MATH. SCAND. 16 (1965), 116–120

ON ENTIRE FUNCTIONS ALMOST PERIODIC IN SEVERAL DIRECTIONS

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1.

An example by R. Petersen [2] proved the existence of entire functions almost periodic in two directions. H. Tornehave [3] proved by further examples that the almost periodicity in the two directions may be of very different type. It was stated without proof in the same paper that an entire function might be almost periodic in three directions. In all these cases almost periodicity in a certain direction means almost periodicity in the whole plane in this direction.

In the present paper we shall present a method for construction of entire functions almost periodic in an arbitrary finite number of directions and even in a denumerable everywhere dense set of direction. In a recent paper P. Asisoff [1] proved that, on the other hand, the set of directions of non-almost-periodicity must be everywhere dense and, for a function of finite order, there exists only a finite set of directions of almost periodicity.

2.

Let ω_1 and ω_2 be complex numbers such that $\omega_1^{-1}\omega_2$ is not real. Let $(p_1,q_1),(p_2,q_2),\ldots$ be a sequence of mutually different pairs of relatively prime integers, the numbers p_ν being non negative. We shall assume that $(p_1,q_1)=(1,0)$ and $(p_2,q_2)=(0,1)$. Under these assumptions we shall prove the following statement:

There exists an entire function almost periodic in the complex plane in each of the directions determined by $p_\nu\omega_1+q_\nu\omega_2$, $\nu=1,2,\ldots$.

The existence will thus be proved for every bundle of directions with the property that every subset consisting of 4 directions has a rational cross-ratio. It is an open question yet, whether there exists an entire function almost periodic in 4 directions with irrational cross ratio.

Received March 13, 1964.

It will be convenient first to prove the theorem for a finite sequence $(p_1,q_1),\ldots,(p_n,q_n)$ and afterwards describe the changes in the procedure necessary in the general case.

3.

The set Ω of all numbers $p\omega_1+q\omega_2$ is a subgroup of the additive group of complex numbers. Let m be a natural number. The set $m\Omega$ is then a subgroup of Ω. We shall call two complex numbers equivalent with respect to $m\Omega$ if they belong to the same coset of $m\Omega$ considered as a subgroup of the additive group of complex numbers. If A is a set of complex numbers, the set $A+m\Omega$ is the set of numbers equivalent to elements of A. In particular, if A is a straight line, the set $A+m\Omega$ will be a system of straight lines parallel to A. If A has one of the directions $p_\nu\omega_1+q_\nu\omega_2$ the set of lines $A+m\Omega$ will be discrete.

The parallelogram

$$P_m = \{m(t_1\omega_1+t_2\omega_2) \mid 0 \leqq t_1 < 1,\ 0 \leqq t_2 < 1\}$$

is divided by the systems of parallels equivalent with respect to $m\Omega$ to the straight lines through 0 in the directions $p_\nu\omega_1+q_\nu\omega_2$, $\nu=1,\ldots,n$, into a finite number of convex polygons. Each parallelogram equivalent to P_m with respect to $m\Omega$ is subdivided by the same system of lines, the subdivision being congruent to the subdivision of P_m. If we replace m by another natural number m' we obtain a subdivision of $P_{m'}$ homothetic to the subdivision of P_m. These phenomena are illustrated in Fig. 1, where we have chosen $\omega_2=i\omega_1$ and the directions $(1,0)$, $(0,1)$, $(3,2)$, $(1,3)$ and $(1,-2)$. The small squares correspond to $m=1$ and are subdivided by all the lines. The thin lines do not belong to the subdivision of the large square corresponding to $m=2$.

4.

As in the paper [2] our entire function will be constructed as the limit of a sequence of elliptic functions. We shall now introduce the particular class of elliptic functions which will be used for our present purpose.

Let $a_{m1},\ldots,a_{mk}$ be the geometric centers of gravity of the polygons of the subdivision of P_m. Let $\boldsymbol{v}=(v_{\varkappa\lambda})$ be a k by l complex matrix. We introduce

$$Q_\varkappa(\boldsymbol{v};z) = v_{\varkappa 1}z^{-3}+\ldots+v_{\varkappa l}z^{-l-2}\,.$$

In [3, § 2] it was proved that we have for $\varrho>0$ an estimate

$$|Q_\varkappa(\boldsymbol{v};z)| \leqq K|z|^{-3}$$

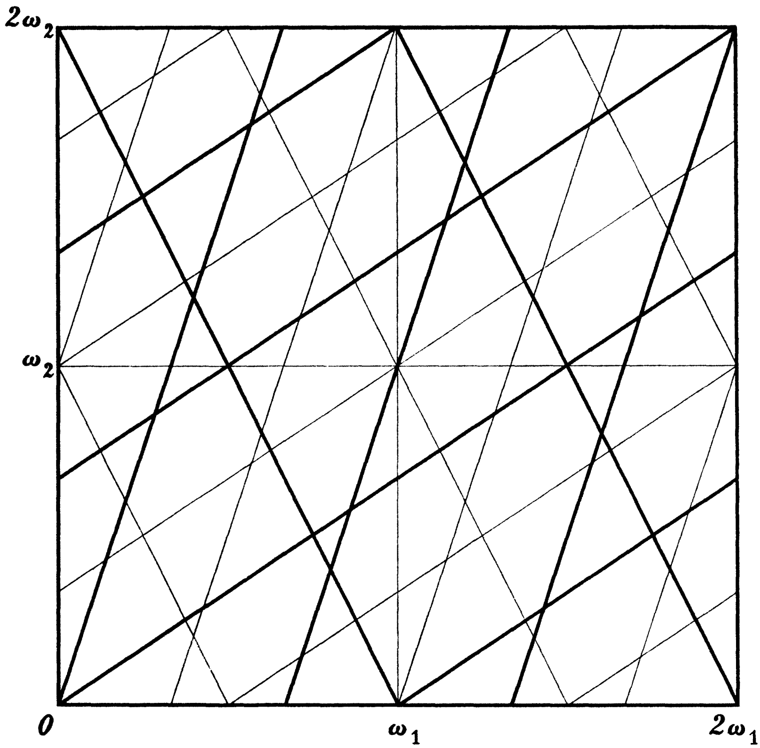

Fig. 1

valid for $|z| \geqq \varrho$. It follows that the function

$$(1) \qquad F_m(v; z) = \sum_{\varkappa=1}^{k} \sum_{\omega \in m\Omega} Q_\varkappa(v; z - a_{m\varkappa} - \omega)$$

is an elliptic function with $m\Omega$ as its group of periods.

Let δ denote the smallest distance between the set of points $a_{1\varkappa}$, $\varkappa = 1, \ldots, k$, and the union S of the straight lines through 0 with directions $p_\nu \omega_1 + q_\nu \omega_2$, $\nu = 1, \ldots, n$. The number δ will be positive. Let S_m denote the set of all points with distance $< m\delta$ from the set S. The set S_m is the union of n strips, and $F_m(v; z)$ is holomorphic and periodic in each of these strips.

We shall rewrite (1) extending the summation with respect to $\varkappa$ over the $4k$ points

$$a_{m\varkappa}, \quad a_{m\varkappa} + m\omega_1, \quad a_{m\varkappa} + m\omega_2, \quad a_{m\varkappa} + m\omega_1 + m\omega_2, \qquad \varkappa = 1, \ldots, k\,.$$

This requires an obvious change of notation, which we indicate by replacing $a'_{m\varkappa}$ by $a'_{m\varkappa}$ and v by v':

$$F_m(\boldsymbol{v}; z) = \sum_{\varkappa=1}^{4k} \sum_{\omega \in 2m\Omega} Q_\varkappa(\boldsymbol{v}'; z - a'_{m\varkappa} - \omega) .$$

For an arbitrary set $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{4k})$ of complex numbers we shall use the notation

$$F_m(\boldsymbol{v}; \boldsymbol{\tau}; z) = \sum_{\varkappa=1}^{4k} \sum_{\omega \in 2m\Omega} Q_\varkappa(\boldsymbol{v}'; z - a'_{m\varkappa} - \tau_\varkappa - \omega) .$$

If every $a'_{m\varkappa} + \tau_\varkappa + \omega$ is outside S_m and $|\tau_\varkappa| \leqq \frac{1}{2}\delta$, $\varkappa = 1, \ldots, 4k$, and η_1 is a positive number, we can choose $\boldsymbol{w}' = (w'_{\varkappa\lambda})_{4k,l}$ such that

$$\left| \sum_{\varkappa=1}^{4k} \big(Q_\varkappa(\boldsymbol{w}'; z - a'_{m\varkappa} - \tau_\varkappa - \omega) - Q_\varkappa(\boldsymbol{v}'; z - a'_{m\varkappa} - \omega)\big) \right| \leqq \eta_1 |z - a'_{m\varkappa} - \omega|^{-3}$$

when $|z - a'_{m\varkappa} - \omega| \geqq \delta$, $\varkappa = 1, \ldots, 4k$, $\omega \in 2m\Omega$. This follows immediately from [3, § 3, lemma 6], since $\boldsymbol{v}'$ is fixed in the present case. Hence, there exists a constant K depending only on m, Ω and k such that

$$(2) \qquad |F_m(\boldsymbol{w}'; \boldsymbol{\tau}; z) - F_m(\boldsymbol{v}; z)| \leqq K\eta_1$$

for every $z \in S_{m-1}$. Since we have used no properties of the points $a'_{m\varkappa}$ except that they and every $a'_{m\varkappa} + \omega$; $\omega \in 2m\Omega$, are outside S_m, we can repeat the procedure N times, and obtain the inequality (2) with $NK\eta_1$ instead of $K\eta_1$; when $|\tau_\varkappa| \leqq \frac{1}{2}N\delta$ and the segment from $a'_{m\varkappa} + \omega$ to $a'_{m\varkappa} + \omega + \tau_\varkappa$ is outside S_m for $\varkappa = 1, \ldots, 4k$; $\omega \in 2m\Omega$.

Each of the numbers $a_{2m,\varkappa} = 2a_{m,\varkappa}$ is the geometric center of gravity of a polygon of the subdivision of P_{2m}, and this polygon contains finitely many of the points $a_{m,\varkappa'}$. For each of these we can choose $\tau_{\varkappa'} = a_{2m,\varkappa} - a_{m,\varkappa'}$. We can then choose the number N such that (2) is satisfied with $NK\eta_1$ instead of $K\eta_1$. Since N and K are independent of η_1, we have proved that for $\eta > 0$ we can choose $\boldsymbol{w} = (w_{\varkappa,\lambda})_{k,l''}$ such that

$$|F_{2m}(\boldsymbol{w}; z) - F_m(\boldsymbol{v}; z)| \leqq \eta \qquad \text{for} \quad |z| \in S_{m-1} .$$

We shall now start with $m = 2$. It does not matter how we choose $\boldsymbol{v} = \boldsymbol{v}^{(1)}$ except that it should have at least one element $\neq 0$. We can then construct the sequence of functions $F_{2m}(\boldsymbol{v}^{(m)}; z)$ such that

$$|F_{2^{m+1}}(\boldsymbol{v}^{(m+1)}; z) - F_{2^m}(\boldsymbol{v}^{(m)}; z)| \leqq 2^{-m}\varepsilon \qquad \text{for} \quad z \in S_{2^m-1} .$$

Here ε is an arbitrary positive number. The sequence of functions is then uniformly convergent in every finite strip in each of the prescribed directions. This implies that the limit function is entire and that it is limit periodic in each strip. If ε is small enough, the limit function will obviously not be constant. This completes the proof.

The case where there is a sequence of directions is treated similarly, except that the periods are not simply multiplied by 2 in each step but by an integer ν chosen separately for each step. Further, the straight lines in the directions $p_\nu\omega_1 + q_\nu\omega_2$, $\nu = 3, 4, \ldots$, are introduced gradually, one new direction in each step. This has the effect that the number k increases with each step, and in each step the distance δ is replaced by a smaller distance δ_1. However, if we multiply the periods by a number greater than $2\delta_1^{-1}\delta$, the width of each of the periodicity strips will be doubled in each step.

The displacement is carried out according to the same rules as in the finite case. It may, however, happen that some of the old poles are situated on sides of polygons in the new subdivision. We may then choose arbitrarily whether we want to consider them as belonging to one or the other of the adjoining polygons. Except for the changes mentioned here, the proof is carried out exactly as in the finite case.

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