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ON A CARTAN FORMULA FOR SECONDARY COHOMOLOGY OPERATIONS

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1. Introduction.

The purpose of the present paper is to prove a Cartan formula for secondary cohomology operations. By a Cartan formula we mean an expansion $\Phi(xy)=\sum\Phi'(x)\,\Phi''(y)$, valid for a certain set of pairs of cohomology classes $\{(x,y)\}$. The existence of such a formula was proved by Adams [1]. Later Adem [2] gave a formula for certain operations Φ.

The present paper is a continuation of the paper [3] in which a definition of secondary operations based on a study of cochain operations was given.

In Section 2 we continue the study of cochain operations which we began in [3]. Let us recall that a cochain operation θ in one variable is a sequence $\{\theta_n\}$ of natural transformations $\theta_n\colon C^n \to C^{n+i}$, where $i=\deg\theta$. Everywhere we shall assume Z_2 coefficients. In [3] we proved the exactness of the sequence

$$\mathcal{O} \xrightarrow{\Delta} Z(\mathcal{O}) \xrightarrow{\varepsilon} A \longrightarrow 0\,,$$

where $\mathcal{O}$ is the set of cochain operations, Δ is the boundary operator $\Delta\theta=\delta\theta+\theta\delta$, $Z(\mathcal{O})=\operatorname{Ker}(\Delta)$ and A is the Steenrod algebra (mod 2).

Here we consider cochain operations G in two variables satisfying

$$\deg G(x,y) \;=\; \deg(x)+\deg(y)+\deg G \tag{1}$$

and

$$G(x,0) \;=\; G(0,y) \;=\; 0\,. \tag{2}$$

The Z_2-module of such operations is denoted by Q. In Q there is a boundary operator ∇ defined by

$$(\nabla G)(x,y) \;=\; \delta G(x,y)+G(\delta x,y)+G(x,\delta y)\,.$$

We prove that the following sequence is exact (Theorem 2.2):

$$Q \xrightarrow{\nabla} Z(Q) \longrightarrow A\otimes A \longrightarrow 0\,, \tag{1.1}$$

where $Z(Q)=\operatorname{Ker}(\nabla)$.

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Sections 3–5 are devoted to the proof of the Cartan formula mentioned above. The main theorem is Theorem 4.1 (in Section 4). The main tool in the proof is the sequence (1.1).

Another application of (1.1) is to cohomology operations associated with Cartan formulas. Operations of this type have been studied by Schweitzer [4]. This application is the topic of a forthcoming paper.

2. Theorem on cochain operations.

In [3] we have considered cochain operations of the following kind. A cochain operation of degree i is a sequence $\theta = \{\theta_n\}$ of natural transformations of the cochain functor into itself

$$\theta_n\colon\ C^n \to C^{n+i}, \qquad -\infty < n < \infty ,$$

preserving the zero element but not in general the additive structure. The integer i is called the degree of θ. If we define $\Delta\theta = \delta\theta + (-1)^i \theta\delta$, then Δ is a boundary operator in the graded Z_2-module $\mathcal{O}$ of all cochain operations. Let $Z(\mathcal{O})$ denote the Δ-cycles, then there is a map ε: $Z(\mathcal{O}) \to A$, where A denotes the mod 2 Steenrod algebra. Composition defines a multiplication in $\mathcal{O}$. This multiplication is left distributive but not right distributive. We shall, however, allow ourselves to say that $\mathcal{O}$ is an algebra. The mapping ε is then an algebra mapping. One can also consider cochain operations in k variables. The set of all these, $\mathcal{O}^{(k)}$, has in a similar way a boundary operator

$$(\Delta\theta)(x_1,\ldots,x_k) = \delta\theta(x_1,\ldots,x_k) + \theta(\delta x_1,\ldots,\delta x_k) .$$

Later in this section we shall consider cochain operations of a different kind. Therefore, in what follows we shall denote the above operations as operations of the first kind. In [3] we proved

THEOREM 2.1. *The sequence*

$$\mathcal{O}^{(k)} \xrightarrow{\Delta} Z(\mathcal{O}^{(k)}) \xrightarrow{\varepsilon} A \oplus A \oplus \ldots \oplus A \to 0, \qquad k \textit{ copies of } A ,$$

is exact.

Now let us consider operations of the second kind. These are operations $H = \{H_{m,n}\}$ of two variables with the properties

$$(2.1) \qquad H_{m,n}\colon\ C^m \oplus C^n \to C^{m+n+i}, \qquad H(x,0) = H(0,y) = 0 ,$$

where i is an integer independent of m and n. This integer is called the degree of H, $\deg(H) = i$. As earlier we do not assume additivity.

Let Q denote the set of all cochain operations of the second kind. Then Q is a graded module over Z_2. If H and K are of degree i, then

$$(H_{m,n}+K_{m,n})(x,y) = H_{m,n}(x,y)+K_{m,n}(x,y) .$$

A boundary operator ∇ in Q is defined by

$$(\nabla H)(x,y) = \delta H(x,y)+H(\delta x,y)+H(x,\delta y) .$$

It is obvious that $\nabla\nabla=0$.

Similar to the case of operations of the first kind there is a mapping from the ∇-cycles $Z(Q)$ into $A\otimes A$,

$$\varepsilon\colon\ Z(Q)\to A\otimes A . \tag{2.2}$$

The mapping ε can be defined on a larger class of operations than $Z(Q)$. This goes as follows.

An operation $F(x,y)$ is in what follows called special if $F(x,y)=0$ on each pair (x,y) of cocycles. If $\delta x=0$ or $\delta y=0$ implies that $F(x,y)=0$, then we call F very special. The mapping ε can be defined on an operation G if ∇G is very special. This is done by considering the Eilenberg–MacLane complexes $K(Z_2,n)$ and $L(Z_2,n)$. For short we shall denote these complexes by K_n and L_n. We recall that in K_n (L_n) there is a basic cocycle z_n (cochain c_n) with the property that to an arbitrary cocycle (cochain) a in a css-complex X there is one and only one mapping $f\colon X\to K_n$ ($f\colon X\to L_n$) such that

$$f\#(z_n) = a \qquad (f\#(c_n) = a) ,$$

$f\#$ denoting the induced cochain transformation. The inclusion of K_n in L_n is denoted by $i\colon K_n\to L_n$ and the projection $L_n\to K_{n+1}$ is denoted by p. They have the properties

$$i\#(c_n) = z_n \qquad \text{and} \qquad p\#(z_{n+1}) = \delta c_n .$$

Since ∇G is very special, we have for each pair (m,n) of integers that $G(z_m,z_n)$ is a cocycle. This means that for each pair (m,n) we have an element

$$\{G(z_m,z_n)\}\in H^*(K_m\times K_n) .$$

We shall examine the connection between these cohomology classes for various pairs (m,n). First, let us consider the pairs (m,n) and $(m+1,n)$. We have

$$\delta G(c_m,z_n) = G(\delta c_m,z_n) = (p\times 1)\#\big(G(z_{m+1},z_n)\big) .$$

This implies that

$$\{G(z_m,z_n)\} = (\sigma\otimes 1)\{G(z_{m+1},z_n)\} ,$$

where σ is the cohomology suspension $H^i(X)\to H^{i-1}(\Omega X)$ and

$$\{G(z_{m+1},z_n)\} \in H^*(K_{m+1}\times K_n) \cong H^*(K_{m+1})\otimes H^*(K_m) .$$

In a similar fashion one sees that

$$\{G(z_m,z_n)\} = (1\otimes\sigma)\{G(z_m,z_{n+1})\} .$$

It is now obvious that there are elements $\alpha_i,\beta_i \in A$ such that

$$\{G(z_m,z_n)\} = \sum\alpha_i(\{z_m\})\,\beta_i(\{z_n\}) . \tag{2.3}$$

The mapping ε of (2.2) is defined by

$$\varepsilon(G) = \sum\alpha_i\otimes\beta_i . \tag{2.4}$$

By the Künneth formula and the well-known structure of the cohomology of K_n (see Serre [5]) the element $\sum\alpha_i\otimes\beta_i \in A\otimes A$ is uniquely determined.

The object of the present section is to establish a theorem similar to Theorem 2.1.

THEOREM 2.2. *The sequence*

$$Q \xrightarrow{\nabla} Z(Q) \xrightarrow{\varepsilon} A\otimes A \to 0$$

is exact. Hence the cohomology of Q with respect to ∇ is $A\otimes A$.

PROOF. First, let us show that $\varepsilon\circ\nabla = 0$. Let $F=\nabla G$. Then by the definition of ε we must consider $\{F(z_m,z_n)\}$. Since $F=\nabla G$, we have

$$\{F(z_m,z_n)\} = \{\delta G(z_m,z_n)+G(\delta z_m,z_n)+G(z_m,\delta z_n)\} = \{\delta G(z_m,z_n)\} = 0 .$$

Next we show that ε is onto. Let $\alpha,\beta\in A$ and let a,b be cochain operations of the first kind representing α and β respectively. It is enough to show that $\alpha\otimes\beta$ is in the image of ε. Define a cochain operation H by

$$H(x,y) = a(x)\,b(y) ;$$

then obviously $\nabla H=0$ and $\varepsilon(H)=\alpha\otimes\beta$.

Finally, we must show that if $F\in Z(Q)$ and $\varepsilon(F)=0$, then there is a cochain operation G with $\nabla G=F$. The construction of G is somewhat more involved than the first part of the proof.

First, let us note that a cochain operation H defines a system of cochains, one cochain $H(c_m,c_n)$ in $L_m\times L_n$ for each pair (m,n). The dimension of $H(c_m,c_n)$ is $m+n+i$, where i is the degree of H, and the restrictions of the cochain to each of the factors L_m and L_n are zero. On the other hand, if we have given a system of cochains satisfying the above conditions, then by naturality there is a unique cochain operation defined on all pairs of cochains such that the associated system of cochains is equal to the given system of cochains. These facts are quite obvious. In what follows we shall go back and forth between the two concepts, cochain operation and system of cochains, quite freely without mentioning so each time.

Before we give the proof, it will be useful to have a look at special examples of cochain operations. First, let us define a cochain operation $G_1 = G_1^{m,n}$ by

$$G_1(c_s, c_t) = \begin{cases} 0 & \text{for } t \neq n, \\ 0 & \text{for } t = n \text{ and } s \leqq m, \\ \alpha(c_s)\beta_1(c_n)\ldots\beta_\nu(c_n) & \text{for } t = n \text{ and } s > n, \end{cases}$$

where $\alpha, \beta_1, \ldots, \beta_\nu \in Z(\mathcal{C})$, $\nu \geqq 1$. Then we get

$$\nabla G_1(c_s, z_t) = \begin{cases} 0 & \text{for } (s,t) \neq (m,n), \\ \alpha(\delta c_s)\beta_1(z_n)\ldots\beta_\nu(z_n) & \text{for } (s,t) = (m,n). \end{cases} \tag{2.6}$$

We see that ∇G_1 is special, that is, $\nabla G_1(z_s, z_t) = 0$ for all (s,t). We therefore have that $\nabla G_1(c_s, z_t)$ is a cocycle in $(L_s, K_s) \times (K_t, *)$ and hence determines a cohomology class in that complex. From (2.6) one easily reads off the cohomology class in question.

Next, let $G_2 = G_2^{m,n}$ be defined by

$$G_2(c_s, c_t) = \begin{cases} 0 & \text{for } (s,t) \neq (m,n), \\ \alpha_1(c_m)\ldots\alpha_\mu(c_m)\beta_1(c_n)\ldots\beta_\nu(c_n) & \text{for } (s,t) = (m,n), \end{cases}$$

where $\alpha_1, \ldots, \alpha_\mu, \beta_1, \ldots, \beta_\nu \in Z(\mathcal{C})$, $\mu > 1$ and $\nu \geqq 1$.

As before, ∇G_2 is special and $\nabla G_2(c_s, z_t)$ determines the following cohomology classes in $(L_s, K_s) \times (K_t, *)$:

$$\{\nabla G_2(c_s, z_t)\} = \begin{cases} 0 & \text{for } (s,t) \neq (m,n), \\ \{\delta[\alpha_1(c_m)\ldots\alpha_\mu(c_m)]\,\beta_1(z_n)\ldots\beta_\nu(z_n) & \text{for } (s,t) = (m,n). \end{cases}$$

Using the facts that ∇ is additive and that $H^{n+i}(K_n)$ is a finite dimensional vector space and that it is generated by products $\hat{\alpha}_1(\{z_n\})\hat{\alpha}_2(\{z_n\})\ldots \hat{\alpha}_\nu(\{z_n\})$, we get the following lemma.

LEMMA 2.3. *For any system* $\{Z_{s,t}\}$ *of cohomology classes in* $H^{s+t+i}((L_s, K_s) \times (K_t, *))$ *there is an operation* G *of degree* $i-1$ *with* ∇G *special such that*

$$\{\nabla G(c_s, z_t)\} = Z_{s,t}$$

for each (s,t).

To prove the lemma we of course have to add together a number of operations G_1 and G_2. Note that although we must add an infinite number of operations, only a finite number of them are non zero in a particular dimension (m,n).

Next, we give examples of operations G with $\nabla G(c_s,z_t)=0$. Let an operation $G_3=G_3{}^n$ be defined by

$$(2.7) \qquad G_3(c_s,c_t) = \begin{cases} \alpha(c_s)\,\beta(c_t) & \text{for } t>n\,, \\ 0 & \text{for } t\leqq n\,, \end{cases}$$

where $\alpha,\beta\in Z(\mathcal{O})$. Then $\nabla G_3(c_s,z_t)=0$ for all (s,t). From (2.7) we get

$$(2.8) \qquad (\nabla G_3)(z_s,c_t) = \begin{cases} 0 & \text{for } t\neq n\,, \\ \alpha(z_s)\,\beta(\delta c_n) & \text{for } t=n\,. \end{cases}$$

Since $\nabla G_3(z_s,z_t)=0$ we see that $(\nabla G_3)(z_s,c_t)$ is a cocycle in $(K_s,*)\times(L_t,K_t)$. The cohomology class determined by $\nabla G_3(z_s,c_t)$ in $(K_s,*)\times(L_t,K_t)$ can be read off from (2.8).

Let G_4 be defined by

$$G_4(c_s,c_t) = \begin{cases} \alpha(c_s)\,\beta_1(c_n)\ldots\beta_\nu(c_n) & \text{for } t=n\,, \\ 0 & \text{for } t\neq n\,, \end{cases}$$

where $\alpha,\beta_1,\ldots,\beta_\nu\in Z(\mathcal{O})$, $\nu>1$. Then $\nabla G_4(c_s,z_t)=0$ for all (s,t), and the cohomology classes in $(K_s,*)\times(L_t,K_t)$ determined by $\nabla G_4(z_s,c_t)$ are

$$\{\nabla G_4(z_s,c_t)\} = \begin{cases} \{\alpha(z_s)\,\delta[\beta_1(c_n)\ldots\beta_\nu(c_n)]\} & \text{for } t=n\,, \\ 0 & \text{for } t\neq n\,. \end{cases}$$

As above, we collect this information in a lemma. First, let us introduce the mappings

$$\mu\colon\quad A\otimes H^*(L_t,K_t) \to H^*(K_s)\otimes H^*(L_t,K_t) \cong H^*\big((K_s,*)\times(L_t,K_t)\big)$$

which lets the first factor act on the basic class in $(K_s,*)$.

LEMMA 2.4. *Let Z_n be a sequence of elements in $A\otimes H^*(L_n,K_n)$, $\deg Z_n=n+i$, with usual grading of $A\otimes H^*(L_n,K_n)$. Then there is an operation G of degree $i-1$ such that, for each pair (s,t), $\nabla G(c_s,z_t)=0$ and*

$$\mu(Z_t) = \{\nabla G(z_s,c_t)\} \in H^*\big((K_s,*)\times(L_t,K_t)\big)\,.$$

Finally, we shall give examples of operations G with ∇G very special, i.e. with $\nabla G(z_s,c_t)=0$, $\nabla G(c_s,z_t)=0$ for all (s,t):

$$G_5(c_s,c_t) = \begin{cases} 0 & \text{for } t\neq n \text{ and for } t=n,\ s\leqq m\,, \\ \alpha(c_s)\,\delta[\beta_1(c_n)\ldots\beta_\nu(c_n)] & \text{for } t=n \text{ and } s>m\,, \end{cases}$$

$$G_6(s_s,c_t) = \begin{cases} 0 & \text{for } s\neq m \text{ and for } s=m,\ t\leqq n\,, \\ \delta[\alpha_1(c_m)\ldots\alpha_\lambda(c_m)]\,\beta(c_t) & \text{for } s=m \text{ and } t>n\,, \end{cases}$$

$$G_7(c_s,c_t) = \begin{cases} 0 & \text{for } (s,t) \neq (m,n)\,, \\ \alpha_1(c_m)\ldots\alpha_\lambda(c_m)\,\delta[\beta_1(c_n)\ldots\beta_\nu(c_n)] & \text{for } (s,t)=(m,n)\,, \end{cases}$$

where $\lambda > 1$ and $\nu > 1$ in the case of G_7. One easily sees that ∇G_j, $j=5,6,7$, is very special. It follows that $\nabla G_j(c_s,c_t)$ is a cocycle in $(L_s,K_s)\times(L_t,K_t)$. The cohomology classes determined by $\nabla G_j(c_j,c_t)$ in the three cases are $\{\nabla G_j(c_s,c_t)\}=0$ for $(s,t)\neq(m,n)$ and

$$\{\nabla G_5(c_m,c_n)\} = \{\alpha(\delta c_m)\,\delta[\beta_1(c_n)\ldots\beta_\nu(c_n)]\}\,,$$
$$\{\nabla G_6(c_m,c_n)\} = \{\delta[\alpha_1(c_m)\ldots\alpha_\lambda(c_m)]\,\beta(\delta c_n)\}\,,$$
$$\{\nabla G_7(c_m,c_n)\} = \{\delta[\alpha_1(c_m)\ldots\alpha_\lambda(c_m)]\,\delta[\beta_1(c_n)\ldots\beta_\lambda(c_n)]\}\,.$$

As before, one derives a lemma.

LEMMA 2.5. *Let $\{Z_{m,n}\}$ be a system of cohomology classes,*

$$Z_{m,n} \in H^{m+n+i}\big((L_m,K_m)\times(L_n,K_n)\big)\,.$$

Then there is an operation G of degree $i-1$ such that

$$\{\nabla G(c_m,c_n)\} = Z_{m,n} \in H^*\big((L_m,K_m)\times(L_n,K_n)\big)$$

for all (m,n).

Now, let us continue the proof of the theorem. We are given F with $\nabla F=0$ and $\varepsilon(F)=0$, and we must determine a G with $\nabla G=F$.

Since $\varepsilon(F)=0$, it follows that for each (m,n)

$$\{F(z_m,z_n)\} = 0 \in H^*(K_m\times K_n, K_m \vee K_n)\,.$$

It follows that there are cochains $\alpha_{m,n}$ in $(K_m\times K_n, K_m\vee K_n)$ with

$$\delta\alpha_{m,n} = F(z_m,z_n) \quad \forall(m,n)\,.$$

Let $\beta_{m,n}$ be a cochain in $(L_m\times L_n, L_m\vee L_n)$ which restricts to $\alpha_{m,n}$ in $(K_m\times K_n, K_m\vee K_n)$. Let H be the cochain operation associated with this system of cochains. Then the cochain operation $F-\nabla H$ is special, that is, $F(z_m,z_n)-\nabla H(z_m,z_n)=0$ for each pair (m,n). This means that to prove the theorem it is enough to show that special operations in the kernel of ε are in the image of ∇. Therefore, in what follows we shall assume that $\nabla F=0$ and that F is special (which implies that $\varepsilon(F)=0$).

For each pair (m,n) the cochain $F(c_m,z_n)$ in $L_m\times K_n$ restricts to zero in $K_m\times K_n$ (since $F(z_m,z_n)=0$). Furthermore, we have that $F(c_m,z_n)$ is a cocycle in $(L_m,K_m)\times(K_n,*)$. This means that F determines a double-sequence of cohomology classes

$$\{F(c_m,z_n)\} \in H^*\big((L_m,K_m)\times(K_n,*)\big)\,.$$

By Lemma 2.3 there is a cochain operation H with ∇H special and with

$$\{\nabla H(c_m, z_n)\} = \{F(c_m, z_n)\} \quad \forall (m,n) .$$

From this we see that besides $\nabla F = 0$ and F special, we can assume that

$$\{F(c_m, z_n)\} = 0 \in H^*\big((L_m, K_m) \times (K_n, *)\big) \quad \forall (m,n) .$$

Let, for all pairs (m,n), $\alpha_{m,n}$ be a cochain in $(L_m, K_m) \times (K_n, *)$ with

$$\delta\alpha_{m,n} = F(c_m, z_n) .$$

Let $H(c_m, c_n)$ be a cochain in $L_m \times L_n$ which restricts to $\alpha_{m,n}$ in $L_m \times K_n$ and to zero in L_n. The cochain operation H defined by this has

$$(\nabla H)(c_m, z_n) = \delta\alpha_{m,n} + (p \times 1)^{\#}(0) = F(c_m, z_n) .$$

This means that in what follows we can assume that $\nabla F = 0$ and $F(c_m, z_n) = 0$ for all pairs (m,n). We see that $F(z_m, c_n)$ is a cocycle in $(K_m, *) \times (L_n, K_n)$. Since

$$\delta F(c_m, c_n) = F(\delta c_m, c_n) = (p \times 1)^{\#} F(z_{m+1}, c_n) ,$$

it follows that

$$(\sigma \otimes 1)\{F(z_{m+1}, c_n)\} = \{F(z_m, c_n)\} \in H^*\big((K_m, *) \times (L_n, K_n)\big) .$$

It follows that there is a $Z_n \in A \otimes H^*\big((L_n, K_n)\big)$ such that

$$\mu(Z_n) = \{F(z_m, c_n)\} \in H^*\big((K_m, *) \times (L_n, K_n)\big) \quad \forall m$$

(see Lemma 2.4). By Lemma 2.4 there is a G such that

$$\{(F - \nabla G)(z_m, c_n)\} = 0 \in H^*\big((K_m, *) \times (L_n, K_n)\big) \quad \forall (m,n) .$$

Therefore, in what follows we can assume that F itself has the properties

$$\nabla F = 0, \qquad F(c_m, z_n) = 0, \qquad F(z_m, c_n) \sim 0 ,$$

in $(K_m, *) \times (L_n, K_n)$. Let $\alpha_{m,n}$ be a cochain in $(K_m, *) \times (L_n, K_n)$ with $\delta\alpha_{m,n} = F(z_m, c_n)$. We extend $\alpha_{m,n}$ to a cochain $H(c_m, c_n)$ in $(L_m, *) \times (L_n, K_n)$. Then

$$\nabla H(c_m, z_n) = 0 \qquad \text{and} \qquad \nabla H(z_m, c_n) = \delta\alpha_{m,n} = F(z_m, c_n) .$$

It follows that $F - \nabla H$ is very special. Therefore, in what follows we shall assume F itself to be very special. In that case $F(c_m, c_n)$ is a cocycle in $(L_m, K_m) \times (L_n, K_n)$. By Lemma 2.5 there is a cochain operation G such that $F - \nabla G$ is very special and $(F - \nabla G)(c_m, c_n) \sim 0$ in $(L_m, K_m) \times (L_n, K_n)$ for each pair (m,n).

Now we see that to prove Theorem 2.2 we only have to show that, if

F is very special, $\nabla F = 0$ and $F(c_m, c_n) \sim 0$ in $(L_m, K_m) \times (L_n, K_n)$ for all (m,n), then F is in the image of ∇. To see this, let $a_{m,n}$ be a cochain in $(L_m, K_m) \times (L_n, K_n)$ with $\delta a_{m,n} = F(c_m, c_n)$. Define $G(c_m, c_n) = a_{m,n}$. Then $\nabla G = F$. This proves the theorem.

3. Some lemmas.

As earlier, let A denote the mod 2 Steenrod algebra. It is well known that A is a Hopf algebra with diagonal $\psi\colon A \to A \otimes A$. If $\hat{a} \in A$ and $\psi(\hat{a}) = \sum \hat{a}' \otimes \hat{a}''$, then for each pair u, v of cohomology classes

$$\hat{a}(uv) = \sum \hat{a}'(u)\, \hat{a}''(v)\,. \tag{3.1}$$

The Cartan formula for primary cohomology operations can then be expressed by

$$\psi(Sq^k) = \sum Sq^i \otimes Sq^j, \qquad i+j = k\,.$$

In what follows we shall be working both with cohomology operations and with cochain operations. We shall use letters a, b, c etc. to denote cochain operations in one variable and commuting with δ. If a is a cochain operation, then $\hat{a}$ is the corresponding cohomology operation.

First, let us consider the equation (3.1) on the level of cochain operations.

Let a, a' and a'' be cochain operations representing $\hat{a}$, $\hat{a}'$ and $\hat{a}''$ respectively. Consider the cochain operation B of the second kind defined by

$$\begin{aligned} B(x,y) = a(xy) + \sum a'(x)\, a''(y) + d(a; \delta x y, x \delta y) + \\ + n(x)\, d(a; x\delta y, x\delta y)\,, \end{aligned} \tag{3.2}$$

where $n(x)$ is one of the functions $n(x) = \deg(x)$ or $n(x) = \deg(x) + 1$, and d is a cochain operation (of the first kind) in two variables defined in [3]. The essential property of d is that it gives the deviation from additivity of the cochain operation a:

$$a(u+v) + a(u) + a(v) = \delta d(a; u,v) + d(a; \delta u, \delta v)\,.$$

For t variables $x_1,, \ldots, x_t$ we have the property

$$d(a; x_1, \ldots, x_t) + d(a; \delta x_1, \ldots, \delta x_t) = a(x_1 + \ldots + x_t) + \sum_{j=1} a(x_j)\,. \tag{3.3}$$

The existence of d can be derived from the exact sequence in Theorem 2.1. The right hand side of (3.3) defines an element of $Z(\mathcal{O}^{(t)})$ in the kernel of ε. Hence there is $d'(a) \in \mathcal{O}^{(t)}$ with

$$(\Delta d'(a))(x_1, \ldots, x_t) = a(\sum x_j) + \sum_j a(x_j)\,.$$

Putting

$$d(a,x_1,\ldots,x_t) = d'(a)(x_1,\ldots,x_t) + \sum_j d'(a)(0,\ldots,0,x_j,0,\ldots,0)\ ,$$

d satisfies (3.3) and has the property

$$d(a;0,\ldots,0,x_j,\ldots,0) = 0\ .$$

This property we shall make use of later in this paper.

It is easy to see that B is a cochain operation of the second kind and that $\nabla B=0$. We can therefore ask about the image of B under the mapping $\varepsilon\colon Z(Q) \to A\otimes A$, discussed in Section 2. We can determine $\varepsilon(B)$ by evaluating B on pairs (x,y) of cocycles. Since by (3.1)

$$B(x,y) = a(xy) + \sum a'(x)\, a''(y) \sim 0\ ,$$

we see that $\varepsilon(B)=0$. By Theorem 2.2 there is a cochain operation $T \in Q$ with

$$\nabla T = B\ .$$

We formulate this in the following lemma.

LEMMA 3.1. *Let $\hat{a} \in A$ and let $\psi(\hat{a})=\sum \hat{a}'\otimes\hat{a}''$. Let a, a' and a'' be cochain operations representing $\hat{a}$, $\hat{a}'$ and $\hat{a}''$ respectively. Then there is a cochain operation T of the second kind with*

$$(\nabla T)(x,y) = a(xy) + \sum a'(x)a''(y) + d(a;\delta x y, x\delta y) + n(x)\, d(a; x\delta y, x\delta y)$$

for each pair (x,y) of cochains.

For the next lemma we need the following setup which shall also be used in Section 4 of this paper. This algebraic machinery was used by Adams [1] in his description of secondary cohomology operations.

Let C_0 be a free (graded) A-module on one generator c_0, and let C_1 be a free A-module on a finite set of generators $\{c_1{}^i \mid i \in I\}$. Let a mapping

$$d\colon\ C_1 \to C_0 \tag{3.4}$$

of graded A-modules be given. We recall that a secondary cohomology operation is determined by an element of the kernel of d.

Let us assume that besides the mapping d of (3.4) we have in the same fashion two other mappings

$$d'\colon\ C_1' \to C_0', \qquad d''\colon\ C_1'' \to C_0'' \tag{3.5}$$

such that $\deg c_0 = \deg c_0' + \deg c_0''$. The tensor product $C_0'\otimes C_0''$ (over Z_2) is an $A\otimes A$-module. Using the diagonal mapping in A, we obtain an induced A-module structure in $C'_0 \otimes C''_0$. There is a unique mapping $\psi_0 \colon C_0 \to C'_0 \otimes C''_0$ with the property $\psi(c_0) = c'_0 \otimes c''_0$. Assume that there is a mapping

$$\psi_1 \colon \; C_1 \to C'_0 \otimes C''_1 + C'_1 \otimes C''_0$$

such that the following diagram is commutative:

$$(3.6) \qquad \begin{array}{ccc} C_1 & \xrightarrow{\quad d \quad} & C_0 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} \\ C'_0 \otimes C''_1 \oplus C'_1 \otimes C''_0 & \xrightarrow{d\otimes} & C'_0 \otimes C''_0 \, , \end{array}$$

where $d\otimes = 1 \otimes d'' + d' \otimes 1$. Let us consider an element $z \in \operatorname{Ker}(d)$ such that

$$(3.7) \qquad \psi_1(z) = \sum \hat{\alpha}'_m c'_0 \otimes z''_m + \sum z'_n \otimes \hat{\alpha}''_n c''_o \, ,$$

where $z'_n \in \operatorname{Ker}(d')$, $z''_m \in \operatorname{Ker}(d'')$ and $\alpha'_m, \alpha''_n \in A$. Let

$$z = \sum \hat{a}_i c_1{}^i, \qquad \hat{a}_i \in A \, ,$$

and let a_i be a cochain operation representing $\hat{a}_i$. Also, let

$$d(c_1{}^i) = \hat{b}_i c_0, \qquad \hat{b}_i = \{b_i\} \in A \, ,$$

where $b_i \in Z(\mathcal{O})$. Then (by Theorem 2.1) there is a cochain operation θ with

$$\Delta\theta = \sum a_i b_i = r \, .$$

In a similar fashion we put

$$d'(c_1'^j) = \hat{b}'_j c'_o, \qquad d''(c_1''^k) = \hat{b}''_k c''_0 \, ,$$

where $\{c_1'^j \mid j \in J\}$ and $\{c_1''^k \mid k \in K\}$ are generators for C'_1 and C''_1 respectively, and $\hat{b}'_j = \{b'_j\} \in A$, $\hat{b}''_k = \{b''_k\} \in A$. Also,

$$z'_n = \sum \hat{a}'_{nj} c_1'^j, \qquad z''_m = \sum \hat{a}''_{mk} c_1''^k \, .$$

Corresponding to these cycles we have

$$\Delta\theta'_n = \sum a'_{nj} b'_j = r'_n, \qquad \Delta\theta''_m = \sum a''_{mk} b''_k = r''_m \, .$$

Finally, we need notation for $\psi_1 \colon C_1 \to C'_0 \otimes C''_1 \oplus C'_1 \otimes C''_0$. Let us put

$$\psi_1(c_1{}^i) = \sum \hat{e}'_{ik} c'_0 \otimes \hat{f}''_{ik} c_1''^k + \sum \hat{f}'_{ij} c_1'^j \otimes \hat{e}''_{ij} c''_0 \, .$$

Let us consider the cochain operation H defined by

$$(3.8) \qquad H(x,y) = \theta(\delta x y + x \delta y) + \theta(\delta x y) + \theta(x \delta y) + \\ + d(r; \delta x y, x \delta y) + n(x)\, d(r; x \delta y, x \delta y) \, .$$

Then $\nabla H = 0$ and $\varepsilon(H) = 0$ as is easily seen. Hence there is a cochain operation $D = D(\theta)$ with

$$\nabla D = H . \tag{3.9}$$

Let us consider a cochain operation E defined by

$$E(x,y) = \theta(xy) + \sum \theta'_n(x)\, \alpha''_n(y) + \sum \alpha'_m(x)\, \theta''_m(y) + \\ + T(r)(x,y) + D(\theta)(x,y) , \tag{3.10}$$

where $T(r)$ is a cochain operation given by Lemma 3.1 with a replaced by r, and

$$\psi(\hat{r}) = \sum \hat{r}'_n \otimes \hat{\alpha}''_n + \sum \hat{\alpha}'_m \otimes \hat{r}''_m$$

($\hat{r}$, $\hat{r}'_n$ and $\hat{r}''_m$ are equal to zero, but the cochain operations r etc. are usually not zero). Since

$$\nabla T(r)(x,y) = r(xy) + \sum r'_n(x)\, \alpha''_n(y) + \sum \alpha'_m(x)\, r''_m(y) + \\ + d(r;\, \delta xy, x\delta y) + n(x) d(r;\, x\delta y, x\delta y) , \tag{3.11}$$

it follows that $\nabla E = 0$. Hence there are cochain operations β'_j and β''_j in $Z(\mathcal{O})$ such that

$$\varepsilon(E) = \sum \hat{\beta}'_j \otimes \hat{\beta}''_j .$$

Applying Theorem 2.3 we get

LEMMA 3.2. *Let notation be as above. There are cochain operations β'_j and β''_j in $Z(\mathcal{O})$ such that for each pair (x,y) of cocycles*

$$\theta(xy) + \sum \theta'_n(x)\alpha''_n(y) + \sum \alpha'_m(x)\theta''_m(y) + D(\theta)(x,y) + T(r)(x,y) \\ \sim \sum \beta'_j(x)\, \beta''_j(y) .$$

This lemma is designed for proving the Cartan formula for secondary cohomology operations. This will be done in Section 4.

4. The Cartan formula.

In this section we shall prove a Cartan formula for secondary operations. This formula gives an expansion of $\Phi(xy)$, where Φ is a secondary operation and x and y are cohomology classes, in a sum of terms $\Phi'(x)\, \Phi''(y)$, where Φ' and Φ'' are cohomology operations. This formula was first treated by Adams [1]. He proved the existence of such an expansion but did not give the actual form of the terms Φ' and Φ''. Later Adem [2] proved that for certain Φ we have $\Phi(xy) = \Phi(x)\, y + x\, \Phi(y)$.

THEOREM 4.1. *Let d, d' and d'' be A-module mappings as described in* (3.4) *and* (3.5) *and, as described by* (3.6), *let a commutative diagram*

$$\begin{array}{ccc} C_1 & \xrightarrow{\quad d \quad} & C_0 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} \\ C'_0\otimes C''_1 \oplus C'_1\otimes C''_0 & \xrightarrow{d^{\otimes}} & C'_0\otimes C''_0 \end{array}$$

be given. Let $z \in \mathrm{Ker}(d)$ *such that*

$$\psi_1(z) = \sum \hat{\alpha}'_m c'_0 \oplus z''_m + \sum z'_n \oplus \hat{\alpha}''_n c''_0$$

with $z'_n \in \mathrm{Ker}\, d'$ *and* $z''_m \in \mathrm{Ker}\, d''$. *Then there are primary operations* $\hat{\beta}'_k$, $\hat{\beta}''_k$ *such that modulo the total indeterminacy*

$$\Phi(\hat{x}\,\hat{y}) = \sum \Phi'_n(\hat{x})\hat{\alpha}''_n(\hat{y}) + \sum \hat{\alpha}'_m(\hat{x})\Phi''_m(\hat{y}) + \sum \hat{\beta}'_k(\hat{x})\hat{\beta}''_k(\hat{y}),$$

where Φ, Φ'_n *and* Φ''_m *are secondary operations associated with* z, z'_n *and* z''_m *respectively, and where* $\hat{x}$ *and* $\hat{y}$ *are arbitrary cohomology classes satisfying that* d'*-operations are defined on* $\hat{x}$ *and* d''*-operations are defined on* $\hat{y}$.

REMARK. Before the proof, let us remark, that in [3] we defined secondary operations associated with relations $\sum \alpha_\nu a_\nu + b = 0$ in A containing an unfactorized term b. These operations are only defined in dimensions less than the excess of b. Hence they are not stable in the usual sense. They are, however, very close to being stable. In the statement of Theorem 4.1 we have only considered stable operations. Minor additions to the proof of Theorem 4.1, however, give a Cartan formula for the almost stable operations mentioned. I believe that in certain applications it will be useful (if not necessary) to take the almost stable operations into consideration.

We hope that the unknown stable primary operations $\hat{\beta}'_k$ and $\hat{\beta}''_k$ can be determined by methods similar to the ones used in this paper.

A simple example where the theorem applies is as follows. Let

$$p_i = Sq^{(0,0,\ldots,1)}, \qquad i-1 \text{ zeroes},$$

be an element in the Milnor basis for the Steenrod algebra. Then

$$p_i p_j + p_j p_i = 0,$$
$$\Phi(p_i) = 1\otimes p_i + p_i \otimes 1.$$

Let $i \neq j$. Let C_1 be generated by $c_1{}^i$ and $c_1{}^j$, and C_0 by c_0. The mapping

$$d\colon\; C_1 \to C_0$$

is given by $d(c_1{}^i) = p_i c_0$ and $d(c_1{}^j) = p_j c_0$. Then

$$\psi_1\colon\; C_1 \to C_0\otimes C_1 \oplus C_1 \otimes C_0$$

is given by

$$\psi_1(c_1{}^k) = c_0 \otimes c_1{}^k + c_1{}^k \otimes c_0, \qquad k = i \text{ or } j\ .$$

The element $z = p_i c_1{}^j + p_j c_1{}^i$ determines the operation Φ. Since

$$\psi_1(z) = z \otimes c_0 + c_0 \otimes z$$

the theorem applies.

Proof of Theorem 4.1. We shall use the algebraic setup from Section 3. If $\hat{x} = \{x\}$ and $\hat{y} = \{y\}$, then $b_j'(x) \sim 0$ and $b_k''(y) \sim 0$. Let us choose bounding cochains

$$\delta w_j' = b_j'(x), \qquad \delta w_k'' = b_k''(y)\ .$$

Since $\psi_1\colon C_1 \to C_0' \otimes C_1'' \oplus C_1' \otimes C_0''$ is given by

$$\psi_1(c_1{}^i) = \sum \hat{e}_{ik}' c_0' \otimes \hat{f}_{ik}'' c_1''^k + \sum \hat{f}_{ij}' c_1'^j \otimes \hat{e}_{ij}'' c_0''\ ,$$

we get

$$\hat{b}_i(\hat{x}\hat{y}) = \sum \hat{e}_{ik}'(\hat{x}) \hat{f}_{ik}'' \hat{b}_k''(\hat{y}) + \sum \hat{f}_{ij}' \hat{b}_j'(\hat{x})\, \hat{e}_{ij}''(\hat{y}) = 0\ .$$

On the cochain level we get

$$\delta w_i = b_i(xy)$$

with w_i given by

$$w_i = \sum e_{ik}'(x) f_{ik}''(w_k'') + \sum f_{ij}'(w_j')\, e_{ij}''(y) + T_i(x,y)\ , \tag{4.1}$$

where T_i is a cochain operation given by Lemma 3.1,

$$\nabla T_i(u,v) = b_i(uv) + \sum e_{ik}'(u) f_{ik}''\, b_k''(v) + \sum f_{ij}'\, b_j'(u)\, e_{ij}''(v) + \\ + d(b_i;\, \delta u\, v, u\, \delta v) + n(u) d(b_i;\, u\, \delta v, u\, \delta v)\ . \tag{4.2}$$

Since $\Delta\theta = \sum a_i b_i$, a cocycle representing $\Phi(\hat{x}\hat{y})$ is given by

$$\theta(xy) + \sum a_i(w_i)\ .$$

To prove the theorem we must examine the expression

$$Z = \theta(xy) + \sum a_i(w_i) + \sum_n [\theta_n'(x) + \sum a_{nj}'(w_j')]\, \alpha_n''(y) + \\ + \sum_m \alpha_m'(x)\, [\theta_m''(y) + \sum a_{mk}''(w_k')]\ .$$

By Lemma 3.2

$$Z \sim \sum a_i(w_i) + \sum a_{nj}'(w_j')\, \alpha_n''(y) + \sum \alpha_m'(x)\, a_{mk}''(w_k'') + \\ + D(\theta)(x,y) + T(r)(x,y) + \sum \beta'(x) \beta''(y)\ . \tag{4.3}$$

By the expression (4.1) for w_i we get

$$(4.4)\qquad a_i(w_i) \sim \sum a_i^{(1)} e'_{ik}(x)\, a_i^{(2)} f''_{ik}(w''_k) + \\ + \sum a_i^{(1)} f'_{ij}(w'_j)\, a_i^{(2)} e''_{ij}(y) + a_i T_i(x,y) + D_i(x,y) + \\ + \sum S_i\big(e'_{ik}(x), f''_{ik} b''_k(y)\big) + \sum S_i\big(f'_{ij} b'_j(x), e''_{ij}(y)\big)\,,$$

where $\Psi(\hat{a}_i) = \sum \hat{a}_i^{(1)} \otimes \hat{a}_i^{(2)}$ and S_i the corresponding cochain operation given by Lemma 3.1,

$$(4.5)\qquad \nabla S_i(u,v) = a_i(uv) - \sum a_i^{(1)}(u)\, a_i^{(2)}(v) + d(a_i;\, \delta u v, u \delta v) + \\ + n(u) d(a_i;\, u\delta v, u \delta v)\,.$$

Also,

$$D_i(x,y) = d\big(a_i;\, \ldots, e'_{ik}(x) f''_{ik} b''_k(y), \ldots, f'_{ij} b'_j(x)\, e''_{ij}(y), \ldots \\ \ldots, b_i(xy) + \sum e'_{ik}(x) f''_{ik} b''_k(y) + \sum f'_{ij} b'_j(x)\, e''_{ij}(y)\big)\,,$$

where $d(a_i; \ldots)$ is deviation from additivity of a_i as defined in Section 3.

Next, we shall compare some of the terms in (4.3) and (4.4). It is clear that

$$\sum \hat{a}_i^{(1)} \hat{e}'_{ik} c'_0 \otimes \hat{a}_i^{(2)} \hat{f}''_{ik} c''^k_1 = \sum \hat{\alpha}'_m c'_0 \otimes \hat{a}''_{mk} c''^k_1\,.$$

If we keep k fixed, this equation is still true. Hence, for each k there is a cochain operation U_k such that

$$(4.6)\qquad (\nabla U_k)(u,v) = \sum a_i^{(1)} e'_{ik}(u)\, a_i^{(2)} f''_{ik}(v) - \sum \alpha'_m(u)\, a''_{mk}(v), \qquad k \text{ fixed}\,,$$

where (u,v) is an arbitrary pair of cochains. Similarly,

$$(4.7)\qquad (\nabla V_j)(u,v) = \sum a_i^{(1)} f'_{ij}(u)\, a_i^{(2)} e''_{ij}(v) - \sum a'_{nj}(u)\, \alpha''_n(v), \qquad j \text{ fixed}\,.$$

This and (4.4) simplify (4.3). We get

$$(4.8)\qquad Z \sim K(x,y)\,,$$

where

$$(4.9)\qquad K(x,y) = \sum U_k\big(x, b''_k(y)\big) + \sum V_j\big(b'_j(x), y\big) + \sum a_i T_i(x,y) + \\ + D(\theta)(x,y) + T(r)(x,y) + \sum S_i\big(e'_{ik}(x), f''_{ik} b''_k(y)\big) + \\ + \sum S_i\big(f'_{ij} b'_j(x), e''_{ij}(y)\big) + \sum D_i(x,y) + \sum \beta'(x) \beta''(y) + \\ + \sum d\big(a_i;\, \delta T_i(x,y), T_i(\delta x, y), T_i(x, \delta y)\big) + \\ + d\big(a_i;\, \nabla T_i(x,y), d(b_i;\, \delta x y, x \delta y), n(x) d(b_i;\, x\delta y, x \delta y)\big).$$

The terms $D(\theta)$ and $T(r)$ are defined in (3.9) and (3.11), respectively. The last two terms we have added for convenience. They are, of course, zero when x and y are cocycles.

It will be convenient to consider x and y as variable cochains. Then the expression (4.9) defines a cochain operation K in Q. We shall examine ∇K. A straightforward computation shows that ∇K is a sum of two terms: a very special operation (i.e. an operation which is zero on (x,y)

if $\delta x = 0$ or $\delta y = 0$) and the following special operation (i.e. zero on (x,y) if $\delta x = 0$ and $\delta y = 0$):

$$n(x)\,[\sum a_i \varkappa(b_i)(x\,\delta y) + \sum \varkappa(a_i)\big(e'_{ik}(x)\,f''_{ik} b''_k(\delta y)\big) + \sum \varkappa(a_i)\,(f'_{ij}\,b'_j(x)\,e''_i(\delta y))]\,,$$

where we have denoted the deviation from additivity $d(a; z,z)$ by $\varkappa(a)(z)$ as it was done in [3], and $n(x) = \deg(x)$. It is now very easy to complete the proof of the theorem. Before doing so we shall, however, give a few relevant remarks about $\varkappa$. These things are proved in the following section. The reason for treating $\varkappa$ in a special section is the feeling that $\varkappa$ is of general interest not only in usual cohomology theory but also in connection with general cohomology theories based on (general) cochain functors.

The defining property of the deviation from additivity at once gives that

$$\delta\varkappa(a)(z) = \varkappa(a)(\delta z)\,,$$

or $\varkappa(a) \in Z(\mathscr{O})$. We therefore get an element $\varepsilon(\varkappa(a)) \in A$. It turns out that $\varepsilon(\varkappa(a))$ depends only on $\varepsilon(a)$ and not on the choices of representatives a for $\varepsilon(a)$ and derivation $d(a; -,-)$. Hence we have a mapping $\hat{\varkappa}: A \to A$,

$$\hat{\varkappa}(\hat{a}) = \varepsilon(\varkappa(a)) \in A\,,$$

where $\hat{a} = \varepsilon(a)$. Furthermore, it follows that $\hat{\varkappa}$ is a derivation in A of degree -1

$$\hat{\varkappa}(\hat{a}\hat{b}) = \hat{\varkappa}(\hat{a})\,\hat{b} + \hat{a}\,\hat{\varkappa}(\hat{b})\,,$$

and that

$$\hat{\varkappa}(Sq^i) = Sq^{i-1}, \qquad i = 0,1,\ldots\,.$$

This last property is not needed in the proof of the theorem, but is added here just to show that $\hat{\varkappa}$ is not zero.

To complete the proof of Theorem 4.1 we must determine K on a pair of cocycles. Since ∇K is special, that is, $(\nabla K)(x,y) = 0$ if $\delta x = 0$ and $\delta y = 0$, it follows that $K(x,y)$ is a cocycle if $\delta x = 0$ and $\delta y = 0$. The $K(z_s,z_t)$ therefore determines an element in $H^*(K(Z_2,s) \times K(Z_2,t))$ for each pair (s,t). As in Section 2 when we defined the mapping ε on very special operations, we shall here compare the classes $\{K(z_s,z_t)\}$ for different pairs (s,t). This, however, is quite easy. First, let us compare $\{K(z_s,z_t)\}$ and $\{K(z_{s+1},z_t)\}$. Since $(\nabla K)(c_s,z_t) = 0$ in $L_s \times K_t$, we have

$$\delta K(c_s,z_t) = (p \times 1)^{\#} K(z_{s+1},z_t)\,,$$

where p is the projection $L(Z_2,s) \to K(Z_2,s+1)$. It follows that

$$\{K(z_s,z_t)\} = (\sigma \otimes 1)\{K(z_{s+1},z_t)\}\,,$$

where σ is the cohomology suspension $H^n(X) \to H^{n-1}(\Omega X)$. To compare $\{K(z_s, z_{t+1})\}$ and $\{K(z_s, z_{t+1})\}$ we note that

$$(\nabla K)(z_s, c_t) = s[\sum a_i \varkappa(b_i)(z_s \delta c_t) + \\ + \sum \varkappa(a_i)(e'_{ik}(z_s) f''_{ik} b''_k(\delta c_t)) + \sum \varkappa(a_i)(f'_{ij} b'_j(z_s)\, e''_i(\delta c_t))]$$

or

$$\delta[K(z_s, c_t) + s[\sum a_i \varkappa(b_i)(z_s c_t) + \sum \varkappa(a_i)(e'_{ik}(z_s) f''_{ik} b''_k(c_t)) + \\ + \sum \varkappa(a_i)(f'_{ij} b'_j(z_s)\, (e''_i(c_t))] = (1 \times p)^{\#} K(z_s, z_{t+1}) .$$

It follows that

$$\begin{aligned}(1 \otimes \sigma)\{K(z_s, z_{t+1})\} &= \{K(z_s, z_t)\} + s[\sum \hat{a}_i \hat{\varkappa}(b_i)\{z_s z_t\} + \\ &\quad + \sum \hat{\varkappa}(\hat{a}_i)(\hat{e}'_{ik})(\{z_s\}) \hat{f}''_{ik} \hat{b}''_k(\{z_t\}) + \sum \hat{\varkappa}(\hat{a}_i)(\hat{f}'_{ij} \hat{b}'_j(\{z_s\}) \hat{e}''_i(\{z_t\})] \\ &= \{K(z_s, z_t)\} + s[\sum \hat{a}_i \hat{\varkappa}(\hat{b}_i)(\{z_s z_t\}) + \sum \hat{\varkappa}(\hat{a}_i)(\hat{b}_i \{z_s z_t\})] \\ &= \{K(z_s, z_t)\} + s\, \hat{\varkappa}(\sum \hat{a}_i \hat{b}_i)(\{z_s z_t\}) \\ &= \{K(z_s, z_t)\} .\end{aligned}$$

This follows from the properties of $\varkappa$ and the fact that $\sum \hat{a}_i \hat{b}_i = 0$. Now, of course, it follows easily that there are cochain operations β'_k and β''_k in $Z(\mathcal{O})$ such that for each pair (x, y) of cocycles $K(x, y) \sim \sum \beta'_k(x)\, \beta''_k(y)$. This completes the proof of the theorem.

5. A derivation in the Steenrod algebra.

As in Section 4, let $a \in Z(\mathcal{O})$ and $d(a; -, -)$ be an operation giving the deviation from additivity of a, satisfying $d(a; x, 0) = 0$ and $d(a; 0, y) = 0$. Then, as before, a cochain operation $\varkappa(a, d) \in Z(\mathcal{O})$ is defined by $\varkappa(a, d)(x) = d(a; x, x)$. We shall show that $\varepsilon(\varkappa(a, d)) \in A$ depends only on $\varepsilon(a)$. Let $d'(a; x, y)$ be another deviation from additivity associated with a. Then

$$D(x, y) = d(a; x, y) - d'(a; x, y)$$

belongs to $Z(\mathcal{O}^{(2)})$. Since $D(x, 0) = D(0, y) = 0$, it follows that $\varepsilon(D) = 0 \in A \oplus A$. Therefore (Theorem 2.1) there is a $B \in \theta^{(2)}$ with $\Delta B = D$. Since

$$\delta B(x, x) + B(\delta x, \delta x) = \varkappa(a, d)(x) - \varkappa(a, d')(x) ,$$

it follows that $\varepsilon(\varkappa(a, d)) = \varepsilon(\varkappa(a, d'))$. Because of that we shall use the notation $\varkappa(a) = \varkappa(a, d)$. Now, let $a, b \in Z(\mathcal{O})$ with $\varepsilon(a) = \varepsilon(b)$ and let $d(a; -, -)$ be associated with a. Since $\varepsilon(a - b) = 0$, there is an operation $\theta \in \mathcal{O}$ with $\Delta\theta = a - b$. Let us define

$$d(b; x, y) = d(a; x, y) + \theta(x + y) - \theta(x) - \theta(y) .$$

Then it is easy to see that $d(b;\ -,-)$ gives the deviation from additivity for b. Using this, we get

$$\varkappa(b)(x) = \varkappa(a)(x)$$

and therefore so much more $\varepsilon(\varkappa(b)) = \varepsilon(\varkappa(a))$. We define $\hat{\varkappa}(\hat{a}) = \varepsilon(\varkappa(a))$, where $\varepsilon(a) = \hat{a} \in A$. It is obvious that $\hat{\varkappa}$ preserves addition in A and that it decreases degrees by one. To see that $\hat{\varkappa}$ is a derivation, we proceed as follows: Let $a,b \in Z(\mathcal{O})$ and let $d(a;\ -,-)$ and $d(b;\ -,-)$ be associated with these; then we define a deviation from additivity for ba by

$$\begin{aligned} d(ba;x,y) = {} & bd(a;x,y) + d(b;a(x),a(y)) + \\ & + d(b;\delta d(a;x,y), d(a;\delta x,\delta y)) + \\ & + d(b;a(x+y), a(x)+a(y)) + \\ & + d(b;a(x),a(x)) + d(b;a(y),a(y)) . \end{aligned}$$

One easily checks that $d(ba;\ -,-)$ has the desired properties. Putting $x=y$ and $\delta x = 0$, we get

$$\varkappa(ba)(x) = b(\varkappa(a)(x)) + \varkappa(b)(a(x))$$

which shows that $\hat{\varkappa}$ is a derivation.

To show that $\hat{\varkappa}(Sq^i) = Sq^{i-1}$, we simply refer to the explicit formula for $d(sq^i, -, -)$ given on p. 60 in [3]:

$$d(sq^i;x,y) = \varphi(e_{n-i+1} \otimes xy + e_{n-i+2} \otimes \delta x\, y), \qquad \deg x = n .$$

If $\delta x = 0$, this shows that

$$\hat{\varkappa}(Sq^i)(\{x\}) = \{\varphi(e_{n-i+1} \otimes x^2)\} = Sq^{i-1}(\{x\}) .$$

It is curious that one can obtain information about the multiplicative structure in the Steenrod algebra in this way. It is, for instance, possible to derive all the Adem relations from the relations $Sq^{2k-1}Sq^k = 0$ by iterated application of $\hat{\varkappa}$; for example

$$Sq^3 Sq^2 = 0 \Rightarrow \hat{\varkappa}(Sq^3 Sq^2) = Sq^2 Sq^2 + Sq^3 Sq^1 = 0 .$$

Using the structure of the dual Hopf algebra A^* of A we can obtain $\hat{\varkappa}$ in a different way. Let $\xi_1 \in A^*$ be the non-zero element of degree one. Multiplication with ξ_1 defines a mapping of A^* into itself of degree one. Since ξ_1 is primitive, it follows that the dual of this mapping is a derivation. This mapping is of degree -1 and it sends Sq^i into Sq^{i-1}. It is therefore equal to $\hat{\varkappa}$.

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