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SOME REMARKS ON CONTINUOUS ORTHOGONAL EXPANSIONS, AND EIGENFUNCTION EXPANSIONS FOR POSITIVE SELF-ADJOINT ELLIPTIC OPERATORS WITH VARIABLE COEFFICIENTS

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0. Introduction.

Let A be a positive self-adjoint realization in the Hilbert space $L_2(\Omega)$ of a formally positive self-adjoint elliptic operator

(0.1)
$$a = a(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \qquad x = (x_1,\ldots,x_n), \quad D = \left(-i\frac{\partial}{\partial x_1},\ldots,-i\frac{\partial}{\partial x_n}\right)$$

with smooth coefficients a_α defined in a domain Ω of R^n. If

$$A = \int_0^\infty \lambda \, dE_\lambda$$

is the corresponding spectral resolution we introduce Riesz means of order α by the formula

$$E_\lambda^{R_\alpha} = \int_0^\lambda (1-\mu/\lambda)^\alpha \, dE_\mu \tag{0.2}$$

and Abel–Laplace means by the formula

$$E_\lambda^L = \int_0^\infty e^{-\mu/\lambda} \, dE_\mu \, . \tag{0.3}$$

We shall study the convergence of $E_\lambda^{R_\alpha} f$ and $E_\lambda^L f$ as $\lambda \to \infty$ when $f \in L_p(\Omega)$, $1 \leqq p \leqq 2$. Our main results read as follows:

1° *For any* $f \in L_p(\Omega)$, $1 \leqq p \leqq 2$, *holds* $E_\lambda^L f(x) \to f(x)$ a.e. (*Abel summability*).

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2° *For any* $f \in L_2(\Omega)$ *holds* $E_\lambda^{R_\alpha} f(x) \to f(x)$ a.e. *for any* $\alpha > 0$ (*Riesz summability*). (An auxiliary rather incomplete result in the case $f \in L_p(\Omega)$, $1 \leqq p \leqq 2$ is also indicated.)

A few comments are necessary. At least if $p = 2$, 1° is strictly speaking a consequence of 2°, in view of known results relating Riesz and Abel summability. However, the point is that we need 1° first for the proof of 2°. Again 1° depends on known estimates for the fundamental solution of the associated "heat" operator $\partial/\partial t - a$. We also want to emphasize that 2° belongs properly to the theory of *continuous orthogonal expansions*. While the theory of *discrete orthogonal expansions* (i.e. the theory of *orthogonal series*) has been given a great attention by many mathematicians (see the books by Kacmarcz–Steinhaus [4] and Alexits [1] as well as the recent survey article by Uljanov [12]), we do not know any single reference dealing with (the summability of) continuous orthogonal expansions. It now turns out that 2° depends essentially on a straight forward extension of results for discrete orthogonal expansions which are due to Kacmarcz and Zygmund (see [1, pp. 101–103]). Indeed our proofs are even somewhat simpler than the ones given in [1]. (Also other similar results can be easily extended to the continuous case, e.g. Kolmogorov's well-known theorem of the convergence of partial sequences (see [1], pp. 111–113), but we shall not enter into details.)

We start in Section 1 by some preliminaries on general means of a function locally of bounded variation. Then we study (Section 2) the summability of general continuous orthogonal expensions, establishing what is necessary for the proof of 2°. Next we specialize (Section 3) to the case of eigenfunction expansions, giving the proof of 1° and completing thus the one of 2°. Finally in Section 4 we discuss briefly Riesz summability in $L_p(\Omega)$, $1 \leqq p \leqq 2$; here our results are most uncomplete.

1. Preliminaries on general means.

Let s_λ be a (possibly vector valued) function locally of bounded variation with $s_\lambda = 0$ if $\lambda \leqq 0$.

Let $\varphi = \varphi(t)$ be a Borel function at least continuous at 0, always normalized by $\varphi(0) = 1$.

We set (whenever the integral exists)

$$s_\lambda^\varphi = \int_0^\infty \varphi(\mu/\lambda)\, ds_\mu \quad \text{(the } \varphi\text{-mean of } s_\lambda) \,. \tag{1.1}$$

If φ has a continuous derivative φ' and s_λ satisfies a proper growth condition, we may integrate by parts and obtain

(1.2) $$s_\lambda^\varphi = -\lambda^{-1}\int_0^\infty \varphi'(\mu/\lambda)\, s_\mu \, d\mu \, .$$

EXAMPLE 1.1. We mention the following special cases

$$\varphi(t) = R_\alpha(t) = \begin{cases} (1-t)^\alpha & \text{if } t \leqq 1 \\ 0 & \text{if } t > 1 \, , \end{cases} \quad \text{(Riesz mean of order } \alpha) \, ,$$

$$\varphi(t) = L(t) = e^{-t} \quad \text{(Abel–Laplace mean)} \, ,$$

$$\varphi(t) = S_\varrho(t) = (1+t)^{-\varrho} \quad \text{(Stieltjes mean of order } \varrho) \, .$$

It is clear what is meant by saying that s_λ is convergent as $\lambda \to \infty$. We say now that s_λ is *φ-summable* if s_λ^φ is convergent as $\lambda \to \infty$ i.e. if, for some s,

$$|s_\lambda^\varphi - s| \to 0 \quad \text{as } \lambda \to \infty \, .$$

We say also that s_λ is *square φ-summable* if, for some s,

$$\lambda^{-1}\int_0^\lambda |s_\mu^\varphi - s|^2 \, d\mu \to 0 \quad \text{as } \lambda \to \infty \, .$$

It is clear that φ-summability implies square φ-summability.

We recall the following result, although strictly speaking it is not needed here.

PROPOSITION 1.1. *If s_λ is convergent then s_λ is φ-summable provided*

(1.3) $$\int_0^\infty |\varphi'(t)| \, dt < \infty \, .$$

PROOF. We write $s_\lambda = s_\lambda^0 + s_\lambda^1$ where

$$s_\lambda^0 = \begin{cases} s_\lambda & \text{if } \lambda < \omega, \\ 0 & \text{if } \lambda \geqq \omega, \end{cases} \qquad s_\lambda^1 = \begin{cases} 0 & \text{if } \lambda < \omega \, , \\ s_\lambda & \text{if } \lambda \geqq \omega \, . \end{cases}$$

It is clear that s_λ^0 converges to 0 as $\lambda \to \infty$; indeed, by (1.3) we have $\varphi(\mu/\lambda) \to 0$ uniformly as $\lambda \to \infty$ in $\mu < \omega$. It sufficies thus to show that $(s_\lambda^1)^\varphi - s$ can be made arbitrarily small. Now by (1.2)

$$(s_\lambda^1)^\varphi - s = -\lambda^{-1}\int_\omega^\infty \varphi'(\mu/\lambda)\, (s_\mu^1 - s) \, d\mu$$

and again by (1.3) the statement readily follows.

COROLLARY 1.1. *If s_λ is convergent then s_λ is R_α-summable for any $\alpha \geqq 0$.*

We need, however, a more precise result relating Riesz means of different order and also square and 'ordinary" summability.

PROPOSITION 1.2. *If s_λ is R_α-summable for some $\alpha > -1$ then s_λ is R_β-summable for any $\beta > \alpha$. If s_λ is square R_α-summable for some $\alpha > -\frac{1}{2}$ then s_λ is R_β-summable for any $\beta > \alpha + \frac{1}{2}$.*

PROOF. The following formula is well-known (see Chandrasekharan-Minakshisundaram [4] p. 3)

$$s_\lambda^{R_\beta} = c_{\alpha\beta}\lambda^{-1}\int_0^\lambda (1-\mu/\lambda)^{\beta-\alpha-1}(\mu/\lambda)^\alpha s_\mu^{R_\alpha}\,d\mu$$

with

$$c_{\alpha\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}.$$

The first part of the proposition follows now as in the proof of the preceeding one. Therefore we may concentrate upon the second part. By Schwarz' inequality we get

$$|s_\alpha^{R_\beta} - s|^2 \leqq c_{\alpha\beta}^2\lambda^{-1}\int_0^\lambda (1-\mu/\lambda)^{2(\beta-\alpha-1)}(\mu/\lambda)^{2\alpha}\,d\mu\;\lambda^{-1}\int_0^\lambda |s_\mu^{R_\alpha} - s|^2\,d\mu\,,$$

where the first integral is finite and independent of λ if $\beta > \alpha + \frac{1}{2}$, $\alpha > -\frac{1}{2}$, and the second one tends to 0 as $\lambda \to \infty$ by assumption. This concludes the proof.

2. Summability of continuous orthogonal expansions.

Let E_λ be any positive spectral family in the Hilbert space $L_2(\Omega)$ where Ω is any domain of R^n (or, more generally, even an abstract measure space). Positive means that $E_\lambda = 0$ if $\lambda \leqq 0$. If $f \in L_2(\Omega)$ we take $s_\lambda = E_\lambda f$ and denote the corresponding φ-means by $s_\lambda^\varphi = E_\lambda^\varphi f$. Under suitable assumptions on E_λ, that will be always fulfilled in the case of eigenfunction expansions—therefore we shall not make them precise—holds

$$E_\lambda^\varphi f(x) = \int_0^\infty \varphi(\mu/\lambda)\,dE_\mu f(x) \quad \text{a.e.}$$

that is, $E_\lambda^\varphi f(x)$ is a.e. the φ-mean of $E_\lambda f(x)$. Also $E_\lambda^\varphi f(x)$ is a measurable function of λ and x. The L_2 norm is, by Parseval's formula, given by

$$\|E_\lambda^\varphi f\|^2 = \int_0^\infty |\varphi(\mu/\lambda)|^2\,d\|E_\mu f\|^2\,.$$

Integrating with respect to $\lambda^{-1} d\lambda$ we get

$$\int_0^\infty \|E_\lambda^\varphi f\|^2 \lambda^{-1}\, d\lambda = \int_0^\infty \int_0^\infty |\varphi(\mu/\lambda)|^2 \|E_\mu f\|^2 \lambda^{-1}\, d\lambda .$$

Applying next Fubini's theorem to each member this yields:

$$\int_\Omega \left(\int_0^\infty |E_\lambda^\varphi f(x)|^2 \lambda^{-1}\, d\lambda \right) dx = \|f\|^2 \int_0^\infty |\varphi(t)|^2 t^{-1}\, dt .$$

It follows that the following result holds, upon which all the following considerations are based.

PROPOSITION 2.1. *Suppose that*

$$\int_0^\infty |\varphi(t)|^2 t^{-1}\, dt < \infty . \tag{2.1}$$

Then for any $f \in L_2(\Omega)$ *holds*

$$\int_0^\infty |E_\lambda^\varphi f(x)|^2 t^{-1}\, dt < \infty \quad \text{a.e.} \tag{2.2}$$

Taking

$$\varphi(t) = L(t) - R_\alpha(t)$$

we see that (2.1) holds if $\alpha > -\frac{1}{2}$. Therefore we have

THEOREM 2.1 (*Kacmarcz*). *Suppose that* $E_\lambda f(x)$ *is* a.e. *square L-summable. Then* $E_\lambda f(x)$ *is* a.e. *square* R_α*-summable for any* $\alpha > -\frac{1}{2}$. *Also* $E_\lambda f(x)$ *is* a.e. R_β*-summable for any* $\beta > 0$.

PROOF. Indeed by (2.2) we get

$$\sup_\lambda \lambda^{-1} \int_0^\lambda |E_\mu^{R_\alpha} f(x) - E_\mu^L f(x)|^2\, d\mu < \infty \quad \text{a.e.}$$

But by assumption

$$\lambda^{-1} \int_0^\lambda |E_\mu^L f(x) - f(x)|^2\, d\mu \to 0 \quad \text{a.e.}$$

Therefore it follows that

$$\sup_\lambda \lambda^{-1} \int_0^\lambda |E_\mu^{R_\alpha} f(x)|^2\, d\mu < \infty \quad \text{a.e.}$$

and, in view of a well-known density argument (Saks' theorem; see Calderòn–Zygmund [3] or Cotlar [5]), this implies

$$\lambda^{-1}\int_0^\lambda |E_\mu^{R_\alpha} f(x) - f(x)|^2 \, d\mu \to 0 \quad \text{a.e.}$$

proving thus the first part of the proposition. The second part follows at once from Proposition 1.2.

REMARK 2.1. A similar result holds with Abel–Laplace means replaced by Stieltjes means of any order $\varrho > 0$ (see Example 1.1).

Next, to give another example of the same technique, taking

$$\varphi(t) = R_\alpha(t) - R_{\alpha_0}(t)$$

we see that (2.1) holds if $\alpha, \alpha_0 > -\frac{1}{2}$. Therefore the same argument yields

THEOREM 2.2 (*Zygmund*). *Suppose that $E_\lambda f(x)$ is* a.e. *R_{α_0}-summable for some $\alpha_0 > -\frac{1}{2}$. Then $E_\lambda f(x)$ is* a.e. *square R_α-summable for any $\alpha > -\frac{1}{2}$. Also $E_\lambda f(x)$ is* a.e. *R_β-summable for any $\beta > 0$.*

3. Summability of eigenfunction expansions.

We now shall prove the results 1° and 2° stated in the Introduction. Thus E_λ will be the spectral family associated with a positive selfadjoint realization of the elliptic operator a (0.1). We may concentrate on proving 1° for then 2° will be an immediate consequence of Theorem 2.1.

We write

$$Ff = F_t f = E_{t^{-1}}^L f, \qquad f \in L_p(\Omega), \quad 1 \leqq p \leqq 2 .$$

(This has obviously a sense even if $p \neq 2$, see Section 4.) Then holds, as is readily seen,

$$\frac{\partial Ff}{\partial t} - aFf = 0 \quad \text{as } t > 0, \qquad Ff = f \quad \text{as } t = 0$$

(at least) in distribution sense; in other words, F is a fundamental solution of the associated "heat" operator $\partial/\partial t - a$. Let now F' be any other fundamental solution and consider the difference $G = F - F'$. Then holds

$$\frac{\partial Gf}{\partial t} - aGf = 0 \quad \text{as } t > 0, \qquad Gf = 0 \quad \text{as } t = 0$$

again in distribution sense. From the hypoellipticity of the operator $\partial/\partial t - a$ (see e.g. Eidelman [6], Chap. II) follows now that

$$G_t f(x) \to 0 \quad \text{a.e.} \quad \text{as} \quad t \to 0 .$$

Therefore the proof of 1° is reduced to shows that

$$(3.1) \qquad F_t'f(x) \to f(x) \quad \text{a.e.} \quad \text{as } t \to 0 .$$

But F' can be choosen (see e.g. Eidelman [6, Chap. I]) of the form

$$(3.1) \qquad (F_t'f)(x) = \int_{\Omega} F_t'(x,y) f(y)\, dy$$

where the kernel satisfies (at least locally) the estimate

$$(3.3) \qquad |F_t'(x,y)| \leqq C t^{-n/m} \exp(-C|x-y|^{m/(m-1)} t^{-1/(m-1)}) ,$$

where C is a constant and m the order of a. It follows readily from (3.2) and (3.3) that

$$(3.4) \qquad |F_t'f(x)| \leqq C(\Lambda f^p(x))^{1/p}$$

with a different C, where

$$\Lambda g(x) = \sup_r r^{-n} \int_{|x-y|\leqq r} |g(y)|\, dy$$

(the Hardy–Littlewood maximal operator). By the Hardy–Littlewood maximal theorem (see [3] or [5]), the inequality (3.4) implies that

$$\sup_t |F_t'f(x)| < \infty \quad \text{a.e.}$$

from which (3.1) follows by a density argument, as in the proof of Theorem 2.1. Thus we have established 1° and 2° of the Introduction.

REMARK 3.1. A result similar to 1° holds probably also with Abel–Laplace means replaced by Stieltjes means of sufficiently large order ϱ (depending on m).

4. Remarks on Riesz summability in the L_p case.

First we say a few words about the definition of Riesz and Abel–Laplace means (as well as of other means) when $p=2$. By a form of "Sobolev's imbedding theorem" we see that $f \in L_p(\Omega)$ implies that $f \in D(A^{-k})$, the domain of the k'th (fractional) power of A, provided $k > (1/p - \frac{1}{2})n/m$. But in $D(A^{-k})$ integrals as (0.2) and (0.3) make sense so that $E_\lambda^{R_\alpha} f$ and $E_\lambda^{L} f$ are now well-defined, indeed they will even belong to $L_2(\Omega)$. (See also Nilsson [9].)

The same idea combined with the technique of Section 2 conveniently adapted leads also to the following estimate for Riesz means

$$(4.1) \qquad \int_0^\infty \left(\frac{|E_\lambda^{R_\alpha} f(x)|}{\lambda^k}\right)^2 \frac{d\lambda}{\lambda} < \infty \quad \text{a.e.}, \quad f \in L_p(\Omega)\,,$$

$$1 \leqq p < 2, \quad k > (1/p - \tfrac{1}{2})\, n/m, \quad \alpha > -\tfrac{1}{2}\,.$$

However, we shall omit the details since this is, as we see below, a very crude result.

It is about only in the quite simple special case of constant coefficients that any more precise results are known (see Bergendal [2] and the references given there; see also Peetre [10]). Let us here just consider the still more special case $\Omega = R^n$ or $\Omega = T^n$ (= the n-dimensional torus), i.e. *non-spherical* summability of multiple Fourier integrals and Fourier series. Then holds the following formula (cf. (3.4)).

$$(4.2) \quad |E_\lambda^{R_\alpha} f(x)| \leqq C\big(\Lambda f^p(x)\big)^{1/p}, \qquad f \in L_p(R^n), \quad 1 \leqq p < 2, \quad \alpha > (n-1)/p\,.$$

Therefore, as in Section 3, holds

$$(4.3) \quad E_\lambda^{R_\alpha} f(x) \to f(x) \quad \text{a.e.}, \qquad f \in L_p(R^n), \quad 1 \leqq p < 2, \; \alpha > (n-1)/p\,.$$

In (4.2) the bound $\alpha > (n-1)/p$ is about the best one (see [10]) but in (4.3) the bound $\alpha > (n-1)/p$ can be replaced by a better one, namely $\alpha > (1/p - \frac{1}{2})\,(n-1)$ (see Stein [11]). (Note that as in (4.1) appears the factor $1/p - \frac{1}{2}$.) It would be very interesting to see to what extent these results can be generalized to the case of variable coefficients and arbitrary Ω.

We conclude by mentioning the work of Levitan [8] who obtains quite complete results in the special case $a = -\Delta + q(x)$, $\Omega = R^3$ (thus $n = 3$). His methods are quite complicated and seem to be bounded, essentially, to that case.

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