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A GENERALIZATION OF A THEOREM OF SYLVESTER ON THE LINES DETERMINED BY A FINITE POINT SET

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In 1893 Sylvester stated without proof: Given any finite set of non-collinear points in the real projective plane, there exists at least one line which contains exactly two of the given points.

This theorem remained unproved for nearly 40 years. In 1933 T. Gallai (Grünwald) gave a proof, and later on several other proofs have been found (for references cf. [3, p. 451] and [2, p. 65]). Here we mention R. Steinberg's (1944, cf. [1, p. 30]) because related ideas are used in the present paper.

In general it is not true that, given a finite point set in a projective space of dimension $d > 2$ which is not contained in a hyperplane, there exists a hyperplane containing exactly d of the given points. A counter-example in 3-space is a set of 6 points, 3 on each of two skew lines. Another one is the Desargues configuration in 3-space [3, p. 452].

The following generalization of Sylvester's theorem has been proved by Th. Motzkin [3] for $d=3$:

Given a finite point set Γ_0 in the d-dimensional real projective space which is not contained in a hyperplane. Then among the hyperplanes determined by points of Γ_0 there is at least one with the property that the points of Γ_0 which it contains, with the exception of precisely one of them, lie in a $(d-2)$-dimensional projective subspace.

It is the aim of this paper to give a proof of this theorem for arbitrary dimension d.

Let Γ_p, $p=0,1,\ldots,d$, denote the set of the projective subspaces of dimension p which are spanned by the points of the given set Γ_0. Since Γ_0 spans the whole d-dimensional space P_d considered, none of these sets is empty, and Γ_d consists of P_d only. The elements of Γ_p will be denoted $A_p,\ldots,P_p,\ldots$. The union

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$$\Gamma = \bigcup_{p=0}^{d} \Gamma_p$$

will be called the *configuration* Γ.

A subspace $A_p \in \Gamma_p$ will be called *elementary* if it contains exactly $p+1$ points of Γ_0. If Γ_0 consists of $d+1$ points, then Γ consists of the subspaces spanned by the vertices of a simplex, and thus all elements of Γ are elementary. In this case the configuration Γ will be called elementary.

If a subspace A_p is spanned by a point B_0 and a subspace C_{p-1}, we shall write

$$A_p = B_0 C_{p-1} .$$

Obviously, $B_0 \in \Gamma_0$ and $C_{p-1} \in \Gamma_{p-1}$ imply $A_p \in \Gamma_p$. If in this case B_0 is the only point of $\Gamma_0 \cap A_p$ outside C_{p-1}, then the subspace A_p is called *ordinary*. (This makes also sense for $p=0$, if, as usual, the (-1)-dimensional projective space is defined to be the empty set.) It is clear that every elementary subspace is ordinary. In this terminology the theorem above states the existence of an ordinary hyperplane.

Every $A_p \in \Gamma$, $p>0$, is divided into polyhedral domains by the subspaces $A_{p-1} \in \Gamma_{p-1}$ contained in A_p. The closures of these domains will be called the *p-dimensional cells* of the configuration Γ. Each cell δ_p in A_p is obviously convex in the sense that, for any two points of it, that segment joining these points which does not intersect an $A_{p-1} \subset A_p$ belongs entirely to δ_p.

We shall prove the following theorem which is slightly stronger than the statement above:

THEOREM. *Suppose that the configuration Γ in the real projective space of dimension d is not elementary, and let δ_d be a d-dimensional cell of Γ. Then there exists an ordinary hyperplane $A_{d-1} = B_0 C_{d-2}$, where $B_0 \in \Gamma_0$ and*

$$A_{d-1} \cap \Gamma_0 \setminus \{B_0\} \subset C_{d-2} \in \Gamma_{d-2} ,$$

such that

$$A_{d-1} \cap \delta_d \subset C_{d-2} .$$

We start by proving two lemmas.

LEMMA 1. *Let σ_d, $d>0$, denote a closed d-dimensional simplex whose vertices belong to Γ_0, and suppose that the point $A_0 \in \Gamma_0$ is not contained in σ_d. Then there is a $(d-2)$-dimensional face σ_{d-2} of σ_d such that the hyperplane $B_{d-1} \in \Gamma_{d-1}$ spanned by A_0 and σ_{d-2} satisfies*

$$B_{d-1} \cap \sigma_d = \sigma_{d-2} .$$

PROOF. Consider the hyperplanes which contain the $(d-1)$-dimensional faces of σ_d. Every pair of distinct such hyperplanes divides the space P_d into two "wedges". The closure of one of these contains σ_d. Since σ_d is the intersection of such closed wedges, there must be at least one which does not contain A_0. The hyperplane spanned by A_0 and the intersection of the hyperplanes bounding such a wedge satisfies the requirement of the lemma.

LEMMA 2. *Let A_0 be a point of Γ_0, and let $C_{d-1} \in \Gamma_{d-1}$ be a hyperplane which does not contain A_0. Let further $\delta_{d-1} \subset C_{d-1}$ be a $(d-1)$-dimensional cell of Γ. If P_0 is a point such that the line A_0P_0 does not meet δ_{d-1} and Q_0 is an interior point of δ_{d-1}, then each of the two segments P_0Q_0 intersects at least one of the hyperplanes, belonging to Γ_{d-1}, which are spanned by A_0 and the $(d-2)$-dimensional faces of δ_{d-1}.*

PROOF. The union of the lines joining A_0 with the points of δ_{d-1} is a polyhedral convex cone. Since P_0 is an exterior point and Q_0 an interior point of this cone, each of the segments P_0Q_0 intersects its boundary. The statement now follows from the fact that every boundary point of the cone is contained in at least one of the hyperplanes spanned by A_0 and the $(d-2)$-dimensional faces of δ_{d-1}.

PROOF OF THE THEOREM. We proceed by induction on the dimension of the space. For $d=1$ the Theorem is obvious. Let $d>1$ be given. We assume the Theorem to be true for spaces of dimension $d-1$.

Let a d-dimensional cell δ_d of Γ be given. Obviously, δ_d is contained in some closed simplex with vertices belonging to Γ_0. If this simplex contains a point of Γ_0 different from its vertices, the hyperplanes spanned by this point and the vertices divide the simplex into smaller simplexes one of which contains δ_d. If this simplex contains a point of Γ_0 different from its vertices, the procedure can be repeated. After finitely many steps a simplex σ_d is obtained which contains δ_d but no point of Γ_0 other than its vertices.

Since Γ is not elementary, there is a point of Γ_0 outside σ_d. By Lemma 1, there exists a hyperplane $B_{d-1} \in \Gamma_{d-1}$ through this point for which

$$B_{d-1} \cap \delta_d \subset B_{d-1} \cap \sigma_d \subset S_{d-2},$$

where $S_{d-2} \in \Gamma_{d-2}$ is the $(d-2)$-dimensional subspace containing one of the $(d-2)$-dimensional faces of σ_d.

If B_{d-1} is elementary, it clearly satisfies the requirement of the Theorem. Hence, in the sequel we may assume that B_{d-1} is not elementary.

We consider a point $P_0 \in \Gamma_0$ which does not lie in B_{d-1}. We choose a line L_1 which joins P_0 with an interior point of δ_d and which has no point different from P_0 in common with any of the $(d-2)$-dimensional subspaces in which two hyperplanes belonging to Γ_{d-1} intersect. (In particular, L_1 does then not meet any subspace belonging to Γ_{d-2} at a point different from P_0.)

By assumption Γ_{d-1} contains non-elementary hyperplanes, for instance B_{d-1}. Since these hyperplanes do not intersect the interior of δ_d, there is at least one among them, Q_{d-1} say, such that the point Q_0 at which it intersects L_1 satisfies the following condition: One of the open segments of L_1 determined by P_0 and Q_0 intersects neither the interior of δ_d nor any of the non-elementary hyperplanes. In other words, traversing L_1 from P_0 in that sense, or one of the senses, in which one meets non-elementary hyperplanes before meeting δ_d, Q_0 is the first point of intersection with a non-elementary hyperplane.

The point Q_0 belongs to the interior of exactly one of the $(d-1)$-dimensional cells of Γ into which Q_{d-1} is divided. The polyhedral cone γ_d consisting of the lines joining P_0 with the points of this cell δ_{d-1} contains δ_d because the hyperplanes which contribute to the boundary of γ_d belong to Γ_{d-1} and the line $L_1 \subset \gamma_d$ intersects δ_d. By the induction hypothesis, there exists in Q_{d-1} an ordinary $(d-2)$-dimensional subspace $C_0 S_{d-3} \in \Gamma_{d-2}$, where

$$C_0 \in \Gamma_0, \qquad C_0 S_{d-3} \cap \Gamma_0 \setminus \{C_0\} \subset S_{d-3} \in \Gamma_{d-3}\,,$$

such that

$$C_0 S_{d-3} \cap \delta_{d-1} \subset S_{d-3}\,.$$

Putting

$$S_{d-2} = P_0 S_{d-3} \in \Gamma_{d-2}\,,$$

we consider the hyperplane

$$C_0 S_{d-2} \in \Gamma_{d-1}\,.$$

Obviously,

(1) $$C_0 S_{d-2} \cap \gamma_d \subset S_{d-2}\,,$$

hence

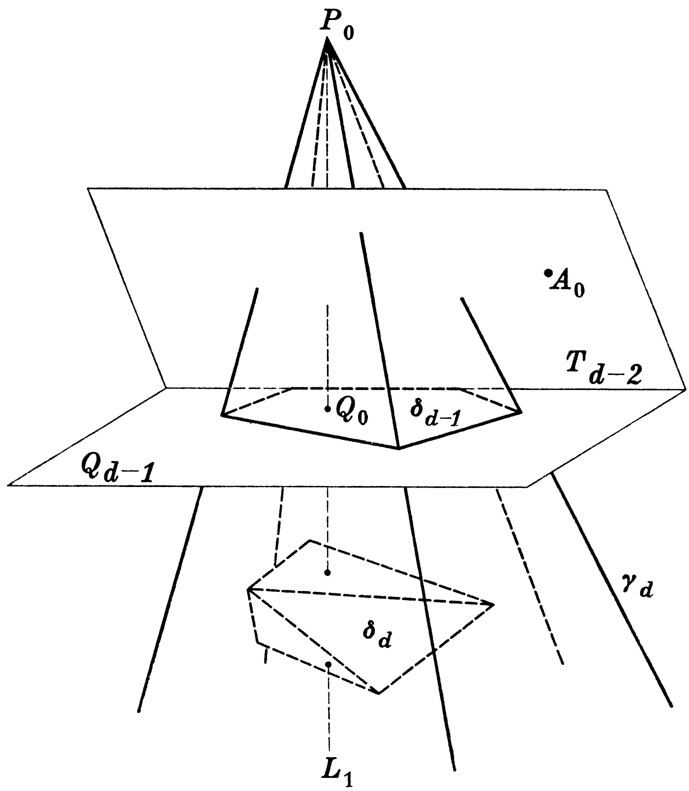
(2) $$C_0 S_{d-2} \cap \delta_d \subset S_{d-2}$$

since $\delta_d \subset \gamma_d$.

We distinguish now between two cases:

1°. In $C_0 S_{d-2}$ there is outside S_{d-2} no other point of Γ_0 than C_0. Then $C_0 S_{d-2}$ is ordinary, and because of (2) this hyperplane satisfies the requirements of the Theorem.

2°. In $C_0 S_{d-2}$ there is a point $A_0 \in \Gamma_0$ which does not lie in S_{d-2}. From the facts that



$$C_0 S_{d-2} \cap Q_{d-1} = C_0 S_{d-3},$$

$C_0 S_{d-3}$ is ordinary, and $S_{d-3} \subset S_{d-2}$ we can conclude that $A_0 \notin Q_{d-1}$. Further, from (1) and $A_0 \notin S_{d-2}$ it follows that $A_0 \notin \gamma_d$. Consequently, $P_0 A_0$ does not meet δ_{d-1}. By Lemma 2, there exists therefore a hyperplane $A_0 T_{d-2} \in \Gamma_{d-1}$ (see fig.) such that T_{d-2} contains a $(d-2)$-dimensional face of δ_{d-1}, and $A_0 T_{d-2}$ intersects that open segment $P_0 Q_0$ which does not meet δ_d. From the way in which Q_0 was determined it follows that $A_0 T_{d-2}$ is elementary, thus ordinary.

It remains to be shown that

$$A_0 T_{d-2} \cap \delta_d \subset T_{d-2}. \tag{3}$$

The hyperplanes Q_{d-1} and $P_0 T_{d-2}$ intersect in T_{d-2}. They belong to Γ_{d-1} and, thus, do not meet the interior of δ_d. Consequently, the closure of one of the two wedges into which they divide the space contains δ_d. Since the hyperplane $A_0 T_{d-2}$ intersects that open segment $P_0 Q_0$ which does not meet δ_d, it can only have T_{d-2} in common with the wedge containing δ_d, and hence (3) holds.

This completes the proof of the Theorem.

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