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SOME OPEN MAPPING THEOREMS IN LF-SPACES AND THEIR APPLICATION TO EXISTENCE THEOREMS FOR CONVOLUTION EQUATIONS

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It is a well known fact that the operation of differentiation can be made continuous if we are prepared to accept the price: the resulting topology has some decidedly unpleasant properties. If E is an inductive limit of a sequence E_n, then some but unfortunately not all topological properties are inherited by E if they are satisfied on each E_n; thus, e.g., a linear mapping of E into a convex space is continuous if (and only if) its restriction to each E_n is continuous or, a hyperplane $H \subset E$ is closed in E if (and only if) its intersection with each E_n is closed in E_n. However if we replace hyperplanes by arbitrary subspaces, the last equivalence is no longer true. Similar complications occur in the study of openness of mappings.

It is the purpose of the present paper to present a few remarks concerning the openness of mappings of one LF-space into another. It turns out that a continuous linear mapping T of an LF-space E into an LF-space F need not be open even if it behaves as one in each F_j. Some additional assumptions are necessary to make T open and the study of this problem is the object of this communication. The main idea of the present remark consists in using quotient mappings (see e.g. proposition (4.5)). Although the results are far from satisfactory we are able to improve an earlier result of B. Malgrange [3] and strenghten slightly a theorem on the range of convolution transforms in distribution spaces obtained by L. Hörmander [2]. Hörmander's work is concerned with existence theorems for equations $Tx=y$ where T is a convolution transform and $x \in \mathscr{D}'(\Omega_2)$, $y \in \mathscr{D}'(\Omega_1)$. Earlier results in this direction, especially those of L. Ehrenpreis, are mentioned in the bibliography of [2].

1. Notation and auxiliary results.

The term "convex space" is used for "locally convex Hausdorff topological linear space". Similarly a convex topology is the topology of a

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convex space. If E is a convex space we shall denote by $U(E)$ the system of all closed absolutely convex neighbourhoods of zero in E. If E and F are two linear spaces, a linear mapping T from E into F has a domain of definition $D(T)$ which is a linear subspace of E; its range will be denoted by $R(T)$. We shall use the preposition "of" to distinguish mappings which are everywhere defined. Thus a mapping T of E into F necessarily has $D(T)=E$ while, for a mapping T from E into F, $D(T)$ may be different from E.

Let E and F be two convex spaces and A a linear mapping from E into F defined on a dense subspace $D(A)$ of E. We define $G(A)$, the graph of A, as the set of all $[x,Ax] \in E \times F$ for $x \in D(A)$. We shall denote by $D(A')$ the set of those $y' \in F'$ for which the scalar product $\langle Ax,y'\rangle$ is continuous on $D(A)$. If $y' \in D(A')$ there exists an $x' \in E'$ such that $\langle Ax,y'\rangle = \langle x,x'\rangle$ for all $x \in D(A)$. Since $D(A)$ is dense in E, there is exactly one such x'; we may thus write $x' = A'y'$. The mapping A', the adjoint of A, is clearly linear. If E' and F' are equipped with the topologies $\sigma(E',E)$ and $\sigma(F',F)$, then $D(A')$ will be dense in F' iff A is closable, i.e. iff the closure of $G(A)$ in $E \times F$ is again the graph of a mapping B from E into F. In this case B is an extension of A and clearly $B' = A'$, hence $D(B') = D(A')$.

The rest of this section is devoted to some simple results concerning weakly continuous and weakly open mappings which will be needed later.

Definition (1,1). *A linear mapping T from a convex space E into a convex space F is said to be weakly continuous if it is continuous as a mapping of $\big(D(T), \sigma(D(T),E')\big)$ into $\big(F, \sigma(F,F')\big)$.*

Proposition (1,2). *Let A be a linear mapping from E into F with $D(A)=E$. Then the following conditions are equivalent:*

1° *if Q is a closed subspace of F, then $A^{-1}(Q)$ is closed in E;*

2° *if Q is a closed hyperplane in F, then $A^{-1}(Q)$ is closed in E;*

3° $D(A') = F'$;

4° *A is continuous as a mapping of $\big(E, \sigma(E,E')\big)$ into $\big(F, \sigma(F,F')\big)$.*

Proof. The implication 1° $\Rightarrow$ 2° being obvious, assume 2° and take an element $y' \in F'$. The linear form $f(x) = \langle Ax,y'\rangle$ defined on E will be continuous iff $f^{-1}(0)$ is closed in E. Since clearly $f^{-1}(0) = A^{-1}(Q)$ when Q is the zero hyperplane of y', $f^{-1}(0)$ will be closed by 2° whence $y' \in D(A')$. It follows that $D(A') = F'$. Suppose now that 3° is satisfied and take $y_1', \ldots, y_n' \in F'$. Since $D(A') = F'$, we may form $x_i' = A'y_i'$. If $|\langle x,x_i'\rangle| \leqq 1$ for $1 \leqq i \leqq n$, we have $|\langle Ax,y_i'\rangle| \leqq 1$ for $1 \leqq i \leqq n$ which proves 4°. The implication 4° $\Rightarrow$ 1° is obvious.

DEFINITION (1,3). *A linear mapping T from a convex space E into a convex space F is said to be weakly open if it is open as a mapping of $(D(T), \sigma(D(T), E'))$ into $(F, \sigma(F, F'))$.*

If T is closed we have the following equivalence.

PROPOSITION (1,4). *Let T be a linear mapping from E into F with $D(T) = E$ the graph of which is closed in $E \times F$. Then the following conditions are equivalent:*

1° $R(T')$ *is closed in* E';

2° $R(T') = T^{-1}(0)^0$;

3° $T^{-1}(0)^0 \subset R(T')$;

4° *if P is a closed subspace of E with $P \supset T^{-1}(0)$ then TP is closed in TE;*

5° *if P is a closed hyperplane in E with $P \supset T^{-1}(0)$ then TP is closed in TE;*

6° *T is open as a mapping of $(E, \sigma(E, E'))$ into $(F, \sigma(F, F'))$.*

PROOF. The inclusion $R(T') \subset T^{-1}(0)^0$ is obvious. Further, if $x \in R(T')^0$, we have $\langle Tx, D(T')\rangle = 0$ whence $Tx = 0$ the subspace $D(T')$ being dense in F'. Hence $R(T)^0 \subset T^{-1}(0)$. It follows that for any mapping T we have

$$R(T') \subset T^{-1}(0)^0 \subset R(T')^{00}.$$

The implication 1° $\Rightarrow$ 2° follows immediately from these inclusions. The implication 2° $\Rightarrow$ 3° is immediate. Now assume 3° and let P be a closed subspace of E containing $T^{-1}(0)$. Let $y \in TE$, $y \operatorname{non} \in TP$. It follows that $y = Tx$ for some x for which $x \operatorname{non} \in P$. Since P is closed there exists an $x' \in E'$ such that $x' \in P^0$ and $\langle x, x'\rangle \neq 0$. Since $x' \in P^0 \subset T^{-1}(0)^0$ we have $x' = T'y'$ for some y', whence

$$\langle TP, y'\rangle = \langle P, x'\rangle = 0 \quad \text{and} \quad \langle y, y'\rangle = \langle Tx, y'\rangle = \langle x, x'\rangle \neq 0\,.$$

The implication 4° $\Rightarrow$ 5° is immediate.

Let us show now that 5° implies 1°. Since $R(T') \subset T^{-1}(0)^0$, it suffices to prove the inclusion $T^{-1}(0)^0 \subset R(T')$ only. To do this, take an $x' \in T^{-1}(0)^0$; if $x' = 0$, we have $x' = T'y'$ for $y' = 0$. We may thus assume $x' \neq 0$ so that there exists an $x_0 \in E$ which $\langle x_0, x'\rangle = 1$. Denote by P the set of all $x \in E$ for which $\langle x, x'\rangle = 0$; we have $T^{-1}(0) \subset P$ since $x' \in T^{-1}(0)^0$ and it follows from 5° that TP is closed in TE. We show next that Tx_0 non $\in TP$. Indeed, if $Tx_0 = Tp$ for some $p \in P$, we have $x_0 - p \in T^{-1}(0)$ whence $\langle x_0 - p, x'\rangle = 0$ so that $1 = \langle x_0, x'\rangle = \langle p, x'\rangle = 0$, a contradiction. Since TP is closed in TE and Tx_0 is outside TP, there exists a $y' \in F$ with $\langle TP, y'\rangle = 0$ and $\langle Tx_0, y'\rangle = 1$. Let us show that $x' = T'y'$. If $x \in E$, we have $x = \langle x, x'\rangle x_0 + p$ with $p \in P$ whence

$$\langle Tx, y'\rangle = \langle\langle x, x'\rangle Tx_0 + Tp, y'\rangle$$
$$= \langle x, x'\rangle\langle Tx_0, y'\rangle + \langle Tp, y'\rangle = \langle x, x'\rangle$$

and the proof is complete.

The equivalence of the first five conditions is thus established. To complete the proof, we intend to prove $1° \Rightarrow 6° \Rightarrow 3°$.

To prove 6°, take $x_1', \ldots, x_n' \in E'$ and denote by U the set of all $x \in E$ for which $|\langle x, x_i\rangle| \leqq 1$, $1 \leqq i \leqq n$. We intend to prove by induction that TU is a neighbourhood of zero in $(TE, \sigma(F, F'))$. If $n=1$, we shall distinguish two cases.

I. $x' \in R(T')$. We have $x' = T'y'$ for some $y' \in D(T')$. Suppose that $y \in TE$ and $|\langle y, y'\rangle| \leqq 1$; $y = Tx$ for some x whence

$$|\langle x, x'\rangle| = |\langle x, T'y'\rangle = |\langle Tx, y'\rangle| \leqq 1$$

so that $x \in U$ and $y \in TU$.

II. x' non $\in R(T')$. The last set being closed in E', there exists an $x_0 \in E$ with $\langle x_0, R(T')\rangle = 0$ and $\langle x_0, x'\rangle = 1$. Since $D(T')$ is dense in E', $\langle x_0, R(T')\rangle = 0$ implies $Tx_0 = 0$. Let us show now that $TU = TE$. If $x \in E$ take $z = x - \langle x, x'\rangle x_0$. We have $Tz = Tx$ and $z \in U$.

Now let $n > 1$ and suppose the statement proved for $n-1$ functionals. Let us denote by M the smallest linear subspace of E' containing $R(T')$ and $x_1' \ldots, x_{n-1}'$. Since $R(T')$ is closed, M is closed as well. We shall distinguish two cases:

I. $x_n' \in M$. We have $x_n' = \beta_1 x_1' + \ldots + \beta_{n-1} x_{n-1}' + T'y'$ for some β_i and a suitable $y' \in D(T')$. Let V be the set of those $x \in E$ for which $|\langle x, x_i'\rangle| \leqq 1$, $1 \leqq i \leqq n-1$. By induction hypothesis TV is a weak neighbourhood of zero in TE.

Take a number $\beta \geqq 1$, $\beta \geqq |\beta_i|$, $1 \leqq i \leqq n-1$. We intend to show that $y \in (n\beta)^{-1}TV$ and $|\langle y, y'\rangle| \leqq (n\beta)^{-1}$ imply $y \in TU$. If $y \in (n\beta)^{-1}TV$ we have, for a suitable x, $y = (n\beta)^{-1}Tx$ and $|\langle x, x_i'\rangle| \leqq 1$, $1 \leqq i \leqq n-1$. At the same time

$$(n\beta)^{-1}|\langle x, T'y'\rangle| = |\langle y, y'\rangle| \leqq (n\beta)^{-1}.$$

We have

$$\langle (n\beta)^{-1}x, x_n'\rangle = (n\beta)^{-1}\sum_1^{n-1} \beta_i\langle x, x_i'\rangle + (n\beta)^{-1}\langle x, T'y'\rangle,$$

whence

$$|\langle (n\beta)^{-1}x, x_n'\rangle| \leqq (n\beta)^{-1}\sum_1^{n-1} |\beta_i| + (n\beta)^{-1} \leqq 1$$

so that $(n\beta)^{-1}x \in U$.

II. x_n' non $\in M$. Since M is closed, there exists an $x_0 \in E$ such that $\langle x_0, M\rangle = 0$ and $\langle x_0, x_n'\rangle = 1$. Again we have $Tx_0 = 0$. We intend to show that $TU=TV$. Indeed if $x \in V$, take $z=x-\langle x,x_n'\rangle x_0$. We have $Tx=Tz$ and $\langle z,x_n'\rangle=0$. If $i<n$, $\langle x_0,x_i'\rangle=0$ so that $\langle z,x_i'\rangle=\langle x,x_i'\rangle$. It follows that $z \in U$ and the proof is complete.

The cycle of implications will be complete if we prove the following proposition.

PROPOSITION (1,5). *Let T be a linear mapping from E into F which is open as a mapping of $(D(T),\sigma(D(T),E'))$ into $(F,\sigma(F,F'))$. If $x' \in T^{-1}(0)^0$ then there exists a $y' \in F'$ so that $\langle x,x'\rangle=\langle Tx,y'\rangle$ for $x \in D(T)$.*

PROOF. Let U be the set of those x for which $|\langle x,x'\rangle| \leqq 1$. Since $T(U \cap D(T))$ is a neighbourhood of zero in $R(T)$, there exist $y_1',\ldots,y_n' \in F'$ so that $x \in D(T)$ and $|\langle Tx,y_i'\rangle| \leqq 1$ imply $Tx \in TU$. Since $x' \in T^{-1}(0)^0$ it is easy to see that $Tx \in TU$ implies $x \in U$. Especially, $x \in D(T)$ and $\langle Tx,y_i'\rangle=0$ imply $\langle x,x'\rangle=0$. It follows that there exist scalars $\lambda_1,\ldots,\lambda_n$ such that

$$\langle x,x'\rangle = \sum \lambda_i \langle Tx,y'_i\rangle = \langle Tx, \sum \lambda_i y'_i\rangle$$

for $x \in D(T)$ which proves the theorem.

PROPOSITION (1,6). *Let E and F be two topological spaces, φ a mapping of E into F. Then the following conditions are equivalent:*

1° *the mapping φ is continuous and open;*

2° *the mapping φ satisfies the following two conditions:*

2.1° *if G is open in E then $\varphi^{-1}\varphi G$ is open in E;*

2.2° *a set $F \subset \varphi E$ is closed in φE iff $\varphi^{-1}F$ is closed in E.*

PROOF. 1° ⇒ 2.1°. Let G be open in E. Since φ is open the set φG is open in φE whence $\varphi G=H \cap \varphi E$ for a suitable H open in F. Since φ is continuous $\varphi^{-1}H$ is open in E. Clearly $\varphi^{-1}H=\varphi^{-1}\varphi G$.

1° ⇒ 2.2°. Let B be closed in φE so that $B=H \cap \varphi E$ for a suitable H closed in F. Since φ is continuous, $\varphi^{-1}H$ is closed in E. Clearly $\varphi^{-1}H=\varphi^{-1}B$. If B is a subset of φE such that $\varphi^{-1}B$ is closed in E, the set $\varphi(E-\varphi^{-1}B)$ will be open in φE since φ is open. Clearly $\varphi E-B=\varphi(E-\varphi^{-1}B)$.

Now assume that 2.1° and 2.2° are satisfied. Let $B \subset F$ be closed in F. Since $\varphi^{-1}B=\varphi^{-1}(B \cap \varphi E)$ and $B \cap \varphi E$ is closed in φE, the set $\varphi^{-1}B$ will be closed in E by 2.2°. It follows that φ is continuous. If G is open in E, we have $\varphi^{-1}\varphi G$ open in E by 2.1° so that $\varphi^{-1}(\varphi E-\varphi G)=E-\varphi^{-1}\varphi G$ is closed in E. It follows from 2.2° that $\varphi E-\varphi G$ is closed in φE hence φG is open in φE. The mapping φ is thus seen to be open.

PROPOSITION (1,7). *Let T be a linear mapping from E into F with $D(T)=E$. Then the following conditions are equivalent:*

1° *if Q is a subspace of TE, then Q is closed in TE iff $T^{-1}(Q)$ is closed in E;*

2° *if Q is a hyperplane in TE, then Q is closed in TE iff $T^{-1}(Q)$ is closed in E;*

3° *T is a homomorphism of $(E,\sigma(E,E'))$ into $(F,\sigma(F,F'))$.*

PROOF. The implication 1° $\Rightarrow$ 2° being immediate, assume 2° and let us prove 3°. If P is a closed hyperplane in F then $T^{-1}(P)=T^{-1}(P\cap TE)$ and the last set is closed by 2° since $P\cap TE$ is closed in TE. It follows that T is weakly continuous. If $T^{-1}(0)\subset H\subset E$ and H is a closed hyperplane in E then by 2°, TH will be closed in TE iff $T^{-1}TH$ is closed in E. We have, however, $T^{-1}TH=H$ since $T^{-1}(0)\subset H$. It follows that T is weakly open.

Now assume 3° and let us prove 1°. If $Q\subset TE$ is closed in TE, the space $T^{-1}(Q)$ will be closed in E since T is weakly continuous. Suppose that $Q\subset TE$ and $T^{-1}(Q)$ is closed in E. Since $T^{-1}(0)\subset T^{-1}(Q)$ and T is weakly open, $Q=TT^{-1}(Q)$ will be closed in TE. The proof is complete.

2. Closed mappings.

This section describes some methods of generating closed mappings. The results, although of an auxiliary character, seem to be interesting enough on their own to be stated separately.

PROPOSITION (2,1). *Let E, E_1, E_2, be convex spaces, h_1 a linear mapping from E into E_1, h_2 a linear mapping from E into E_2. Suppose that*

1° *h_1 is weakly open and $R(h_1)=E_1$;*

2° *h_2 is closed in $E\times E_2$;*

3° *$h_1^{-1}(0)\subset h_2^{-1}(0)$.*

Let H be the subset of $E_1\times E_2$ consisting of $[x_1,x_2]$ such that $x_1=h_1x$ and $x_2=h_2x$ for some $x\in D(h_1)\cap D(h_2)$.

Then H is closed in $E_1\times E_2$.

PROOF. For the proof we shall adopt the following convention: if T is a linear mapping from E into F, not necessarily densely defined, T' is not uniquely defined. We shall nevertheless use the symbol $D(T')$ for the set of those $y'\in F'$ for which $\langle Tx,y'\rangle$ is continuous on $D(T)$.

First we shall prove the following statement: if $x_2'\in D(h_2')$ and $z'\in E'$ is such that $\langle h_2x,x_2'\rangle=\langle x,z'\rangle$ for $x\in D(h_2)$ then there exists an $x_1'\in D(h_1')$ with $\langle h_1x,x_1'\rangle=\langle x,z'\rangle$ for $x\in D(h_1)$; the functional $[-x_1',x_2']$ annihilates H.

Indeed, since $h_1^{-1}(0)\subset h_2^{-1}(0)$, we have $\langle h_1^{-1}(0),z'\rangle=0$ so that, h_1 being weakly open, there exists by Theorem (1,5) an $x_1' \in D(h_1')$ with $\langle h_1 x, x_1' \rangle = \langle x, z' \rangle$ for $x \in D(h_1)$; if $x \in D(h_1) \cap D(h_2)$, we have

$$\langle [h_1 x, h_2 x][-x_1', x_2'] \rangle = -\langle h_1 x, x_1' \rangle + \langle h_2 x, x_2' \rangle = 0 .$$

Suppose now that $[x_1, x_2]$ non $\in H$. Since $R(h_1) = E_1$ we have $x_1 = h_1 x_0$ for some $x_0 \in D(h_1)$. Then $[x_0, x_2]$ non $\in G(h_2)$ hence there exists $[x', x_2']$ such that $\langle [x, h_2 x], [x', x_2'] \rangle = 0$ for $x \in D(h_2)$ and $\langle [x_0, x_2], [x', x_2'] \rangle \neq 0$. It follows that $\langle x, x' \rangle + \langle h_2 x, x_2' \rangle = 0$ for $x \in D(h_2)$ so that $x_2' \in D(h_2')$. According to what has been proved above there exists an $x_1' \in D(h_1')$ with $\langle h_1 x, x_1' \rangle = -\langle x, x' \rangle$ for $x \in D(h_1)$. Now $[-x_1', x_2']$ annihilates H and

$$\begin{aligned} \langle [x_1, x_2][-x_1', x_2'] \rangle &= -\langle x_1, x_1' \rangle + \langle x_2, x_2' \rangle \\ &= -\langle h_1 x_0, x_1' \rangle + \langle x_2, x_2' \rangle \\ &= \langle x_0, x' \rangle + \langle x_2, x_2' \rangle = \langle [x_0, x_2], [x', x_2'] \rangle \neq 0 . \end{aligned}$$

The proof is complete.

The meaning of the preceding theorem is obvious: the set H is the graph of a mapping from E_1 into E_2. Indeed, if $[0, x_2] \in H$, we have $0 = h_1 x$ and $x_2 = h_2 x$ for some $x \in D(h_1) \cap D(h_2)$. It follows that $x \in h_1^{-1}(0) \subset h_2^{-1}(0)$, whence $x_2 = h_2 x = 0$.

The following result is, in a certain sense, dual to the preceding one.

Proposition (2,2). *Let E_1, E_2, E be convex spaces, f_1 a linear mapping from E_1 into E, f_2 a linear mapping from E_2 into E. Suppose that*

1° f_1 is weakly continuous and $D(f_1) = E_1$,

2° f_2 is closed in $E_2 \times E$.

Let F be the subset of $E_1 \times E_2$ consisting of $[x_1 x_2]$ such that $f_1 x_1 = f_2 x_2$.

Then F is closed in $E_1 \times E_2$.

Proof. Define a mapping f from $E_1 \times E_2$ into E in the following manner: $D(f)$ will be the set of all $[x_1, x_2]$ where

$$x_2 \in D(f_2) \quad \text{and} \quad f(x_1, x_2) = f_1 x_1 - f_2 x_2 .$$

We intend to show that f is closed in $E_1 \times E_2 \times E$; this also proves our theorem since clearly $F = f^{-1}(0)$.

Take a point $[x_1^0 \, x_2^0 \, x^0]$ which does not belong to $G(f)$. It follows that $[x_2^0, f_1(x_1^0) - x^0]$ non $\in G(f_2)$ so that there exists a point $[x_2', x'] \in E_2' \times E'$ with the following properties:

$$\langle [x_2, f_2(x_2)], [x_2', x'] \rangle = 0 \quad \text{for all } x_2 \in D(f_2)$$

and

$$\langle [x_2^0, f_1(x_1^0) - x^0], [x_2', x'] \rangle \neq 0 .$$

It follows that

$$\langle x_2, x_2'\rangle + \langle f_2(x_2), x'\rangle = 0 \quad \text{for all } x_2 \in D(f_2)\,.$$

Since f_1 is weakly continuous, $D(f_1') = E'$ and hence $f_1'(x')$ is defined. If $p = [f_1'(x'), x_2', -x']$, we have

$$\begin{aligned}\langle G(f), p\rangle &= \langle x_1, f_1'(x')\rangle + \langle x_2, x_2'\rangle - \langle f_1(x_1) - f_2(x_2), x'\rangle \\ &= \langle x_1 f_1'(x')\rangle - \langle f_1(x_1), x'\rangle + \langle x_2, x_2'\rangle + \langle f_2(x_2), x'\rangle = 0\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle [x_1^0, x_2^0, x^0], p\rangle &= \langle x_1^0, f_1'(x')\rangle + \langle x_2^0, x_2'\rangle - \langle x^0, x'\rangle \\ &= \langle x_2^0, x_2'\rangle + \langle f_1(x_1^0) - x^0, x'\rangle \\ &= \langle [x_2^0, f_1(x_1^0) - x^0], [x_2', x']\rangle \neq 0\,.\end{aligned}$$

This completes the proof.

The set F is the graph of a mapping from E_1 into E_2 if f_2 is injective. Indeed, if $[0, x_2] \in F$ we have $f_2 x_2 = 0$ whence $x_2 = 0$. The domain of this mapping will be the whole of E_1 if $R(f_1) \subset R(f_2)$.

PROPOSITION (2,3). *Let h be a linear mapping of (E, u) onto (B, v). Suppose that (E, u) is continuously imbedded in a space (E^*, u^*). Suppose that*

1° h is weakly open;

2° $h^{-1}(0)$ is closed in (E, u^*).

Then the closure of $G(h)$ in $(E^*, u^*) \times (B, v)$ is again the graph of a mapping from (E^*, u^*) onto (B, v).

PROOF. Suppose that $[0, y_0]$ belongs to the closure of $G(h)$ in $(E^*, u^*) \times (B, v)$. Since $R(h) = B$, $y_0 = h(x_0)$ for some $x_0 \in E$. Suppose that $y_0 \neq 0$. It follows that x_0 non $\in h^{-1}(0)$. The last set being closed in (E, u^*) there exists a $z' \in (E, u^*)$ such that $\langle h^{-1}(0), z'\rangle = 0$ and $\langle x_0, z'\rangle \neq 0$. The mapping h being weakly open there exists by (1,5) an element $b' \in (B, v)'$ such that $\langle hx, b'\rangle = \langle x, z'\rangle$ for all $x \in E$. If $x \in E$, we have

$$\langle [x, h(x)], [z', -b']\rangle = \langle x, z'\rangle - \langle h(x), b'\rangle = 0$$

and

$$\langle [0, y_0], [z', -b']\rangle = -\langle y_0, b'\rangle = -\langle h(x_0), b'\rangle = -\langle x_0, z'\rangle \neq 0\,.$$

This contradiction proves the theorem.

3. Sequentially open mappings.

Let E and F be two LF-spaces and T a continuous linear mapping of E into F. Let $U \in \boldsymbol{U}(E)$ and consider the set TU. If $R(T) = F$ it is not difficult to show [1] that TU will be a neighbourhood of zero in F. If $R(T)$ is different from F, the situation is much more complicated:

TU need not be a relative neighbourhood of zero in $R(T)$ even if $TU \cap F_j$ is a relative neighbourhood of zero in $R(T) \cap F_j$ for each j. In the present section we intend to study mappings which are open only in this weakened sequential sense.

PROPOSITION (3,1). *Let E and F be two LF-spaces and T a continuous linear mapping of E into F. Then for each i there exists a j such that $TE_i \subset F_j$.*

PROOF. Since $E_i = \bigcup_j (E_i \cap T^{-1}(F_j))$ it follows from Baire's theorem that $E_i \subset T^{-1}(F_j)$ for some j whence $TE_i \subset F_j$.

THEOREM (3,2). *Let E and F be two LF-spaces and T a continuous linear mapping of E into F. Then the following conditions are equivalent:*

1° *for each j the intersection $TE \cap F_j$ is closed;*

2° *for each j the intersection $TE \cap F_j$ is closed and there exists a defining sequence H_j of E such that $TH_j = TE \cap F_j$;*

3° *for each neighbourhood of zero U in E and each j the set TU is a neighbourhood of zero in $TE \cap F_j$ and there exists a defining sequence H_r of E such that for each r the restriction of T to H_r is open and $H_r + T^{-1}(0)$ closed in E.*

PROOF. 1° $\Rightarrow$ 2°. Let j be fixed. The set $TE \cap F_j$ is complete since it is closed in F_j. Since it may be represented as the union of $TE_i \cap F_j$ the set $TE_p \cap F_j$ will be of the second category in $TE \cap F_j$ for some p. Consider the mapping T of $E_p \cap T^{-1}(F_j)$ into $TE \cap F_j$. Its range $TE_p \cap F_j$ is of the second category in $TE \cap F_j$, so that we have $TE_p \cap F_j = TE \cap F_j$ by the open mapping theorem.

We see thus that for each j there exists a p such that $T^{-1}(F_j) \subset E_p + T^{-1}(0)$. Let us denote by $p(j)$ the minimal p of this property. We shall distinguish two cases:

I. The sequence $p(j)$ is bounded: $p(j) \leqq q$ for all j. It follows that $E = E_q + T^{-1}(0)$ whence $TE = TE_q$ so that $TE = TE_q \subset F_k$ for some k. Since TE is closed in F_k, the mapping T considered as a mapping of E_q onto $TE_q = TE$ is open. Hence T is open as a mapping of E onto TE. If $j \geqq \max(q, k)$ we have

$$TE_j = TE = TE \cap F_j$$

so that E_j for $j \geqq \max(q, k)$ is a defining sequence of the required property.

II. The sequence $p(j)$ is unbounded. Clearly $p(j) \leqq p(j+1)$, hence $p(j)$ tends to infinity. If j is given, put

$$H_j = E_{p(j)} \cap T^{-1}(F_j) .$$

Let us show first that H_j is a defining sequence.

(1) If j is given, we have $H_j \subset E_{p(j)}$ by definition.

(2) Let i be given. By (3,1) there exists a j such that $E_i \subset T^{-1}(F_j)$. Since $p(r)$ tends to infinity, there exists a $k > j$ such that $p(k) > i$. It follows that

$$E_i \subset T^{-1}(F_j) \subset T^{-1}(F_k) \quad \text{and} \quad E_i \subset E_p(k) .$$

Combining these inclusions we obtain

$$E_i \subset E_{p(k)} \cap T^{-1}(F_k) = H_k .$$

The sequence H_j is thus seen to be a defining sequence for E.

To show that $TH_j = TE \cap F_j$ let us observe first that $T^{-1}(F_j) \subset E_{p(j)} + T^{-1}(0)$. Each $x \in T^{-1}(F_j)$ may thus be written in the form $x = y + z$ with $y \in E_{p(j)}$ and $z \in T^{-1}(0)$. Since $y = x - z \in T^{-1}(F_j)$ we have $y \in H_j$ so that $T^{-1}(F_j) \subset H_j + T^{-1}(0)$. It follows that

$$H_j \subset T^{-1}(F_j) \subset H_j + T^{-1}(0) ,$$

whence $TH_j = TE \cap F_j$.

$2° \Rightarrow 3°$. Let H_j be a defining sequence of E such that $TH_j = TE \cap F_j$; we have $H_j + T^{-1}(0) = T^{-1}(F_j)$ so that $H_j + T^{-1}(0)$ is closed in E for each j.

Further, consider the mapping T as a mapping of H_j onto $TE \cap F_j$. The last set being closed in F_j and hence complete, it follows that T is an open mapping of H_j into F_j. If U is an arbitrary neighbourhood of zero in E, we have $TU \cap F_j \supset T(U \cap H_j) \cap F_j$ and the last set is a neighbourhood of zero in $TE \cap F_j$.

$3° \Rightarrow 1°$. Let H_j be a defining sequence for E such that $H_j + T^{-1}(0)$ is closed in E and the restriction of T to H_j is open for each j. Let j be given and let us show first that

$$TE \cap F_j = TH_p \cap F_j \quad \text{for some } p .$$

Suppose that for each p there exists an x_p such that $Tx_p \in F_j$ and Tx_p non $\in TH_p$. Taking, if necessary, suitable multiples of the x_j, we may assume the sequence Tx_p to be bounded. Since Tx_p non $\in TH_p$, we have x_p non $\in H_p + T^{-1}(0)$ so that there exists an $x_p' \in E'$ such that $\langle H_p + T^{-1}(0), x_p' \rangle = 0$ and $\langle x_p, x_p' \rangle = 1$. Let U_p be the set of all $x \in E$ for which $|\langle x, x_p' \rangle| \leqq 1/p$. Clearly $U = U_1 \cap U_2 \cap \ldots$ is a neighbourhood of zero in E. Let $m > 0$ be given and suppose that $x_p \in mU + T^{-1}(0)$ for all p. If $p > m$, we have $x_p \in mU_p + T^{-1}(0)$ whence $1 = \langle x_p, x_p' \rangle \leqq m/p < 1$ which is a contradiction. It follows that, given m, the inclusion $Tx_p \in mTU$ can never be satisfied for all p. Since TU is a relative neighbourhood of zero in F_j and Tx_p is bounded, we arrive at a contradiction. We have thus shown that

$$TE \cap F_j = TH_p \cap F_j \quad \text{for some } p .$$

Since $TH_p \subset F_k$ for some k by (3,1), the set TH_p is closed in F_k, the mapping T being open when considered as a mapping of H_p. It follows that $TE \cap F_j = TH_p \cap F_j$ is closed in E and the proof is complete.

If $R(T) = F$, the situation is very simple. Indeed, condition 1° of theorem (3,2) is automatically satisfied. It follows that, for each $U \in \boldsymbol{U}(E)$ and each j, the set $TU \cap F_j$ is a neighbourhood of zero in F_j so that TU is a neighbourhood of zero in F. A continuous linear mapping of an LF-space E onto an LF-space F is thus seen to be open. This result has been obtained first by J. Dieudonné and L. Schwartz [1].

Theorem (3,3). *Let E and F be two LF-spaces and T a continuous linear mapping of E onto F. Then T is open.*

Proof. If j is fixed we have $F_j = \bigcup T(E_p \cap T^{-1}(F_j))$ so that the set $T(E_p \cap T^{-1}(F_j))$ will be of the second category in F_j for some p. It follows from the open mapping theorem that $F_j = T(E_p \cap T^{-1}(F_j))$ for this p and that the restriction of the mapping T to $E_p \cap T^{-1}(F_j)$ is open. If U is a neighbourhood of zero in E, the set $U \cap E_p \cap T^{-1}(F_j)$ is a neighbourhood of zero in $E_p \cap T^{-1}(F_j)$, hence

$$TU \cap F_j \supset T(U \cap E_p \cap T^{-1}(F_j))$$

is a neighbourhood of zero in F_j. Since j was arbitrary this shows that T is open.

4. Open mappings.

A sequentially open mapping need not be open. Indeed, suppose T is a continuous and sequentially open mapping of an LF-space E into an LF-space F with range R. If U is a neighbourhood of zero in E we know that, for each j, there exists a neighbourhood of zero V_j in F such that $V_j \cap R \cap F_j \subset TU$. It does not follow that the V_j can be put together somehow to form a neighbourhood V of zero in F so as to have $V \cap R \subset TU$. It is necessary to indroduce additional assumptions to be able to do this; in the present section we intend to discuss some sufficient conditions for T to be open.

First let us mention the following rather strong condition due to B. Malgrange ([3, Prop. 6, p. 315]).

THEOREM (4,1). *Let F be an LF-space and M an absolutely convex subset of F. Let $Z \subset M$ be absolutely convex and such that for each j the set $Z \cap F_j$ is an open neighbourhood of zero in $M \cap F_j$. Suppose that $M \cap F_j$ is compact for each j.*
Then Z is a neighbourhood of zero in M.

PROOF. Let $V_j \in \boldsymbol{U}(F)$ be such that

$$V_j \cap M \cap F_j \subset Z\,.$$

We intend to show that there exists a $V_{j+1} \in \boldsymbol{U}(F)$ such that
1° $V_{j+1} \cap M \cap F_{j+1} \subset Z$;
2° $V_j \cap F_j \subset V_{j+1} \subset V_j$.
Denote by $\boldsymbol{V}$ the system of all $V \in \boldsymbol{U}(F)$ which satisfy $V_j \cap F_j \subset V \subset V_j$ and suppose that no $V \in \boldsymbol{V}$ satisfies $V \cap M \cap F_{j+1} \subset Z$. It follows that the set $V \cap M \cap F_{j+1} \cap (F-Z)$ is nonvoid for each $V \in \boldsymbol{V}$. Since $\boldsymbol{V}$ has the finite intersection property and the sets under consideration are compact, there exists a point x_0 which satisfies

$$x_0 \in V \cap M \cap F_{j+1} \cap (F-Z)$$

for each $V \in \boldsymbol{V}$. Since $V_j \in \boldsymbol{V}$, we have $x_0 \in V_j$.

Take now a point y non $\in F_j$. There exists a point $x' \in F'$ such that $\langle F_j, x'\rangle = 0$ and $\langle y, x'\rangle = 1$. Let S be the set of those $x \in F$ for which $|\langle x, x'\rangle| \leqq 1$. Clearly $V_j \cap S \in \boldsymbol{V}$ and $V_j \cap S$ does not contain y. It follows that $x_0 \in F_j$. We have thus $x_0 \in V_j \cap M \cap F_j \subset Z$ which is a contradiction.

Now take a $V_1 \in \boldsymbol{U}(F)$ such that $V_1 \cap M \cap F_1 \subset Z$ and construct by induction a sequence $V_j \in \boldsymbol{U}(F)$ with the properties 1° and 2°. It is easy to see that $V = \cap V_j$ belongs to $\boldsymbol{U}(F)$ and satisfies $V \cap M \subset Z$. This completes the proof.

The preceding compactness condition may be somewhat loosened. For that purpose a slightly more refined construction is required the idea of which is due to B. Malgrange (l.c. p. 316).

LEMMA (4,2). *Let F be an LF-space and $W \subset R$ two absolutely convex subsets of F. Suppose that, for each j, there exists a $V_j \in \boldsymbol{U}(F)$ such that*

$$V_j \cap R \cap F_j \subset W\,.$$

Suppose that the following condition is satisfied: Given j, ε, V_j such that $0 < \varepsilon < 1$, $V_j \in \boldsymbol{U}(F)$ and $V_j \cap R \cap F_j \subset W$ there exists a $V_{j+1} \in \boldsymbol{U}(F)$ such that
1° $V_{j+1} \cap R \cap F_{j+1} \subset W$;
2° $(1-\varepsilon)(V_j \cap F_j) \subset V_{j+1} \subset V_j$.
Then there exists a $V \in \boldsymbol{U}(F)$ so that $V \cap R \subset W$.

PROOF. Take a sequence $0<\varepsilon_j<1$ such that $\eta=\prod(1-\varepsilon_j)$ is positive. Choose a $V_1 \in \boldsymbol{U}(F)$ so that $V_1 \cap R \cap F_1 \subset W$ and construct by induction a sequence $V_j \in \boldsymbol{U}(F)$ so that

$$V_j \cap R \cap F_j \subset W \quad \text{and} \quad (1-\varepsilon_j)(V_j \cap F_j) \subset V_{j+1} \subset V_j$$

Put $V=\bigcap V_j$ so that $R \cap V \subset W$. To see that V is a neighbourhood of zero in F it suffices to show that $\eta(V_j \cap F_j) \subset V$ or, in other words, that $\eta(V_j \cap F_j) \subset V_k$ for each k. With view so the inclusions $V_1 \supset V_2 \supset \ldots$ it is sufficient to prove this for $k>j$ only. Put $H_j=V_j \cap F_j$; we have

$$(1-\varepsilon_j)H_j \subset V_{j+1} \cap F_{j+1} = H_{j+1}$$

so that $\eta_j H_j \subset \eta_{j+1} H_{j+1}$ where $\eta_j=(1-\varepsilon_j)(1-\varepsilon_{j+1})\ldots$. It follows that, for each $p>0$,

$$\eta H_j \subset \eta_j H_j \subset \eta_{j+p} H_{j+p} \subset V_{j+p},$$

and the proof is complete.

PROPOSITION (4,3). *Let F be an LF-space and R a subspace of F. Let $H \in \boldsymbol{U}(F)$ and put $M=R \cap H$. Let $Z \subset M$ be an absolutely convex set such that for each j the set $Z \cap F_j$ is an open neighbourhood of zero in $M \cap F_j$. Suppose that, for each j, the following condition is satisfied: if $X \subset M \cap F_{j+1}$ and X/F_j is bounded, then $X/R \cap F_j$ is relatively compact in $M/R \cap F_j$. Then Z is a neighbourhood of zero in M.*

PROOF. Let $V_j \in \boldsymbol{U}(F)$ be such that $V_j \subset H$ and

$$V_j \cap M \cap F_j \subset Z$$

and let $0<\varepsilon<1$ be given. We intend to show that then exists a $V_{j+1} \in \boldsymbol{U}(F)$ such that

1° $V_{j+1} \cap M \cap F_{j+1} \subset Z$;

2° $(1-\varepsilon)(V_j \cap F_j) \subset V_{j+1} \subset V_j$.

Denote by $\boldsymbol{V}$ the system of all $V \in \boldsymbol{U}(F)$ which satisfy

$$(1-\varepsilon)(V_j \cap F_j) \subset V \subset V_j$$

and suppose that no $V \in \boldsymbol{V}$ satisfies $V \cap M \cap F_{j+1} \subset Z$. Clearly there exists a convex topology w on F with the following properties:

(1) w is coarser than the topology v of F;

(2) w coincides with v on F_{j+1};

(3) w has a countable complete system of neighbourhoods of zero.

Let us denote by Q the canonical quotient mapping of F modulo F_j and put $\tilde{w}=Qw$. The topology $\tilde{w}$ has a countable complete system of neighbourhoods of zero $\tilde{W}_n \in \boldsymbol{U}(F/F_j,\tilde{w})$. For each n put

$$V^{(n)} = (1-\varepsilon)V_j \cap Q^{-1}\tilde{W}_n .$$

Let us show first that $V^{(n)} \in \mathbf{V}$ for each n. Since $\tilde{W}_n \in \mathbf{U}(F/F_j, \tilde{w})$ we have $Q^{-1}\tilde{W}_n \in \mathbf{U}(F)$ whence $V^{(n)} \in \mathbf{U}(F)$. The inclusion

$$V^{(n)} = (1-\varepsilon)V_j \cap Q^{-1}\tilde{W}_n \subset (1-\varepsilon)V_j \subset V_j$$

being obvious, take an $x \in (1-\varepsilon)(V_j \cap F_j)$. Since

$$x \in F_j = Q^{-1}(0) \subset Q^{-1}\tilde{W}_n ,$$

we have $x \in V^{(n)}$ which completes the proof of $V^{(n)} \in \mathbf{V}$.

Let $y_n \in V^{(n)} \cap M \cap F_{j+1} \cap (F-Z)$. Since $Qy_n \in \tilde{W}_n$, $y_n \in F_{j+1}$ and w coincides with v on F_{j+1}, the set of the y_n is bounded in F_{j+1} modulo F_j. It follows from our assumption that there is a subsequence z_n of y_n, a point $y_0 \in M \cap F_{j+1}$ and a sequence $r_n \in R \cap F_j$ such that $z_n - r_n \to y_0$. Since

$$Qy_n \to 0 \quad \text{and} \quad Qz_n = Q(z_n - r_n) \to Qy_0 ,$$

we have $Qy_0 = 0$ so that $y_0 \in F_j$. Hence $y_0 \in R \cap F_j$ and $q_n = r_n + y_0 \in R \cap F_j$. Now $q_n = z_n + (q_n - z_n)$ and $(q_n - z_n) \to 0$ so that

$$q_n \in (q_n - z_n) + (1-\varepsilon)V_j \subset (1-\tfrac{1}{2}\varepsilon)V_j \subset (1-\tfrac{1}{2}\varepsilon)H \quad \text{for large } n .$$

It follows that

$$q_n \in (1-\tfrac{1}{2}\varepsilon)(V_j \cap H \cap R \cap F_j) \subset (1-\tfrac{1}{2}\varepsilon)Z$$

for large n.

Now both z_n and q_n belong to $R \cap F_{j+1}$ and z_n do not belong to Z. The last set being a relative neighbourhood of zero in $R \cap F_{j+1}$, it follows that, for large n, $q_n \notin (1-\tfrac{1}{2}\varepsilon)Z$ which is a contradiction.

Theorem (4,4). *Let (E,u) and (F,v) be two LF-spaces. A continuous linear mapping T of E into F will be open if the following condition is satisfied:*

There exist defining sequences E_j and F_j such that $T^{-1}F_j \subset E_j + T^{-1}(0)$ and an $H \in \mathbf{U}(F)$ with the following properties: if $x_n \in E_{j+1}$, $Tx_n \in H$ and $Tx_n \to 0 \bmod F_j$, then $x_n \to 0 \bmod E_j + T^{-1}(0)$.

Proof. Denote by R_j the set TE_j and by Q the canonical mapping of F onto F/F_j. Let w be a convex topology on F which is coarser than v, coincides with v on R_{j+1} and has a countable complete system of neighbourhoods of zero. Put $\tilde{w} = Qw$ and take a countable complete system of neighbourhoods of zero $\tilde{W}_n \in \mathbf{U}(F/F_j, \tilde{w})$. Let $U \in \mathbf{U}(E)$ and suppose we are given a $V_j \in \mathbf{U}(F)$ such that $V_j \subset H$ and $R_j \cap V_j \subset TU$. Let $0 < \varepsilon < 1$ be given. Put

$$V^{(n)} = (1-\varepsilon)V_j \cap Q^{-1}\tilde{W}_n$$

and let us show that at least one $V^{(n)}$ satisfies the following two conditions:

1° $(F_j \cap V^{(n)}) = (1-\varepsilon)(F_j \cap V_j)$;

2° $R_{j+1} \cap V^{(n)} \subset TU$.

First of all, $V^{(n)} \in \mathfrak{U}(F)$ since Q is continuous and w is coarser than v. Further it is easy to see that 1° is satisfied for any n. Indeed, $V^{(n)} \subset (1-\varepsilon)V_j$ whence

$$F_j \cap V^{(n)} \subset (1-\varepsilon)(F_j \cap V_j)\,.$$

On the other hand, if $y \in (1-\varepsilon)(F_j \cap V_j)$ we have $y \in (1-\varepsilon)V_j$ and $Qy = 0$ since $y \in F_j$. Hence $y \in Q^{-1}\tilde{W}_n$ so that $y \in V^{(n)}$ and $y \in F_j \cap V^{(n)}$.

Suppose now that the inclusion 2° is not satisfied for any n. It follows that there exists a sequence $x_n \in E_{j+1}$ such that

$$Tx_n \in (1-\varepsilon)V_j \subset H, \qquad QTx_n \in \tilde{W}_n, \qquad Tx_n \notin TU\,.$$

Since $Tx_n \in R_{j+1}$ and w coincides with v on R_{j+1}, we have $Tx_n \to 0$ $\operatorname{mod} F_j$. It follows from our assumption that we have $x_n - z_n \to 0$ for a suitable sequence $z_n = p_n + q_n$ with $p_n \in E_j$ and $q_n \in T^{-1}(0)$. Since

$$Tp_n = T(x_n - q_n) - T(x_n - p_n - q_n) = Tx_n - T(x_n - z_n)\,,$$

$Tp_n \in (1-\frac{1}{2}\varepsilon)V_j$ for large n. Since $Tp_n \in R_j$, it follows that, for large n,

$$Tp_n \in (1-\tfrac{1}{2}\varepsilon)(V_j \cap R_j) \subset (1-\tfrac{1}{2}\varepsilon)TU$$

so that there exist $r_n \in T^{-1}(0)$ such that $p_n + r_n \in (1-\frac{1}{2}\varepsilon)U$. Hence

$$x_n - q_n + r_n = p_n + r_n + (x_n - z_n) \in U$$

for large n, so that $Tx_n = T(x_n - q_n + r_n) \in TU$ for large n, which is a contradiction.

The conclusion follows from lemma (4,2).

THEOREM (4,5). *Let (E,u) and (F,v) be two LF-spaces and T a continuous linear mapping of E into F. The mapping T will be open if the following two conditions are satisfied:*

1° *there exist defining sequences E_j and F_j such that $T^{-1}(F_j) \subset E_j + T^{-1}(0)$ for each j;*

2° *there exists an $H \in \mathfrak{U}(F)$ such that, for each j, the mapping $Q_j \circ T_{j+1}$ is open in H (here Q_j is the canonical quotient mapping of F $\operatorname{mod} F_j$ and T_{j+1} is the restriction of T to E_{j+1}).*

PROOF. An immediate consequence of Proposition (4,4).

5. Convolution operators.

In this section we intend to give an application of Theorem (4,4) to the case of a convolution operator in a space of type $\mathscr{D}(\Omega)$. Let Ω_1 and Ω_2 be two open subsets of the euclidean space E_n and let $S \in \mathscr{E}'(E_n)$ be such that $\operatorname{supp}(S*\varphi) \subset \Omega_2$ for each $\varphi \in \mathscr{D}(\Omega_1)$. The following facts are well known: if $\varphi \in \mathscr{D}(\Omega_1)$ then $S*\varphi \in \mathscr{D}(\Omega_2)$ and the mapping T defined by $T\varphi = S*\varphi$ is an injective linear and continuous mapping of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$. The mapping T will be called a convolution operator of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$.

The operator T may be extended in an obvious manner to a mapping A of $\mathscr{E}'(\Omega_1)$ into $\mathscr{E}'(\Omega_2)$. Let us recall now a result of Hörmander [2]: if T^{-1} is sequentially continuous then $T'\mathscr{E}(\Omega_2) = \mathscr{E}(\Omega_1)$. Especially it follows from this result that, for a sequentially open convolution operator T, the mapping A is an injective continuous and open extension of T to $\mathscr{E}'(\Omega_1)$. We shall apply this result in the proof of Theorem (5,1).

THEOREM (5,1). *Let Ω_1 and Ω_2 be two open subsets of the euclidean space E_n and let T be a convolution operator of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$. Suppose that the following two conditions are satisfied:*

1° *T is sequentially open;*

2° *for each compact $K_2 \subset \Omega_2$ there exists a compact $K_1 \subset \Omega_1$ with the following property: if $\varphi \in \mathscr{E}'(\Omega_1)$, $A\varphi$ is continuous on Ω_2 and infinitely differentiable outside K_2, then φ is infinitely differentiable outside K_1.*

Then T is open.

PROOF. We intend to show that the assumptions of Theorem (4,4) are satisfied.

1. Let K_2^j be an increasing sequence of compact sets with union Ω_2. It follows from Theorem (3,2) and condition 2° of the present theorem that there exists an increasing sequence of compact sets $K_1^j \subset \Omega_1$ with union Ω_1 such that (a) if $\varphi \in \mathscr{D}(\Omega_1)$ and $\operatorname{supp} T\varphi \subset K_2^j$ then $\operatorname{supp}\varphi \subset K_1^j$ (b) if $\varphi \in \mathscr{E}'(\Omega_1)$, $A\varphi$ is continuous on Ω_2 and infinitely differentiable outside K_2^j then φ is infinitely differentiable outside K_1^j. Now take an arbitrary j and keep it fixed. We shall denote by h_{j+1} the natural imbedding of $C(K_2^{j+1})$ into $\mathscr{E}'(\Omega_2)$. Let q_2^j be the canonical mapping of $\mathscr{D}(K_2^{j+1})$ modulo $\mathscr{D}(K_2^j)$ and let Q_2^j be its natural extension to $\mathscr{E}'(\Omega_2)$ in the sense of Proposition (2,3). The mappings q_1^j and Q_1^j are defined in a similar manner.

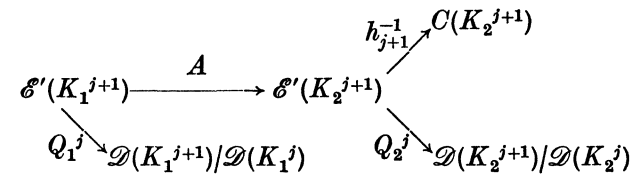
2. Consider now the space $\mathscr{E}'(K_2^{j+1})$; clearly h_{j+1}^{-1} is an open mapping from $\mathscr{E}'(K_2^{j+1})$ onto $C(K_2^{j+1})$. By its definition, Q_2^j is a mapping from $\mathscr{E}'(K_2^{j+1})$ onto $\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^j)$ the graph of which is closed in

$$\mathscr{E}'(K_2^{j+1}) \times \big(\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^{j})\big)$$

It follows from theorem (2,1) that the set

$$P \subset C(K_2^{j+1}) \times \big(\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^{j})\big)$$

of those pairs $[x,y]$ for which there exists a $\xi \in D(h_{j+1}^{-1}) \cap D(Q_2^{j})$ with $h_{j+1}^{-1}\xi = x$ and $Q_2^{j}\xi = y$ is closed in $C(K_2^{j+1}) \times \big(\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^{j})\big)$.



3. Clearly the mapping p from $\mathscr{E}'(K_2^{j+1})$ into P defined for $\xi \in D(p) = D(h_{j+1}^{-1}) \cap D(Q_2^{j})$ as

$$p(\xi) = [h_{j+1}^{-1}\xi,\, Q_2^{j}\xi]$$

is open and $R(p) = P$. Further, we have assumed T to be sequentially open so that A is an open mapping of $\mathscr{E}'(\Omega_1)$ into $\mathscr{E}'(\Omega_2)$. Using these facts, it is easy to see that the mapping $p \circ A$ defined on $A^{-1}(D(p))$ with range P is open as well. Since $D(p) \subset \mathscr{E}'(K_2^{j+1})$, we have $A^{-1}(D(p)) \subset \mathscr{E}'(K_1^{j+1})$. We have thus an open mapping $p \circ A$ from $\mathscr{E}'(K_1^{j+1})$ into P with domain $A^{-1}(D(p))$ and range P. Further, the graph of the mapping Q_1^{j+1} is closed in $\mathscr{E}'(K_1^{j+1}) \times \big(\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^{j})\big)$. It follows from theorem (2,1) that the set M of those $[u,v] \in P \times \big(\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^{j})\big)$ for which there exists a $z \in A^{-1}(D(p)) \cap D(Q_1^{j})$ with $p(A(z)) = u$ and $Q_1^{j}z = v$, is closed in $P \times \big(\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^{j})\big)$.

4. Since $p \circ A$ is one-to-one, the set M is the graph of a mapping g from P into $\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^{j})$. Now we use assumption 2°. It follows that $A^{-1}(D(p)) \subset D(Q_1^{j})$ so that g is defined on the whole of P. By the closed graph theorem the mapping g is continuous.

5. We intend to show now that the assumption of theorem (4,4) is satisfied. Denote by H the set of those $y \in \mathscr{D}(\Omega_2)$ for which

$$\max |y(t)| \leqq 1, \qquad t \in \Omega_2 .$$

It follows that $H \in U(\mathscr{D}(\Omega_2))$. Suppose now that we have a sequence $x_n \in \mathscr{D}(K_1^{j+1})$ such that $Tx_n \in H$ and $Q_2^{j}Tx_n \to 0$. It follows that pTx_n is bounded so that $Q_1^{j}x_n = gpTx_n$ is bounded as well.

Since pTx_n is bounded, the sequence Tx_n is bounded in $\mathscr{E}'(K_2^{j+1})$ so that, A being open, x_n is bounded in $\mathscr{E}'(K_1^{j+1})$. Since $Q_1^{j}x_n$ is bounded, there exists a sequence $y_n \in \mathscr{D}(K_1^{j+1})$ such that $Q_1^{j}y_n = Q_1^{j}x_n$ and y_n is bounded in $\mathscr{D}(K_1^{j+1})$, a Montel space. Let $y_0 \in \mathscr{D}(K_1^{j+1})$ be an arbitrary limit point of the sequence y_n and let W be an infinite set of natural numbers such that $\lim y_n = y_0$ for $n \in W$. Since x_n is bounded in $\mathscr{E}'(K_1^{j+1})$ there exists an infinite $S \subset W$ and an $x_0 \in \mathscr{E}'(K_1^{j+1})$ such that $\lim x_n = x_0$ for $n \in S$ in $\mathscr{E}'(K_1^{j+1})$. It follows that, for $n \in S$, we have $\lim Tx_n = Ax_0$. Since $Q_2^j Tx_n \to 0$, we have $Ax_0 \in \mathscr{E}'(K_2^j)$ whence $x_0 \in \mathscr{E}'(K_1^j)$. Since $Q_1^j(y_n - x_n) = 0$ and $y_n - x_n$ converges for $n \in S$ to $y_0 - x_0$ in the topology of $\mathscr{E}'(K_1^{j+1})$, we have $Q_1^j(y_0 - x_0) = 0$. Since $\mathscr{E}'(K_1^j) \subset D(Q_1^j)$ it follows that

$$Q_1^j y_0 = Q_1^j x_0 + Q_1^j(y_0 - x_0) = 0 .$$

Hence, for $n \in W$,

$$\lim Q_1^j x_n = \lim Q_1^j y_n = 0 .$$

Since y_0 was arbitrary, we have $\lim Q_1^j x_n = 0$.

Theorem (5,1) represents a slight improvement of a result of Hörmander [2]. Hörmander's theorem requires that, for each $K_2 \subset \Omega_2$ there exists a $K_1 \subset \Omega_1$ with the following property: if $\varphi \in \mathscr{E}'(\Omega_1)$ and $A\varphi$ is infinitely differentiable outside K_2 then φ is infinitely differentiable outside K_1. In (5,1) we require this for continuous $A\varphi$ only. This (very mild) generalization can be pushed a little further.

THEOREM (5,2). *Let Ω_1 and Ω_2 be two open subsets of the euclidean space E_n and let T be a convolution operator of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$. Suppose that T is sequentially open. Suppose we have two sequences of compact sets $K_2^j \subset \Omega_2$ and $K_1^j \subset \Omega_1$ and a nondecreasing sequence of nonnegative integers $n(i)$ with the following properties:*

1° *$K_2^0 = 0$, the sequence K_2^j is increasing with union Ω_2, the sets K_2^j are closures of their interiors; similarly for K_1^j;*

2° *if $\varphi \in \mathscr{E}'(\Omega_1)$, $A\varphi$ has continuous derivatives up to order $n(i)$ outside K_2^i with continuous extensions to the set $\Omega_2 - \operatorname{int} K_2^i$ and if $A\varphi$ is infinitely differentiable outside K_2^j then φ is infinitely differentiable outside K_1^j.*

Then T is open.

PROOF. We shall introduce first, for each j, a space C_{j+1} defined as follows. The elements of C_{j+1} will be those $x \in C(K_2^{j+1})$ which possess the following properties: for $i = 0, 1, \ldots, j$ the function x has continuous derivatives up to the order $n(i)$ on $\operatorname{int} K_2^{j+1} - K_2^i$ and these have continuous extensions to $K_2^{j+1} - \operatorname{int} K_2^i$. The topology of C_{j+1} will be defined by means of the following $j+1$ pseudonorms:

$$p_i(x) = \max_{|D| \leq n(i)} \{\max |x^D(t)|,\ t \in K_2^{j+1} - \operatorname{int} K_2^i\}$$

for $i = 0, 1, \ldots, j$. Clearly C_{j+1} is complete.

Further, let us denote by h_{j+1}^* the natural imbedding of C_{j+1} into $\mathscr{E}'(\Omega_2)$ so that h_{j+1}^* is continuous. We shall denote by H^* the set of those $y \in \mathscr{D}(\Omega_2)$ which satisfy the following inequalities

$$\max_{|D| \leqq n(i)} |x^D(t)| \leqq 1, \qquad t \in \Omega_2 - \operatorname{int} K_2{}^i$$

for $i = 0, 1, 2, \ldots$. It follows that $H \in \boldsymbol{U}(\mathscr{D}(\Omega_2))$.

The proof of the present theorem is identical with the proof of Theorem (5,1). It suffices to replace $C(K_2{}^{j+1})$ by C_{j+1}, h_{j+1}, by h_{j+1}^* and H by H^*.

BIBLIOGRAPHY

1. J. Dieudonné et L. Schwartz, *La dualité dans les espaces* F *et* LF, Ann. Inst. Fourier Grenoble 1 (1949), 61–101.
2. L. Hörmander, *On the range of differential and convolution operators*, Ann. of Math. 76 (1962), 148–170.
3. B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier Grenoble 6 (1956), 271–355.

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