin P \operatorname{lin}^\beta C = \operatorname{lin}^\beta (PC) \quad \text{for} \quad \beta<\alpha .$$

However, applying (3.14) and Lemma 2.3 we find

$$X = \langle \bigcup \{Re_n : n \in I'\} \rangle \subset P \operatorname{lin}^\alpha C \subset \operatorname{lin}^\alpha (PC) \subset X .$$

Therefore PC has order α and $\operatorname{lin}^\alpha(PC)=X$. This completes the proof in the case that α is a successor ordinal.

Case II. *α is a limit ordinal.* Again let

$$X = X_0 \oplus \textstyle\sum^d \{X_n^i :\ i=1,2 \wedge n \in I'\}$$

where for $n \in I'$ and $i=1,2$, $\dim X_n^i=\aleph_0$. Let $\alpha_1,\alpha_2,\alpha_3,\ldots$ be a strictly increasing sequence of successor ordinals greater than zero with $\lim \alpha_n=\alpha$. For each $i=1,2$ and each $n \in I'$ let C_n^i be an α_n-delineated set of order α_n contained in the natural cone of a basis $\mathscr{B}_n^i$ for X_n^i with

$$\operatorname{lin}^{\alpha_n} C_n^i = \langle \{\theta\} \cup \operatorname{lin}^{\alpha_n - 1} C_n^i \rangle .$$

Let $\mathscr{B}_0$ be a basis for X_0 and let $e_1,e_2,e_3,\ldots$ be a basis for X with

$$e_n \in \mathscr{B}_0 \cup \bigcup \{\mathscr{B}_k^i :\ i=1,2 \wedge k=1,2,\ldots,n-1\} .$$

Let

$$C = \langle \bigcup \{(-1)^i e_n + C_n^i \ :\ i=1,2 \wedge n \in I'\} \rangle .$$

The hypotheses of Lemma 3.3 are satisfied where

$$C_0^\beta = \langle \bigcup \{[-1,1] e_n \ : \alpha_n \leqq \beta\} \rangle .$$

It is clear that for $\beta<\alpha$, C_0^β is finite dimensional and linearly bounded. Letting G_0 be the function on the ordinals $<\alpha$ defined by

$$G_0(\beta) = \{[-1,1] e_n : \alpha_n \leqq \beta\}$$

we see that for $\beta<\alpha$, $G_0(\beta)$ is a finite family of segments, that $C_0^\beta=\langle \sigma G_0(\beta) \rangle$ and that for $\gamma<\beta<\alpha$, $G_0(\gamma) \subset\subset G_0(\beta)$. Since, furthermore, each C_n^i is α-delineated, Lemma 3.4 assures us that C is α-subdelineated.

Now Lemma 3.3 yields for $\beta<\alpha$

$$(3.15)\qquad \mathrm{lin}^{\beta}C = \langle\langle\bigcup\{[-1,1]e_n : \alpha_n \leqq \beta\}\rangle \cup \cup \bigcup\{(-1)^i e_n + \mathrm{lin}^{\beta} C_n^{\,i} : i=1,2 \wedge n\in I'\}\rangle .$$

Furthermore

$$\mathrm{lin}^{\alpha}C = \bigcup_{\beta<\alpha}\mathrm{lin}^{\beta}C \supset \langle\bigcup\{[-1,1]e_n : n\in I'\}\rangle .$$

Now by Lemma 2.3

$$\mathrm{lin}^{\alpha}(PC) \supset P\,\mathrm{lin}^{\alpha}C \supset \langle\bigcup\{Re_n : n\in I'\}\rangle = X .$$

It now only remains to be shown that

$$\mathrm{lin}^{\beta}(PC) \neq X \quad \text{for} \quad \beta<\alpha .$$

To this end we will first show that for $r>0$, $-re_{n+1} \notin \mathrm{lin}^{\alpha_n}C$. Suppose on the contrary that for some $r>0$, $-re_{n+1}\in \mathrm{lin}^{\alpha_n}C$. Then by (3.15) we find

$$(3.16)\qquad -re_{n+1} = t_0 y_0 + \sum\{t_{hi}((-1)^i e_h + y_h^{\,i}) : i=1,2 \wedge h\in I'\}$$

where $y_0\in\langle\bigcup\{[-1,1]e_h : h=1,2,\ldots,n\}\rangle$ and $y_h^{\,i}\in \mathrm{lin}^{\alpha_n}C_h^{\,i}\subset X_h^{\,i}$ and $t\in K$ with domain $t=\{0\}\cup(I'\times\{1,2\})$.

Recalling that $C_h^{\,i}$ (and hence by Lemma 2.7 also $\mathrm{lin}^{\alpha_n}C_h^{\,i}$) is contained in the natural cone of $\mathscr{B}$, we see that when the $y_h^{\,i}$ are expressed as linear combinations of the basis elements e_m, the coefficients are therefore $\geqq 0$. Consequently (3.16) can hold only if $t_{n+1,1}>0$. Thus, letting k be the largest of the integers h for which $t_{h1}+t_{h2}>0$ we see that $k>n$. And since $\theta\notin \mathrm{lin}^{\alpha_n}C_k^{\,i}$ we have $y_k^{\,i}\neq\theta$. But $y_k^{\,i}$ when expressed in terms of the basis elements e_m involves only e_m for which $m>k\geqq n+1$. (This is because $y_k^{\,i}\in X_k^{\,i}$ and

$$e_m \in X_0 \oplus \textstyle\sum^d \{X_h^{\,i} : i=1,2 \wedge h=1,2,\ldots,m-1\}.)$$

These terms cannot drop out of the sum on the right hand side of (3.16) since all the terms with negative coefficients involve only e_h with $h\leqq k$. This contradiction shows that

$$-re_{n+1} \notin \mathrm{lin}^{\alpha_n}C .$$

Using the fact that C is β-delineated for each $\beta<\alpha$ we have by Lemma 2.6 that for $\beta<\alpha$

$$\mathrm{lin}^{\beta}(PC) = P\,\mathrm{lin}^{\beta}C .$$

Thus $-Pe_{n+1}\cap \mathrm{lin}^{\alpha_n}C=\emptyset$ for $n\in I'$. As we have already seen that

$$\mathrm{lin}^{\alpha}(PC) = X$$

we see that PC is a convex set of order α for which $\mathrm{lin}^{\alpha}(PC)=X$. This completes the proof.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.
2. V. L. Klee, Jr., *Decomposition of an infinite-dimensional linear system into ubiquitous convex sets*, Amer. Math. Monthly 57 (1950), 540–541.
3. V. L. Klee, Jr., *Convex sets in linear spaces*, Duke Math. J. 18 (1951), 443–466.
4. V. L. Klee, Jr., *Iterations of the "lin" operation for convex sets*, Math. Scand. 4 (1956), 231–238.
5. O. M. Nikodym, *Sur la clôture faible des ensembles convexes dans l'espace réel linéaire ou aucune topologie n'est admise*, C. R. Acad. Sci. Paris 234 (1952), 1727–1728.
6. O. M. Nikodym, *Sur les clôtures faible et forte des ensembles convexes dans les espaces linéaires réels abstraits*, C. R. Acad. Sci. Paris 234 (1952), 1831–1833.
7. O. M. Nikodym, *On transfinite iterations of the weak linear closure of convex sets in linear spaces*. Part A. *Two notions of closure*, Rend. Circ. Mat. Palermo (2) 2 (1953), 85–105.
8. O. M. Nikodym, *On transfinite iterations of the weak linear closure of convex sets in linear spaces*. Part B. *An existence theorem in weak linear closure*, Rend. Circ. Mat. Palermo (2) 3 (1954), 5–75.
9. W. Stenberg, *On the separation of convex sets*, Thesis, University of California at Berkeley, 1955.

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