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A CHARACTERIZATION OF REFLEXIVITY OF BANACH SPACES

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1. Introduction.

A Banach-space X is reflexive if and only if the natural embedding of X into X^{**} is onto. A large number of necessary and sufficient conditions for reflexivity are known, one of them being that a Banach-space X is reflexive if and only if its unitball B is weakly compact. In this paper we prove an extension of this criterion. We shall construct a topology on a normed linear space X which is in general weaker than the weak topology and show that the space is reflexive if and only if the unitball is compact in this weaker topology.

The problem was originally related to the notion of a semi-inner-product space [5]. Our topology is defined in terms of a mapping $W\colon X \to X^*$ satisfying two simple conditions (see section 2). One can show that W satisfies these conditions if and only if $[x,y] = W_y(x)$, where W_y is the image of y under W, is a semi-inner-product consistent with the norm on X. We shall not make any use of this relationship.

The statement and proof of our theorem are contained in section 4. Section 2 lists the notation and terminology necessary for the remainder of the paper and section 3 mentions some known results that we use in section 4. We conclude with an example showing that the topology in question is generally weaker than the weak topology.

2. Notation and terminology.

If X is a vector space we denote the space of all linear functionals on X by X^+. If $F \subset X^+$ is a linear subspace with the property that if $f(x)=0$ for all f in F then $x=0$, we shall say that F is a total subspace of X^+ or that F is total over X.

Suppose that X is a vector space and that F is a total subspace over X. Following [2] we define the F-topology of X (or the weak topology in X induced by F) as the topology with base consisting of all sets of the form

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$$N(x_0, A, \varepsilon) = \{x \in X;\ |f(x) - f(x_0)| < \varepsilon \text{ for all } f \in A\}$$

where A is a finite subset of F and $\varepsilon > 0$.

Now, let X be a normed linear space (in the following abbreviated NLS) with dual X^*. Let $W: X \to X^*$ be a mapping satisfying the conditions (we write W_x for the image of x under W)

i) $\|W_x\| = \|x\|$,
ii) $W_x(x) = \|x\|^2$ for all x in X.

(It follows from the Hahn–Banach Theorem (cf. [2, p. 65] and the proof of Theorem 1) that such a mapping exists. In fact, there will in general be infinitely many such mappings on any given NLS X).

Using the notation $\mathrm{sp}(S)$ for the linear span (i.e. finite linear combinations of elements from S) of a subset S of a vector space we define the W-topology as the topology defined above with $F = \mathrm{sp}(W(X))$. This topology is Hausdorff since it is easily seen that $\mathrm{sp}(W(X))$ is a total subspace of X^+.

We end this section with a few definitions that will be needed in section 3:

Let X be a vector space and Y a subspace of X^+. If $S \subset X$ then S°, the polar of S in Y is defined as

$$S^\circ = \{f \in Y;\ |f(x)| \leqq 1 \text{ for all } x \text{ in } S\}.$$

S is convex if $x, y \in S$, $t \in [0,1]$ imply $tx + (1-t)\,y \in S$. S is radial at 0 if for every $x \in X$ there exists $t > 0$ such that $0 \leqq s < t$ implies $sx \in S$.

3. Some known results.

We list here two very powerful theorems which are crucial in our proof of the main theorem. They are Smulian's necessary and sufficient condition for compactness of a closed, circled convex subset of a linear space in the weak topology defined in section 2 by a total subspace of the algebraic dual (cf. [4, pp. 142–143]) and R. C. James' characterization of reflexivity of a Banach-space [3].

The Smulian Compactness Criterion:

Let X be a vector space and F a total subspace of X^+. Let B be a circled, convex subset of X such that B is closed in the weak topology induced by F. Then B is compact in this topology if and only if $B^\circ \subset F$ is radial at 0 and for each linear functional f on F which is bounded on B° there is an x in X such that $f(y) = y(x)$ for all $y \in F$.

James' characterization of reflexivity:

A Banach-space X is reflexive if and only if every $x^* \in X^*$ attains its norm on the unitball B of X, i.e. for every x^* in X^* there is an x in X of norm 1 such that $x^*(x) = \|x^*\|$.

The following results—found in [1]—are all needed in section 5:

Let S be an index set and let Y be a Banach-space of real-valued functions on S. If for each s in S a NLS N_s is given, let $P_Y N_s$ be the space of all functions x on S satisfying

i) x_s is in N_s for all s in S,
ii) if h is the real-valued function defined by $h(s) = \|x_s\|$ for all s in S then $h \in Y$.

LEMMA 1. *If Y satisfies the condition that whenever $h \in Y$ and $|k(s)| \leqq |h(s)|$ for all s then $k \in Y$ and $\|k\| \leqq \|h\|$, then $P_Y N_s$ is a normed linear space and is complete if all N_s are complete.*

LEMMA 2. *If $1 < p < \infty$ and $Y = \ell^p(S)$, then*

$$(P_Y N_s)^* = P_{Y^*} N_s^* .$$

From this follows directly

LEMMA 3. *If $1 < p < \infty$ and $Y = \ell^p(S)$ then $P_Y N_s$ is reflexive if and only if each N_s is reflexive.*

Since every finite dimensional NLS is reflexive ([2, p. 246]) it follows that with Y defined as in Lemma 3 $P_Y N_s$ is reflexive if every N_s is finite dimensional.

4. The main theorem.

THEOREM 1. *Let X be a Banach-space. Then X is reflexive if and only if there exists a mapping $W: X \to X^*$ satisfying the conditions*

$$\text{i)} \quad W_x(x) = \|x\|^2 ,$$
$$\text{ii)} \quad \|W_x\| = \|x\|$$

for all $x \in X$ such that the unitball B of X is compact in the weak topology induced by $\mathrm{sp}(W(X))$.

REMARK. Since all such W-topologies are weaker than the weak topology and since reflexivity implies weak compactness of B, it follows that if B is compact in one W-topology, it is compact in them all.

Proof of Theorem 1. Suppose first that X is a reflexive Banach-space. By the Hahn–Banach Theorem for each x in X, $x \neq 0$, choose V_x in X^* such that $\|V_x\| = 1$ and $V_x(x) = \|x\|$. Define $W_x = \|x\| V_x$. Since each W_x is bounded, it follows that $W(X) \subset X^*$, i.e. $\mathrm{sp}(W(X)) \subset X^*$. From this it is clear that the W-topology induced by sp(W)) is weaker than the weak topology. Since the unitball B is weakly compact, we conclude that B is compact in the topology induced by W.

Conversely, let $W\colon X \to X^*$ be a mapping satisfying the conditions i) and ii). Let $F = \mathrm{sp}(W(X)) \subset X^*$. F is a NLS with the norm inherited from X^*. The polar of B in F is exactly the unitball of F, i.e. $B^\circ = B^* \cap F$, where B^* is the unitball of X^*. It is trivial that B is convex and circled and since the W-topology is Hausdorff, W-compactness of B implies that B is W-closed. The Smulian compactness criterion therefore implies that every linear functional on F which is bounded on B° is evaluation at a point of X. But since F is a NLS, the set of linear functionals bounded on B° is precisely F^*, i.e. $f \in F^*$ implies that there is $x_f \in X$ such that $f(z) = z(x_f)$ for every $z \in F$.

Since every x in X by the natural embedding of X into X^{**} is a continuous linear functional on X^* and hence (by restriction) on F, we have a linear mapping Q of X into F^* such that $(Qx)(z) = z(x)$ for all $z \in F$. Given any $f \in F^*$, $(Qx_f)(z) = z(x_f) = f(z)$ for all z in F so $Qx_f = f$. This proves that Q is onto.

We shall show that Q is an isometry, i.e. $\|Qx\| = \|x\|$ for all $x \in X$.

$$\begin{aligned} \|Qx\| &= \sup\{|(Qx)(Z)|;\ Z \in B^\circ,\ \|Z\| = 1\} \\ &\leqq \sup\{|Z(x)|;\ Z \in B^*,\ \|Z\| = 1\} = \|x\|, \end{aligned}$$

since $B^\circ \subset B^*$. But

$$(Qx)(W_x/\|x\|) = W_x(x)/\|x\| = \|x\|$$

by condition i). This gives us the reverse inequality and thus that $\|Qx\| = \|x\|$ for all $x \in X$. The above computation also shows that every continuous functional on F attains its norm. The completion $\bar{F}$ of F in X^* has the same dual as F. Using the James characterization of reflexivity we conclude that $\bar{F}$ is reflexive. This implies that $\bar{F}^* = X$ is reflexive (cf. [2, p. 67]).

The following corollary is immediate.

Corollary. *If X is a reflexive Banach-space and W is a mapping as defined in Theorem 1, then the weak topology and the W-topology coincide on bounded sets.*

5. An example.

Theorem 1 is not particularly interesting unless it is known that there exists a reflexive Banach-space with a W-topology which is weaker than the weak topology. There are many examples of reflexive Banach-spaces in which the W-topology coincides with the weak topology. It can for instance be shown that if X has a smooth unitball (every point of the unit sphere has a unique supporting hyperplane) then W is uniquely determined and its induced topology equals the weak topology.

We shall not prove these facts here, but proceed immediately to our example.

It suffices to find a reflexive Banach-space X and a mapping W satisfying i) and ii) in Theorem 1 such that $\mathrm{sp}(W(X)) \neq X^*$ (cf. [2, p. 421 and p. 436]).

Let S be the set of natural numbers and $Y = \ell^2(S)$ be the space of square summable real sequences with the usual norm. For $s \in S$ let $N_s = \ell^\infty(s)$ be the space of real s-tuples with the max-norm and let $X = P_Y N_s$. We note that $x = (A_1, A_2, A_3, \ldots)$ where $A_i = (a_{i1}, a_{i2}, \ldots, a_{ii})$ for $i = 1, 2, \ldots$ is an element of X if and only if

$$\sum_{i=1}^{\infty} \|A_i\|^2 < \infty, \qquad \text{where} \quad \|A_i\| = \max_{j \leq i} |a_{ij}| \, .$$

Furthermore

$$\|x\| = \left(\sum_{i=1}^{\infty} \|A_i\|^2 \right)^{\frac{1}{2}} .$$

It follows from the discussion in section 3 that X is reflexive and $X^* = P_Y N_s^*$ where $N_s^* = \ell^1(s)$; if $B = (b_1, b_2, \ldots b_s) \in \ell^1(s)$, then $\|B\| = \sum_{k=1}^{s} |b_k|$. If $x^* = (B_1, B_2, \ldots) \in X^*$ with $B_i = (b_{i1}, b_{i2}, \ldots, b_{ii})$, then $\|x^*\| = (\sum_{i=1}^{\infty} (\sum_{j=1}^{i} |b_{ij}|^2))^{\frac{1}{2}}$ and

$$x^*(x) = \sum_{i=1}^{\infty} B_i(A_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij} a_{ij} \, .$$

To construct $W : X \to X^*$ we can proceed as follows:

Let $x = (A_1, A_2, \ldots)$ with $A_i = (a_{i1}, a_{i2}, \ldots, a_{ii})$ be an element of X. For each i let $j(i)$ be the smallest index such that $|a_{ij(i)}| = \|A_i\|$. Choose $x^* \in X^*$ such that $x^* = (B_1, B_2, \ldots)$ where for each i, $B_i = (b_{i1}, b_{i2}, \ldots, b_{ii})$ is defined by

$$b_{ij(i)} = a_{ij(i)} \qquad \text{and} \qquad b_{ij} = 0 \quad \text{if } j \neq j(i) \, .$$

Then

$$\|x^*\| = \left(\sum_{i=1}^{\infty} \|B_i\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |b_{ij(i)}|^2 \right)^{\frac{1}{2}} = \|x\|$$

and

$$x^*(x) = \sum_{i=1}^{\infty} B_i(A_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij} a_{ij} = \sum_{i=1}^{\infty} a_{ij(i)}^2 = \sum_{i=1}^{\infty} \|A_i\|^2 = \|x\|^2 .$$

Thus if for each x in X we let W_x be defined as indicated above we see that W satisfies the conditions i) and ii) of Theorem 1.

It remains to find an x_0^* in X^* which is not in $\mathrm{sp}(W(X))$. From the way we have constructed W we see that if $x^* \in W(X)$, $x^* = (B_1, B_2, \ldots)$ then each B_i has exactly one coordinate which is different from 0. This implies that if

$$y^* = \sum_{k=1}^{p} \delta_k x_k^* = (C_1, C_2, \ldots)$$

is an element of $\mathrm{sp}(W(X))$ then for $i > p$, C_i has at most p coordinates different from 0. Thus any element in X for which all coordinates are different from 0 is not in $\mathrm{sp}(W(X))$. One such $x_0^* = (B_1, B_2, \ldots)$ is defined by $B_i = (b_{i1}, b_{i2}, \ldots, b_{ii})$ with $b_{ij} = (i^2 2^i)^{-\frac{1}{2}}$ for each $j = 1, 2, \ldots, i$. We have

$$\|x_0^*\| = \left(\sum_{i=1}^{\infty} \|B_i\|^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} |b_{ij}|\right)^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} 2^{-i}\right)^{\frac{1}{2}} = 1 .$$

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