MATH. SCAND. 16 (1965), 94–96

ON THE DIRECTIONS OF ALMOST PERIODICITY OF ENTIRE FUNCTIONS

PER ASISOFF

The purpose of this note is to prove two theorems concerning entire functions almost periodic in several directions. The theorems state, roughly speaking, that there cannot exist too many such directions. The existence of entire functions almost periodic in several directions is proved in the paper by H. Tornehave [4].

Let $f(z)$ be an entire function and let S denote the set of numbers φ in the interval $0 \leqq \varphi < \pi$, for which $f(z)$ is almost periodic in every strip $-\infty < a < \mathrm{Re}(ze^{-i\varphi}) < b < \infty$. With these notations we have the following theorems:

THEOREM 1. *If the entire function $f(z)$ is not a constant, the complementary set of S with respect to the interval $0 \leqq \varphi < \pi$ is everywhere dense.*

THEOREM 2. *If the entire function $f(z)$ is of finite order and not a constant, the set S is finite.*

Let α and β be real numbers satisfying $0 < \beta - \alpha < 2\pi$. Let P denote the point set

$$\{z = re^{i\varphi} \mid r \geqq 0, \alpha \leqq \varphi \leqq \beta\} .$$

With these notations we have the following theorem:

THEOREM 3. *Let $f(z)$ be analytic in a domain containing P. Let $f(z)$ be bounded in P and let for some γ in the interval $\alpha < \gamma < \beta$ the function $f_\gamma(r) = f(re^{i\gamma})$ be the restriction to the half line $r \geqq 0$ of an almost periodic function $g(r)$, $-\infty < r < \infty$. Then $f(z)$ is a constant.*

We shall first prove that the theorems 1 and 2 follow from theorem 3. Afterwards, we shall prove theorem 3.

THEOREM 3 IMPLIES THEOREM 1. If theorem 1 is not true, we can choose α_1 and β_1 such that $0 < \beta_1 - \alpha_1 < 2\pi$, and such that for every φ satisfying $\alpha_1 \leqq \varphi \leqq \beta_1$ the function $f_\varphi(r) = f(re^{i\varphi})$ is almost periodic and,

Received March 13, 1964.

hence, bounded for $-\infty < r < \infty$. From a classical theorem of W. F. Osgood [3] applied to the family of functions $f_\varphi(r)$ for $r \geqq 0$, $\alpha_1 \leqq \varphi \leqq \beta_1$ it follows that there exist numbers α_2, β_2 satisfying $\alpha_1 \leqq \alpha_2 < \beta_2 \leqq \beta_1$ such that the family is uniformly bounded for $\alpha_2 \leqq \varphi \leqq \beta_2$. Then theorem 3 implies that $f(z)$ is a constant in contradiction to the assumptions in theorem 1.

THEOREM 3 IMPLIES THEOREM 2. If theorem 2 is not true, we can for every $\delta > 0$ choose real numbers α, β, γ such that $\alpha < \gamma < \beta$, $\beta - \alpha < \delta$, and such that each of the functions $f(re^{i\alpha})$, $f(re^{i\beta})$, $f(re^{i\gamma})$ is almost periodic. If $f(z)$ is not a constant, theorem 3 implies that $f(z)$ is not bounded for $r \geqq 0$, $\alpha \leqq \varphi \leqq \beta$. According to Phragmén–Lindelöf's theorem the order of $f(z)$ is then at least $\pi(\beta-\alpha)^{-1}$. Since $\beta - \alpha$ can be chosen arbitrarily small, this contradicts the assumption that the order of $f(z)$ is finite.

PROOF OF THEOREM 3. We may assume that $\gamma = 0$, $-\frac{1}{2}\pi < \alpha < 0$ and $0 < \beta < \frac{1}{2}\pi$. We introduce the Laplace transforms

$$F_\varphi(s) = \int_0^\infty f(re^{i\varphi})\, e^{-sre^{i\varphi}}\, dre^{i\varphi}$$

for $\varphi = 0, \alpha, \beta$. Since $f(z)$ is bounded in P, these Laplace transforms are analytic when $\operatorname{Re}(se^{i\varphi}) > 0$. The almost periodic function $g(r)$ has a Fourier series

$$g(r) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n r}, \quad \lambda_1 = 0\,, \tag{1}$$

where

$$a_n = \lim_{T\to\infty} \frac{1}{T} \int_0^T g(r)\, e^{-i\lambda_n r}\, dr\,.$$

By means of a theorem of abelian type [2, p. 193], we conclude that $(s - i\lambda_n)F_0(s) \to a_n$ for $s = \sigma + i\lambda_n$ and $\sigma \to 0$ through positive values. Hence $i\lambda_n$ is a singular point for $F_0(s)$ if $a_n \neq 0$. (As a matter of fact the closure of the set $\{i\lambda_n \mid a_n \neq 0\}$ is the set of singular points for $F_0(s)$ on the imaginary axis, see [1]).

Let $\varkappa$ denote the boundary of the triangle

$$\{z \mid 0 \leqq \operatorname{Arg} z \leqq \beta,\ 0 \leqq \operatorname{Re} z \leqq T\}$$

orientated in the positive direction. From Cauchy's theorem it follows that

$$\int_\varkappa f(z)\, e^{-sz}\, dz = 0\,.$$

If s is a positive number, the integral along the vertical side tends to 0 when $T \to \infty$ since $f(z)$ is bounded. Hence we have $F_\beta(s) = F_0(s)$ for positive values of s. According to the theorem on uniqueness of analytic continuation this implies that the functions $F_\beta(s)$ and $F_0(s)$ are identical. Similarly, we prove that $F_\alpha(s)$ and $F_0(s)$ are identical. However, $F_\alpha(s)$ is analytic for $\mathrm{Re}(se^{i\alpha}) > 0$ and $F_\beta(s)$ is analytic for $\mathrm{Re}(se^{i\beta}) > 0$. We have thus proved that no points of the imaginary axis except 0 are singular points for $F_0(s)$. According to the preceding remark this implies that the Fourier series (1) consists only of the constant term. Hence, $g(r)$ is constant, and this implies that $f(z)$ is constant. This completes the proof of theorem 3.

BIBLIOGRAPHY

1. S. Bochner and F. Bohnenblust, *Analytic functions with almost periodic coefficients*, Ann. of Math. 35 (1934), 152–161.
2. G. Doetsch, *Theorie und Anwendung der Laplacetransformation*, Berlin, 1937.
3. W. F. Osgood, *Note on the functions defined by infinite series whose terms are analytic functions of a complex variable; with corresponding theorems for definite integrals*, Ann. of Math. (2) 3 (1901), 25–34.
4. H. Tornehave, *On entire functions almost periodic in several directions*, Math. Scand. 16 (1965), 116–120.

ORDRUPVEJ 98 C4, CHARLOTTENLUND, DENMARK