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CONVEX SETS AND CHEBYSHEV SETS

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Introduction.

A subset M of a normed linear space E is called a Chebyshev set if each point in E has a unique nearest point in M. We shall deal with the relationship between the closed convex sets and the Chebyshev sets in certain finite dimensional Banach spaces. All spaces are taken as linear spaces over the reals; i.e. every complex space is identified with its underlying real space.

The following results are well known:

(I) *A finite dimensional Banach space E is rotund if and only if every non-empty closed convex set in E is a Chebyshev set.*

(II) *A finite dimensional Banach space E is rotund and smooth if and only if the Chebyshev sets in E are identical with the non-empty closed convex sets in E.*

(III) *A 2-dimensional Banach space E is smooth if and only if every Chebyshev set in E is convex.*

(IV) *If E is a finite dimensional smooth Banach space, then every Chebyshev set in E is convex.*

We note that (IV) is an extension of the *only if* part of (III). It has been believed (e.g. V. Klee [7]) that the *if* part be true, too, for spaces of any finite dimension. The purpose of the present note is to prove that this is not the case. In fact, one can find counter-examples for any dimension $\geqq 3$.

For proofs of (I)–(III) and other related results, see for instance L. N. H. Bunt [1], H. Busemann [2], N. V. Efimov and S. B. Stečkin [3], B. Jessen [6], Th. Motzkin [8], [9], and F. A. Valentine [10].

For proofs of (IV), see V. Klee [7, Theorem 2.2] and L. P. Vlasov [11]. The proofs use the Brouwer–Tychonov fixed point theorem; no elemen-

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tary proof of (IV) seems to be known, except for 2-dimensional spaces. Both in [7] and [11], theorem (IV) is actually obtained as a special case of a more general result concerning convexity of Chebyshev sets in spaces of arbitrary dimension. Other such results can be found in N. V. Efimov and S. B. Stečkin [4], [5], and V. Klee [7].

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Terminology.

The real number field is denoted by R.

A ball in a normed linear space E is a set of the form

$$\{x \in E: \ \|x-y\| \leqq r\},$$

where $y \in E$ and $r > 0$. The unit ball of E is the ball with $y=o$ (where o is the zero element in E) and $r=1$.

A point x contained in the boundary of a closed convex subset M of a space E is said to be an exposed point of M if there exists a closed supporting hyperplane H of M such that $H \cap M = \{x\}$. A point $x \in \mathrm{bd}\, M$ is said to be a smooth point of M if M has a unique supporting hyperplane at x. The space E is said to be rotund (or strictly convex) if all points in the boundary of the unit ball K are exposed. If all points in $\mathrm{bd}\, K$ are smooth, then E is said to be smooth.

The supporting cone of a closed convex set M at a point $x \in \mathrm{bd}\, M$ is the intersection of all closed halfspaces containing M and bounded by supporting hyperplanes of M at x.

A flat in E is a translate of a subspace. A convex body in E is a bounded closed convex set with a non-empty interior.

A point $x \in E$ is said to be separated from a set M by a closed hyperplane H if x is contained in one of the two open halfspaces determined by H, and M is contained in the complement of this open halfspace.

Preliminaries.

Clearly, any Chebyshev set is a closed set. A subset M of a normed linear space E is said to be boundedly compact if every ball in E has a compact (possibly empty) intersection with M. It is easy to verify that if M is a boundedly compact Chebyshev set in any normed linear space E, then the metric projection of E onto M (i.e. the mapping which carries each point in E onto the unique nearest point on M) is continuous. Hence, *in a finite dimensional space every Chebyshev set has a continuous metric projection.*

Let M be a Chebyshev set in a space E, and let π be the metric projection of E onto M. Then M is said to be a sun if $\pi(z)=\pi(x)$ for every $x \in E \setminus M$ and every z on the halfline emanating from $\pi(x)$ and passing through x. *In a finite dimensional space every Chebyshev set is a sun.* In fact, V. Klee [7, Lemma 2.1] has proved that if M is a Chebyshev set in a reflexive Banach space E, and if each point in $E \setminus M$ has a neighbourhood on which the restriction of the metric projection is continuous and weakly continuous, then M is a sun. Alternatively, L. P. Vlasov [11] has proved that in any Banach space every boundedly compact Chebyshev set is a sun.

No example seems to be known of a Chebyshev set which is not a sun or does not have a continuous metric projection. It is unknown whether there exist any infinite dimentional spaces in which every Chebyshev set is convex. However, it follows from proposition 1 below that in any smooth space every sun is convex. In particular, by either of the two theorems quoted above, every Chebyshev set in a finite dimensional smooth space is convex.

PROPOSITION 1. *Let E be a normed linear space with unit ball K. Let M be a Chebyshev set in E which is a sun, and let π be the metric projection of E onto M. For $x \in E \setminus M$, let*

$$K_x = x + \|\pi(x)-x\| K$$

and let T_x be the supporting cone of K_x at the point $\pi(x)$. Then

$$(\operatorname{int} T_x) \cap M = \emptyset$$

for every $x \in E \setminus M$.

PROOF. Let $x \in E \setminus M$, and let $z \in \operatorname{int} T_x$. We claim that

$$z \in \operatorname{int} K_y$$

for some y contained in the open halfline

$$L = \{(1-t)x+t\pi(x):\ t \in\,]-\infty, 1[\}\,.$$

Since trivially

$$(\operatorname{int} K_y) \cap M = \emptyset\,,$$

this will prove the proposition. Suppose that z is not in $\operatorname{int} K_y$ for any $y \in L$, that is,

$$\|z-((1-t)x+t\pi(x))\| \geqq \|\pi((1-t)x+t\pi(x))-((1-t)x+t\pi(x))\|$$

for every $t \in\,]-\infty, 1[$. Since M is a sun,

$$\pi((1-t)x + t\pi(x)) = \pi(x)$$

for every $t \in]-\infty, 1[$, whence

$$\left\| \left(\frac{1}{1-t} z + \frac{-1}{1-t} \pi(x) \right) - x \right\| \geqq \|\pi(x) - x\|$$

for every $t \in]-\infty, 1[$. But this says that the halfline emanating from $\pi(x)$ and passing through z is disjoint from $\operatorname{int} K_x$. Therefore the whole line determined by $\pi(x)$ and z is disjoint from $\operatorname{int} K_x$, whence, by the Hahn–Banach theorem, some supporting hyperplane of K_x at $\pi(x)$ contains z. This, however, contradicts that $z \in \operatorname{int} T_x$.

Main result.

We are going to prove:

THEOREM. *For every integer $n \geqq 3$ there exists a non-smooth n-dimensional Banach space E with the property that every Chebyshev set in E is convex.*

The proof is divided into two parts. We first construct a symmetric convex body K in R^n, $n \geqq 3$, and establish some properties of K. Then secondly we prove that if we let E be the Banach space obtained by attaching to R^n the norm in which K is the unit ball, then E has the properties stated in the theorem.

The elements of R^n will be denoted either by symbols like x or like (α_i), the latter being an abbreviation for $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We denote $(0,0,\ldots,0,1)$ by e_n, and $(0,0,\ldots,0)$ by o. Furthermore, we let

$$R_{n-2} = \{(\alpha_i) \in R^n:\ \alpha_{n-1} = \alpha_n = 0\},$$
$$R_{n-1} = \{(\alpha_i) \in R^n:\ \alpha_n = 0\},$$

and

$$C = \left\{(\alpha_i) \in R^n:\ \sum_{i=1}^{n} \alpha_i^2 \leqq 1\right\}.$$

By *horizontal* we mean *parallel to R_{n-1}*, and by *vertical* we mean *parallel to the line Re_n*.

Now, letting

$$K_{n-2} = C \cap R_{n-2}, \qquad K_{n-1} = C \cap R_{n-1},$$

and, for $t \in [-1, 1]$,

$$\alpha(t) = (1-t^2)^{\frac{1}{2}} + 1, \qquad \beta(t) = (\tfrac{4}{3} - t^2)^{\frac{1}{2}} - 3^{-\frac{1}{2}},$$

we define

$$K = \bigcup \{te_n + \alpha(t) K_{n-2} + \beta(t) K_{n-1}:\ t \in [-1, 1]\},$$

and

$$S = \operatorname{bd} K.$$

PROPOSITION 2. *With K and S as above, we have:*

(i) *The set K is a symmetric convex body in R^n.*

(ii) *The projection in vertical direction of R^n onto R_{n-1} maps K onto $K \cap R_{n-1}$. If H is a supporting hyperplane of K at a point in R_{n-1}, then H is vertical.*

(iii) *The hyperplanes*

$$H_1 = \{(\alpha_i) \in R^n:\ \alpha_n = 1\}, \qquad H_2 = \{(\alpha_i) \in R^n:\ \alpha_n = -1\}$$

are supporting hyperplanes of K, and

$$H_1 \cap K = e_n + K_{n-2}, \qquad H_2 \cap K = -e_n + K_{n-2}\,.$$

(iv) *All points in $S \setminus \big((e_n + K_{n-2}) \cup (-e_n + K_{n-2})\big)$ are smooth points of K.*

(v) *All points in $(e_n + K_{n-2}) \cup (-e_n + K_{n-2})$ are non-smooth points of K.*

(vi) *If H is a supporting hyperplane of K, and*

$$H \cap (e_n + K_{n-2}) \neq \emptyset\,,$$

then

$$H \cap K = e_n + K_{n-2}\,.$$

The union of all closed halfspaces bounded by such hyperplanes and not containing K is the union of just two of the halfspaces in question. Similarly for $-e_n + K_{n-2}$.

(vii) *If H is a supporting hyperplane of K, and H is not parallel to R_{n-2}, then $H \cap K$ contains just one point.*

PROOF OF PROPOSITION 2. (i). It is obvious that K as a subset of R^n is a symmetric bounded set with a non-empty interior. Using the concavity of the functions α and β, and the convexity of K_{n-2} and K_{n-1}, it is easy to prove that K is convex. Finally, let the mapping f from

$$[-1,1] \times K_{n-2} \times K_{n-1}$$

into R^n be defined by

$$f(t,y,z) = te_n + \alpha(t)y + \beta(t)z\,.$$

Since the domain is compact, and f is continuous, the image K is compact, and hence closed.

(ii) and (iii) are obvious.

(iv). Let

$$x \in S \setminus \big((e_n + K_{n-2}) \cup (-e_n + K_{n-2})\big)\,.$$

Then

$$x = t_0 e_n + \alpha(t_0) y_0 + \beta(t_0) z_0$$

for some

$$t_0 \in \,]-1,1[, \qquad y_0 \in K_{n-2}, \qquad \text{and} \qquad z_0 \in K_{n-1}\,.$$

It is easy to verify that the curve

$$\{te_n + \alpha(t)y_0 + \beta(t)z_0 \colon \ t \in [-1,1]\} \tag{1}$$

lies in S and has a non-horizontal tangent at the point x. Since every supporting hyperplane of K at x contains this tangent, it follows that if H' and H'' are two different supporting hyperplanes of K at x, then $H' \cap H''$ is non-horizontal. The intersection of K with the horizontal hyperplane H through x is smooth in H, for a non-smooth point in $H \cap K$ would be a non-smooth point in a translate of $\beta(t_0)K_{n-1}$. But this implies that $H' \cap H''$ must be horizontal. Hence we conclude that x is a smooth point.

(v) and (vi). Let x be a point in the relative boundary of K_{n-2}, and let G be the subspace spanned by x and e_n. It is easy to see that $S \cap G$ consists of two curves of the form (1) with $y_0 = z_0 = x$ and $y_0 = z_0 = -x$, respectively, and the two horizontal segments determined by the endpoints of the curves. Since the curves have horizontal tangents at the endpoints, it follows that $K \cap G$ is smooth in G. This implies that if H is a supporting hyperplane of K such that

$$H \cap (e_n + K_{n-2}) \neq \emptyset ,$$

then

$$e_n + K_{n-2} \subset H .$$

But then it is easy to see that H is of the form

$$\{(\alpha_i) \in R^n \colon \ \alpha_{n-1}\delta + \alpha_n = 1\} \tag{2}$$

for some $\delta \in R$. Furthermore, a $\delta \in R$ determines such a hyperplane if and only if

$$\pm \beta(t)\delta + t \leqq 1 \qquad \text{for all} \quad t \in [-1,1] . \tag{3}$$

An elementary calculation shows that (3) is equivalent to

$$\delta \in [-3^{-\frac{1}{2}}, 3^{-\frac{1}{2}}] ,$$

and that we only have equality in (3) for $t=1$. This proves (v) and (vi).

(vii). Let

$$H = \left\{(\alpha_i) \in R^n \colon \ \sum_{i=1}^{n} \alpha_i \beta_i = 1\right\}$$

be a supporting hyperplane of K, and assume that $\beta_j \neq 0$ for at least one $j \in \{1,2,\ldots n-2\}$. We want to prove that the expression

$$F(t,(y_i),(z_i)) = t\beta_n + \alpha(t) \sum_{i=1}^{n} y_i \beta_i + \beta(t) \sum_{i=1}^{n} z_i \beta_i$$

attains its maximum value at just one point $(t,(y_i),(z_i))$ for

$$(t,(y_i),(z_i)) \in [-1,1]\times K_{n-2}\times K_{n-1}\,.$$

The maximum value is actually attained, and by (vi) we have $t\in\,]-1,1[$ at any point $(t,(y_i),(z_i))$ where it is attained. Now, let $t_0\in\,]-1,1[$ be fixed. Since $\alpha(t_0)>0$, $\beta(t_0)>0$, and $\beta_j\neq 0$ for at least one $j\leqq n-2$, it follows that $F(t_0,(y_i),(z_i))$ is maximum for

$$(y_i) = \left(\sum_{i=1}^{n-2}\beta_i^2\right)^{-\frac12}(\beta_1,\beta_2,\dots\beta_{n-2},0,0)\,, \tag{4}$$

and

$$(z_i) = \left(\sum_{i=1}^{n-1}\beta_i^2\right)^{-\frac12}(\beta_1,\beta_2,\dots\beta_{n-2},\beta_{n-1},0)\,, \tag{5}$$

and that any other choice of (y_i) and (z_i) yields a smaller value of $F(t_0,(y_i),(z_i))$. Let y_0 be the point (y_i) in (4), and z_0 the point (z_i) in (5). Then $F(t,(y_i),(z_i))$ can only attain its maximum for

$$t\in\,]-1,1[,\qquad (y_i) = y_0,\qquad \text{and}\qquad (z_i) = z_0\,.$$

Since

$$F(t,y_0,z_0) = t\beta_n+\alpha(t)\left(\sum_{i=1}^{n-2}\beta_i^2\right)^{\frac12}+\beta(t)\left(\sum_{i=1}^{n-1}\beta_i^2\right)^{\frac12},$$

it is now easy to verify that $F(t,(y_i),(z_i))$ attains its maximum at just one point.

We next pass to:

Proof of the Theorem. By proposition 2(i), the set K is the unit ball of a norm on R^n. Let E be the n-dimensional Banach space thus obtained. By proposition 2(v), E is non-smooth. We want to prove that every Chebyshev set in E is convex. So, let $M\subset E$ be a Chebyshev set, and assume that M is not convex.

If, for $x\in E\setminus M$, the point $\pi(x)$ is a smooth point of the ball

$$K_x = x + \|\pi(x)-x\|\,K\,,$$

then x is said to be a point of type 1; and if $\pi(x)$ is a non-smooth point, x is said to be of type 2. No type is ascribed to points in M. If x is a point of type 2, we let L_x be the flat $\pi(x)+R_{n-2}$. For any point $x\in E$ we let H_x be the vertical hyperplane through x which is parallel to R_{n-2}. Hence, if x is a point of type 2, then H_x is the hyperplane spanned by x and L_x.

Since M is a sun, proposition 1 yields:

For each $x \in E \setminus M$ of type 1, the set M is contained in the closed halfspace bounded by the supporting hyperplane of K_x at $\pi(x)$ and not containing x. (We shall call such a halfspace a set of type 1). For each $x \in E \setminus M$ of type 2, the set M is contained in the union U of all closed halfspaces bounded by supporting hyperplanes of K_x at $\pi(x)$ and not containing x. By proposition 2(vi), the set U is in fact the union of just two of the halfspaces in question. (We shall call such a set U a set of type 2).

Now, by means of these observations we shall prove the following two statements:

(A) *The set M is not contained in any hyperplane.*
(B) *The set M is contained in some hyperplane.*

The contradiction expressed by (A) and (B) proves the theorem.

The proof of (A) goes as follows. Suppose that M is contained in a hyperplane H. Let $x \in H \setminus M$ be a point of type 2; we are going to prove that x can be separated from M in H by an $(n-2)$-dimensional flat. Without loss of generality we may assume that $x=o$ and $\|\pi(x)\|=1$, whence $K_x = K$. Then $\pi(x)$ is in $e_n + K_{n-2}$ or in $-e_n + K_{n-2}$; let us assume $\pi(x) \in e_n + K_{n-2}$. If all of $e_n + K_{n-2}$ is contained in H, then $H = H_x$. In this case x can be separated from M in H by the $(n-2)$-dimensional flat L_x; this follows from the fact that L_x is the intersection of the bounding hyperplanes of the two halfspaces whose union is the set of type 2 belonging to x. If $e_n + K_{n-2}$ is not contained in H, then H is not parallel to R_{n-2}. Let H' be one of the two supporting hyperplanes of K parallel to H. By proposition 2(vii), $H' \cap K$ contains just one point z, and clearly z is a smooth point of K. It is easy to see that for a sufficiently large positive real number t, the point $-tz$ is of type 1. Hence, the sets K_{-tz} and M are separated by the supporting hyperplane J of K_{-tz} at $\pi(-tz)$. Since x is the only point which is nearest to $-tz$ in H, it follows that x is an interior point of K_{-tz}. From this we deduce that J intersects H in a $(n-2)$-dimensional flat which separates x from M. Thus, we have proved that every point in H of type 2 can be separated from M by a $(n-2)$-dimensional flat. Since clearly every point in H of type 1 has this property, we conclude that M must be convex. This contradicts our assumption on M, and so we have proved (A).

The proof of (B) is divided into three steps.

Since M is closed and non-convex, there exist two different points p and q in M such that

$$]p,q[\cap M = \emptyset .$$

As the first step in the proof of (B) we shall prove:

(B_1) *For every* $x \in]p,q[$, *we have* $M \cap H_x = \{\pi(x)\}$.

Let $x \in]p,q[$. Then $x \in (\operatorname{conv} M) \setminus M$, whence clearly x is a point of type 2. Let H' and H'' be the bounding hyperplanes of the two closed halfspaces whose union is the set of type 2 belonging to x. Then $H' \cap H'' = L_x$, and hence x is separated from $M \cap H_x$ in H_x by L_x. Furthermore, $M \cap L_x = \{\pi(x)\}$. For if y were a point in $M \cap L_x$ different from $\pi(x)$, then some points in H_x, situated on the same side of L_x as x, would have both y and $\pi(x)$ as nearest points in M. Finally, suppose that some point $z \in H_x$ situated on the opposite side of L_x as x belongs to M. The complement of $H' \cup H''$ consists of four disjoint open connected sets A_i, $i = 1,2,3,4$. From the fact that the triangle spanned by p, q, and z is contained in $\operatorname{conv} M$ it follows that

$$(\operatorname{conv} M) \cap A_i \neq \emptyset, \qquad i = 1,2,3,4\,. \tag{6}$$

Let M_1 denote the intersection of all sets of type 1; then M_1 is a closed convex set containing M. Hence, (6) yields

$$M_1 \cap A_i \neq \emptyset, \qquad i = 1,2,3,4\,. \tag{7}$$

Since furthermore, by (A), M_1 is not contained in any hyperplane, we deduce from (7) that $M_1 \cap L_x$ contains more than one point. Thus, the set

$$(M_1 \cap L_x) \setminus \{\pi(x)\}$$

is non-empty; let y be some point in this set. Then clearly y is of type 2, and so L_x is contained in the complement of the set of type 2 belonging to y. In particular, the sets L_x and M are disjoint. This is obviously a contradiction, and the proof of (B_1) is completed.

Let

$$P = \operatorname{int} \operatorname{conv}(H_p \cup H_q)\,,$$

where H_p and H_q are the vertical hyperplanes through p and q, respectively, parallel to R_{n-2}. It follows from (B_1) that H_p and H_q are different, and that

$$M \cap P = \pi(]p,q[)\,.$$

Since π is continuous, we may describe the set

$$(M \cap P) \cup \{p,q\}$$

as a continuous curve from p to q. As the next step in the proof of (B) we shall prove:

(B_2) *The set* $M \cap P$ *is contained in every vertical hyperplane through* p *and* q.

Let H be a vertical hyperplane containing p and q, and suppose that $M \cap P$ is not contained in H. Let φ denote the projection in vertical direction of E onto R_{n-1}. Then there exists a point $x \in M \cap P$ such that $\varphi(x) \notin \varphi(H)$. Let G be the 2-dimensional flat spanned by $\varphi(p)$, $\varphi(q)$, and $\varphi(x)$. The hyperplane H_x intersects G in a line through $\varphi(x)$, and this line intersects the segment $[\varphi(p), \varphi(q)]$ in an interior point y. Now, let z be a point in $]y, \varphi(x)[$ so close to $\varphi(x)$ that for some $\varepsilon > 0$ we have

(8) $$\varphi(x) \in z + \varepsilon K \subset P$$

and

(9) $$(z + \varepsilon K) \cap H = \emptyset .$$

The set

$$\varphi((M \cap P) \cup \{p, q\})$$

is compact, being the image of the compact set $[p, q]$ under the continuous mapping $\varphi \circ \pi$. Hence, z has at least one nearest point in this set. Let w be such a point; then by (8) and (9)

$$w \in \varphi(M \cap P) \setminus H .$$

Let u be the point in $M \cap P$ which by φ is mapped onto w. Then, by proposition 2(ii),

$$u = \pi(u + z - w)$$

and the supporting hyperplane J of K_{u+z-w} at u is vertical. But then the $(n-2)$-dimensional flat $\varphi(J)$ separates the point

$$z = \varphi(u + z - w)$$

from $\varphi(M)$ in R_{n-1}. This is, however, impossible, since $\varphi(p)$, $\varphi(q)$, and $\varphi(x)$ belong to $\varphi(M)$, and z is an interior point of the triangle spanned by $\varphi(p)$, $\varphi(q)$, and $\varphi(x)$. Hence, (B_2) is proved.

We shall complete the proof of (B)—and the proof of the theorem—by proving:

(B_3) *The set M is contained in every vertical hyperplane through p and q.*

Let H be a vertical hyperplane containing p and q. It follows from (B_1) that H is not parallel to R_{n-2}. Let $x \in M \cap P$; then, by (B_2), x is in H. From proposition 2(vii) we deduce that the set of points $y \in E$ with the property that H is a supporting hyperplane of the ball

$$y + \|x - y\| K$$

at the point x is a horizontal line N through x. Clearly, if $z \in N$ and z is close to x, then $\pi(x) = x$. But then $\pi(z) = x$ for all $z \in N$, since M is a sun. And since x is a smooth point in K_z for $z \in N$, we conclude by proposition 1 that $M \subset H$.

Final remarks.

REMARK 1. The theorems in the introduction labelled (I) and (II) suggest the following two problems (which should be compared to (III), (IV), and the Theorem on p. 8):

1) *Is it possible to characterize those finite dimensional spaces in which every Chebyshev set is convex,—in terms of geometrical properties of the unit ball?*

2) *Is it possible to characterize those finite dimensional spaces which are smooth, – in terms of the Chebyshev sets?*

REMARK 2. The notion of a Chebyshev set may be extended in the following manner. Let E be a topological vector space, C a convex body in E with the zero element o contained in $\operatorname{int} C$, and M a non-empty subset of E. For any $x \in E$ we let

$$\varrho_x = \sup\{\sigma \geqq 0:\ (x+\sigma C) \cap M = \emptyset\},$$

where $\varrho_x = 0$ if $(x+\sigma C) \cap M \neq \emptyset$ for every $\sigma \geqq 0$. Then M is called a Chebyshev set (with respect to C) if for every $x \in E$ the set

$$(x+\varrho_x C) \cap M$$

contains exactly one point.

If C is the unit ball of a Banach space, we get the previous definition of a Chebyshev set. We note that if C is a convex body containing o in its interior, and $-y \in \operatorname{int} C$, then $y+C$ produces the same Chebyshev sets as C; thus, we may speak of the Chebyshev sets produced by any convex body.

An inspection of the proofs of (I)–(IV) shows that these results are still valid when we, so to say, allow non-symmetric unit balls.

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