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ON INFINITELY DIVISIBLE ONE-SIDED DISTRIBUTIONS

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1. Introduction.

The characteristic function $f(x)$ of the distribution function $F(u)$ is for all real x defined by

$$f(x) = \int_{-\infty}^{\infty} e^{ixu}\, dF(u) . \tag{1.1}$$

The characteristic function $f(x)$ and its distribution are called infinitely divisible if to every integer $n \geqq 1$ there corresponds a characteristic function $g_n(x)$ such that $f(x) = (g_n(x))^n$. In order for $f(x)$ to be infinitely divisible it is necessary and sufficient that it have the following canonical representation (Lévy):

$$\begin{aligned} \log f(x) = iax - \tfrac{1}{2}\sigma^2 x^2 + \int_{-\infty}^{0} \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right) dM(u) \; + \\ + \int_{0}^{\infty} \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right) dN(u) , \end{aligned} \tag{1.2}$$

where

(i) $M(u)$ and $N(u)$ are defined and non-decreasing on $(-\infty,0)$ and $(0,+\infty)$, respectively,

(ii) $M(-\infty) = N(+\infty) = 0$,

(iii) $\int_{-\varepsilon}^{0} u^2 dM(u)$ and $\int_{0}^{\varepsilon} u^2 dN(u)$ are finite for every $\varepsilon > 0$,

(iv) the constants a and σ are real.

Furthermore, the representation (1.2) is unique.

The characteristic function $f(x)$ defined by (1.1) is called analytic if there exists an analytic function $\varphi(z)$, $z = x + iy$, regular at $z = 0$, such that $f(x) = \varphi(x)$ on $-\Delta < x < \Delta$ for some $\Delta > 0$. Then it can be shown

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that $\varphi(z)$ is regular in a horisontal strip $-a<y<b$, a, $b>0$, containing the real axis and that $\varphi(z)$ in this strip has the representation

$$\varphi(z) = \int_{-\infty}^{\infty} e^{izu}\, dF(u), \qquad -a<y<b\,. \tag{1.3}$$

The integral in (1.3) is absolutely convergent in the strip. On account of (1.1) and (1.3) we may extend the definition of $f(x)$ to complex values of the independent variable, setting

$$f(z) = \int_{-\infty}^{\infty} e^{izu}\, dF(u), \qquad -a<y<b\,.$$

If further $f(x)$ is infinitely divisible, it can be shown (see for instance Lukacs [4, p. 189]) that the Lévy representation (1.2) is valid in the whole strip of regularity of $f(z)$.

In this paper we shall consider characteristic functions which are boundary values of analytic functions. This class of functions contains the analytic characteristic functions but is more extensive. Our starting point is the following theorem due to Marcinkiewicz [5]:

THEOREM 1. *Let $f(x)$ be the characteristic function of the distribution function $F(u)$. Then $f(x)$ is the boundary value on $-\Delta<x<\Delta$ of an analytic function $\varphi(z)$, $z=x+iy$, regular in $-\Delta<x<\Delta$, $0<y<b$ and continuous on $-\Delta<x<\Delta$, $0\leqq y<b$, if and only if*

$$\int_{-\infty}^{\infty} e^{-yu}\, dF(u) < \infty \qquad \textit{for} \quad 0\leqq y<b\,.$$

If $f(x)$ is the boundary value of $\varphi(z)$ in the sense of Theorem 1 it is easily seen that $\varphi(z)$ is unique and regular in the whole strip $0<y<b$ with the representation

$$\varphi(z) = \int_{-\infty}^{\infty} e^{izu}\, dF(u), \qquad 0\leqq y<b\,,$$

and that $f(x)=\varphi(x)$ for all real x. The definition of $f(z)$ can thus be extended to complex z by means of

$$f(z) = \int_{-\infty}^{\infty} e^{izu}\, dF(u), \qquad 0\leqq y<b\,, \tag{1.4}$$

where the integral converges absolutely in the strip. Thus $f(z)$ is regular in the interior of the strip but not necessarily on the real axis. Let us denote by (M) the class of characteristic functions for which a representation of the form (1.4) is valid for some $b>0$. The class (M) and the class of characteristic functions that are boundary values of analytic functions in the sense of Theorem 1 are obviously identical. Incidentally, we mention the following property of (M): if $f_1(z)$ and $f_2(z)$ both belong to (M) and $f_1(x)=f_2(x)$ on a set of positive Lebesgue measure situated on the real axis, then $f_1(z) \equiv f_2(z)$.

While the class of analytic characteristic functions has been extensively studied, the class (M) has been much less often treated. Many theorems on analytic characteristic functions are still true for the class (M). It was mentioned that if a characteristic function is analytic and infinitely divisible, then the Lévy representation (1.2) is valid in the whole strip of regularity. In Section 2 we shall prove that this theorem still holds for the class (M). In Section 3 we make an application of this result giving a new proof of a theorem due to Baxter and Shapiro [1] on one-sided infinitely divisible distributions.

2. Infinitely divisible characteristic functions belonging to (M).

In this section we shall prove the following theorem.

THEOREM 2. *Let $f(z)$ belong to (M) and be regular in $0<y<b$ and let $f(z)$ be infinitely divisible. Then the Lévy canonical representation is valid in the domain $0 \leqq y < b$.*

The proof of this theorem will be similar to that of the corresponding theorem for analytic characteristic functions. The proof is based on two auxiliary theorems which are known in the analytic case.

By $F=F_1 * F_2$ we denote the convolution of the two distribution functions F_1 and F_2, that is,

$$F(u) = \int_{-\infty}^{\infty} F_1(u-v)\, dF_2(v) \,.$$

Then $F(u)$ is also a distribution function and $f(x)=f_1(x)f_2(x)$ for real values of x, where f, f_1 and f_2 are the characteristic functions of F, F_1 and F_2, respectively.

LEMMA 2.1. *Let F, F_1 and F_2 be distribution functions, $F=F_1 * F_2$, and y a real number. If*

$$\int_{-\infty}^{\infty} e^{-yu}\, dF(u) < \infty \,,$$

then

$$\int_{-\infty}^{\infty} e^{-yu}\, dF_k(u) < \infty, \qquad k=1,2\,,$$

and

$$\int_{-\infty}^{\infty} e^{-yu}\, dF(u) = \int_{-\infty}^{\infty} e^{-yu}\, dF_1(u) \int_{-\infty}^{\infty} e^{-yu}\, dF_2(u)\,.$$

For a proof of Lemma 2.1 see Loève [3, p. 214].

LEMMA 2.2. *Let $f(z)$ belong to (M) and be regular in $0<y<b$. If $f(x)=f_1(x)f_2(x)$ for all real values of x where f_1 and f_2 are characteristic functions, then $f_k \in (M)$, $k=1,2$, and f_k has at least the same strip of regularity as f. Further $f(z)=f_1(z)f_2(z)$ in the strip $0 \leqq y<b$.*

This lemma corresponds to a result due to Raikov [7] in the analytic case. Denote by F, F_1 and F_2 the distribution functions of f, f_1 and f_2, respectively. Then $F=F_1 * F_2$. Since $f(z) \in (M)$, the integral

$$\int_{-\infty}^{\infty} e^{-yu}\, dF(u) < \infty \quad \text{for} \quad 0 \leqq y<b\,.$$

From Lemma 2.1 it follows that

$$\int_{-\infty}^{\infty} e^{-yu}\, dF_k(u) < \infty, \qquad k = 1,2, \quad 0 \leqq y < b\,.$$

Thus f_k belongs to (M) and has at least the same strip of regularity as f. Further, we get from Lemma 2.1 that

$$f(iy) = f_1(iy)\, f_2(iy) \qquad \text{for} \quad 0 \leqq y<b\,.$$

Hence, by analytic continuation, $f(z)=f_1(z)f_2(z)$ in the same strip.

LEMMA 2.3. *Let $f(z)$ belong (M) and be regular in $0<y<b$. If $f(z)$ is infinitely divisible, then $f(z)$ has no zeros in the domain $0 \leqq y<b$.*

This lemma is also well known in the analytic case (see for instance Lukacs [4, p. 187]). Since an infinitely divisible characteristic function has no real zeros it is sufficient to prove that $f(z)$ has no zeros in the interior of the regularity strip. By the definition of an infinitely divisible characteristic function there corresponds to every integer $n \geqq 1$ a characteristic function $g_n(x)$ such that $f(x)=(g_n(x))^n$ for all real values of x. From Lemma 2.2 it follows that $g_n \in (M)$ and that $g_n(z)$ is regular in $0<y<b$. If, however, $f(z)$ has a zero at $z=z_0$, $0<y_0<b$, then $g_n(z)$ can not be regular at $z=z_0$ for a sufficiently large n. This is a contradiction and thus $f(z)$ has no zeros in the regularity strip.

PROOF OF THEOREM 2. From Lemma 2.3 it follows that $\log f(z)$ is regular in $0<y<b$ and continuous in $0 \leqq y<b$. Let $f(x)$ have the Lévy representation (1.2) for real values of x. Set

$$(2.1) \qquad \varphi_1(z) = \int_0^\infty \left(e^{izu}-1-\frac{izu}{1+u^2}\right) dN(u), \qquad y \geqq 0 .$$

As is easily seen $\varphi_1(z)$ is regular in the upper half plane $y>0$ and continuous in $y \geqq 0$. Consider

$$(2.2) \qquad \psi(z) = \log f(z) - iaz - \tfrac{1}{2}\sigma^2 z^2 - \varphi_1(z) .$$

The function $\psi(z)$ is regular in the strip $0<y<b$ and continuous in $0 \leqq y < b$. Further, let

$$(2.3) \qquad \varphi_2(z) = \int_{-\infty}^0 \left(e^{izu}-1-\frac{izu}{1+u^2}\right) dM(u) , \qquad y \leqq 0 .$$

This function is regular in the lower half plane $y<0$ and continuous in $y \leqq 0$.

From (1.2) it follows that $\psi(x)=\varphi_2(x)$ for real values of x. Applying a well-known theorem on analytic continuation we find that $\psi(z)$ is the analytic continuation of $\varphi_2(z)$ and that $\varphi_2(z)$ is regular in $y<b$. Thus $\varphi_2''(z)$ is regular in $y<b$. If $y<0$ it is easily seen from (2.3) that

$$(2.4) \qquad \varphi_2''(z) = -\int_{-\infty}^0 e^{izu}\, u^2\, dM(u), \qquad y<0 .$$

Defining the non-decreasing function $L(u)$ by

$$L(u) = \int_0^u v^2\, dM(v), \qquad u<0 ,$$

we can write (2.4) in the form

$$\varphi_2''(z) = -\int_{-\infty}^0 e^{izu}\, dL(u), \qquad y<0 .$$

It has already been proved, however, that $\varphi_2''(z)$ is regular in $y<b$. Since $L(u)$ is non-decreasing it follows from a property of the Laplace integral (see for instance Widder [8, p. 58]) that

$$\int_{-\infty}^0 e^{izu}\, dL(u) = \int_{-\infty}^0 e^{izu}\, u^2\, dM(u)$$

is absolutely convergent in $y<b$. It is then easily seen that the representation (2.3) of $\varphi_2(z)$ is valid in the half plane $y<b$. Since $\psi(z)=\varphi_2(z)$ in the strip $0\leqq y<b$, we find from (2.1), (2.2) and (2.3) that $\log f(z)$ has the Lévy canonical representation in this strip.

3. On infinitely divisible one-sided distribution functions.

A distribution function $F(u)$ is called bounded to the left if there exists a finite h such that for every $\varepsilon>0$,

$$F(h-\varepsilon) = 0 \quad \text{and} \quad F(h+\varepsilon) > 0 .$$

Following Pólya [6] we shall use the notation $\operatorname{lext}[F]=h$. Boundedness to the right is defined in a similar way. A distribution function which is bounded to the left or to the right is called one-sided. If $F(u)$ is bounded to the left by h then $\int_h^\infty e^{izu}\,dF(u)$ converges absolutely in $y\geqq 0$ and represents a regular function in $y>0$. Obviously the characteristic function f corresponding to F belongs to (M) and we may set

$$f(z) = \int_h^\infty e^{izu}\,dF(u), \qquad y\geqq 0 ,$$

where the strip of regularity of $f(z)$ is at least the upper half plane $y>0$.

It is known that an infinitely divisible distribution function can not be bounded both to the left and to the right except when it is degenerate (see Lukacs [3, p. 188] or, for a different proof, Chatterjee and Pakshirajan [2]). There exist, however, one-sided infinitely divisible distributions, for instance the Poisson distribution, the Γ-distribution and the stable distribution with the characteristic function.

$$\exp\{-|x|^{\frac{1}{2}}(1-i\operatorname{sign}x)\} \tag{3.1}$$

and the frequency function

$$\begin{cases} 0 & \text{for } u<0 \\ (2\pi)^{-\frac{1}{2}}\,u^{-\frac{3}{2}}\,e^{-1/(2u)} & \text{for } u>0 . \end{cases}$$

Since an explicit determination of the distribution function or the frequency function corresponding to an infinitely divisible characteristic function can only seldom be obtained it is of interest to give conditions which assure the one-sidedness of the distribution function and which are expressed by quantities and functions present in the canonical representation of the characteristic function. The following theorem yields the desired information; it is, in a slightly different form, due to Baxter and Shapiro [1].

THEOREM 3. *Let $f(x)$ be the characteristic function of the infinitely divisible distribution function $F(u)$ with the canonical representation* (1.2). *In order that $F(u)$ be bounded to the left it is necessary and sufficient that*

(i) $\sigma^2 = 0$,
(ii) $M(u) = 0$ for $u < 0$,
(iii) $\int_0^1 u \, dN(u) < \infty$.

If the conditions (i), (ii) *and* (iii) *are satisfied, then*

(iv) $\text{lext}[F] = a - \int_0^\infty (u/(1+u^2)) \, dN(u)$.

REMARK 1. A theorem on boundedness to the right can be stated in a corresponding way.

REMARK 2. Instead of (iii) Baxter and Shapiro use the condition $\int_0^1 |N(u)| \, du < \infty$. These two conditions are equivalent.

REMARK 3. Baxter and Shapiro only proved that $\text{lext}[F] \geqq$ the right hand side of (iv).

The proof of Baxter and Shapiro is based on a theorem, due to Gnedenko and Kolmogorov, in the theory of limits of sums of independent infinitesimal random variables. In this section we shall give a new proof of Theorem 3. In this proof Theorem 2 is fundamental.

Before proceeding to the proof we state some auxiliary theorems which will be needed later.

LEMMA 3.1. *Let the characteristic function $f(z)$ belong to (M) and be regular in the upper half plane $y > 0$. A necessary and sufficient condition for the distribution function $F(u)$ of $f(z)$ to be bounded to the left is that*

$$\overline{\lim_{y \to +\infty}} \frac{1}{y} \log f(iy) < +\infty .$$

If this condition is satisfied, then

$$\text{lext}[F] = - \overline{\lim_{y \to +\infty}} \frac{1}{y} \log f(iy) .$$

A corresponding theorem was proved by Pólya [6] for distribution functions bounded to the left and to the right. The characteristic functions are then entire functions. Pólya's method of proof can be used practically without any change in our case.

REMARK. It can be shown that if F is bounded to the left then

$$\lim_{y\to+\infty} \frac{1}{y} \log f(iy)$$

exists and is finite. Thus $\overline{\lim}$ in Lemma 3.1 can be replaced by lim.

The following lemma is elementary and is stated without proof.

LEMMA 3.2. *If x is real, then*

(a) $e^x - 1 - x \geqq 0$ *for all* x,
(b) $e^x - 1 - (x/(1+u^2)) \geqq \frac{1}{2}x^2$ *for* $x \geqq 0$, $-\infty < u < +\infty$,
(c) $(e^{-x} - 1 + x)/x$ *is positive and non-decreasing in* $(0, +\infty)$.

PROOF OF THEOREM 3. We recall that the functions $M(u)$ and $N(u)$ in the Lévy representation are defined and non-decreasing in $(-\infty, 0)$ and $(0, +\infty)$, respectively, that $M(-\infty) = N(+\infty) = 0$, and that

$$\int_{-\varepsilon}^{0} u^2\, dM(u) \quad \text{and} \quad \int_{0}^{\varepsilon} u^2\, dN(u)$$

are finite for every $\varepsilon > 0$.

A. *Proof of necessity.* (i) $\sigma^2 = 0$. Suppose, on the contrary, that $\sigma^2 > 0$. From (1.2) it follows that

$$f(x) = \exp\{-\tfrac{1}{2}\sigma^2 x^2\} g(x) ,$$

where $g(x)$ is the characteristic function of a distribution function $G(u)$ and $\exp\{-\frac{1}{2}\sigma^2 x^2\}$ is the characteristic function of the normal distribution function $\Phi(u/\sigma)$. Then $F(u) = \Phi(u/\sigma) * G(u)$, $F(u)$ is absolutely continuous and

$$F'(u) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}\sigma^{-2}(u-v)^2\} dG(v) > 0$$

for all u. Thus $F(u)$ cannot be bounded to the left, which is contradictory to the assumption.

(ii) $M(u) = 0$ for $u < 0$. Since $F(u)$ is supposed bounded to the left its characteristic function $f(z)$ belongs to (M) and is regular in the upper half plane. By Lemma 3.1

$$\overline{\lim_{y\to+\infty}} \frac{1}{y} \log f(iy) < +\infty . \tag{3.2}$$

From Theorem 2 it follows that the Lévy representation of $f(z)$ is valid in the upper half plane. Thus on account of (i)

$$
\begin{aligned}
\log f(iy) &= -ay + \int_{-\infty}^{0}\left(e^{-yu}-1+\frac{yu}{1+u^2}\right)dM(u) + \\
&\quad + \int_{0}^{\infty}\left(e^{-yu}-1+\frac{yu}{1+u^2}\right)dN(u) \qquad (3.3)\\
&= -ay+J_1+J_2, \qquad y \geqq 0,
\end{aligned}
$$

where J_1 and J_2 denote the first and second integral, respectively, of the right hand side of the first equality.

Suppose that $M(u)$ is not constant for $u<0$. We shall show that this is contradictory to (3.2). Since $M(u)$ is not constant it has a point of increase ξ, $\xi<0$, such that $M(\xi+\varepsilon)-M(\xi-\varepsilon)>0$ for any $\varepsilon>0$. We choose a fixed $\varepsilon>0$ such that $\xi+\varepsilon<0$. Then

$$
p = M(\xi+\varepsilon)-M(\xi-\varepsilon) > 0 . \tag{3.4}
$$

Since $yu \leqq 0$ in J_1 we obtain from (3.3), Lemma 3.2 (b), and (3.4), that

$$
J_1 \geqq \int_{\xi-\varepsilon}^{\xi+\varepsilon} \tfrac{1}{2}y^2 u^2\, dM(u) \geqq \tfrac{1}{2}p(\xi+\varepsilon)^2 y^2 . \tag{3.5}
$$

We write J_2 in the following way:

$$
\begin{aligned}
J_2 &= \int_0^1 (e^{-yu}-1+yu)\, dN(u) - y\int_0^1 \frac{u^3}{1+u^2}\, dN(u) + \\
&\quad + \int_1^{\infty}\left(e^{-yu}-1+\frac{yu}{1+u^2}\right)dN(u) . \qquad (3.6)
\end{aligned}
$$

Applying Lemma 3.2 (a) we get

$$
J_2 \geqq -y\int_0^1 \frac{u^3}{1+u^2}\, dN(u) - \int_1^{\infty} dN(u) . \tag{3.7}
$$

Hence from (3.3), (3.5) and (3.7)

$$
\frac{1}{y}\log f(iy) \geqq -a + \tfrac{1}{2}p(\xi+\varepsilon)^2 y - \int_0^1 \frac{u^3}{1+u^2}\, dN(u) - \frac{1}{y}\int_1^{\infty} dN(u), \qquad y>0 ,
$$

or

$$
\overline{\lim_{y\to+\infty}}\, \frac{1}{y}\log f(iy) = +\infty .
$$

This contradicts (3.2) and thus $M(u)=0$ for $u<0$.

(iii) $\int_0^1 u dN(u) < +\infty$. Let us suppose that

$$\int_0^1 u dN(u) = +\infty . \tag{3.8}$$

We shall show that (3.8) and (3.2) are contradictory. Since (i) and (ii) necessarily hold we get from (3.3) and (3.6)

$$\log f(iy) = -ay + \int_0^1 (e^{-yu} - 1 + yu)\, dN(u) - y\int_0^1 \frac{u^3}{1+u^2} dN(u) + $$
$$+ \int_1^\infty \left(e^{-yu} - 1 + \frac{yu}{1+u^2}\right) dN(u)$$
$$\geqq -ay + yR(y) - y\int_0^1 \frac{u^3}{1+u^2} dN(u) - \int_1^\infty dN(u) ,$$

where

$$R(y) = \int_0^1 \frac{e^{-yu} - 1 + yu}{y} dN(u)$$

and $y>0$. Thus if we can show that $R(y) \to +\infty$ as $y \to +\infty$, then we have a contradiction of (3.2), and the necessity of (iii) is proved.

Let $H(u)$ be defined by

$$H(u) = \int_1^u u\, dN(u), \qquad u>0 .$$

Then $H(u)$ is non-decreasing and by (3.8)

$$H(+0) = -\infty . \tag{3.9}$$

Further, on account of Lemma 3.2 (c) and for $y>1$,

$$R(y) = \int_0^1 \frac{e^{-yu} - 1 + yu}{yu} dH(u) \geqq \int_1^y \frac{e^{-v} - 1 + v}{v} dH(vy^{-1})$$
$$\geqq e^{-1}\big(H(1) - H(y^{-1})\big) .$$

From (3.9) it follows that $R(y) \to +\infty$ as $y \to +\infty$ and the necessity of (iii) is proved.

B. *Proof of sufficiency.* Let the conditions (i), (ii) and (iii) of Theorem

3 be satisfied. Then for real x the function $\log f(x)$ has the Lévy representation

$$\log f(x) = iax + \int_0^\infty \left(e^{ixu} - 1 - \frac{ixu}{1+u^2}\right) dN(u) .$$

Since the integral, with x replaced by z, is a regular function in the upper half plane $y > 0$ and continuous in $y \geqq 0$ we find that $f(z)$ belongs to (M) and is regular in $y > 0$. From Theorem 2 it follows that the Lévy representation is valid in $y \geqq 0$. Thus

$$\log f(iy) = -ay + \int_0^\infty \left(e^{-yu} - 1 + \frac{yu}{1+u^2}\right) dN(u), \qquad y \geqq 0 .$$

By (iii) we are allowed to write

$$\frac{1}{y}\log f(iy) = -a + \int_0^\infty \frac{u}{1+u^2} dN(u) + Q(y), \qquad y > 0 ,$$

where

$$Q(y) = \int_0^\infty \frac{e^{-yu} - 1}{y} dN(u) .$$

Applying the condition (iii) once more, we find easily that $\lim_{y\to+\infty} Q(y) = 0$. Thus

$$\lim_{y\to+\infty} \frac{1}{y}\log f(iy) = -a + \int_0^\infty \frac{u}{1+u^2} dN(u) < +\infty .$$

From Lemma 3.1 it follows that the distribution function $F(u)$ is bounded to the left and that

$$\operatorname{lext}[F] = a - \int_0^\infty \frac{u}{1+u^2} dN(u) ,$$

and the sufficiency of (i), (ii) and (iii) is proved.

Finally we make an application of Theorem 3. The characteristic functions of the stable distributions have the representation (Loève [3, p. 327])

$$\log f(x) = i\alpha x - b|x|^\gamma \left\{1 + ic\frac{x}{|x|}\,\omega(\gamma, x)\right\}$$

with real x, $-\infty < \alpha < \infty$, $b \geqq 0$, $0 < \gamma \leqq 2$, $-1 \leqq c \leqq 1$, and

$$\omega(\gamma,x) = \begin{cases} \operatorname{tg} \frac{1}{2}\pi\gamma & \text{if } \gamma \neq 1 , \\ 2\pi^{-1} \log|x| & \text{if } \gamma = 1 . \end{cases}$$

The stable distributions are infinitely divisible. Which stable distributions are bounded to the left? If $0<\gamma<2$ we know from the theory of the stable distributions that

$$M'(u) = \beta |u|^{-(1+\gamma)}, \qquad N'(u) = \beta' u^{-(1+\gamma)} ,$$

where $\beta \geqq 0$ and $\beta' \geqq 0$ are constants, $\beta+\beta'>0$ and $c=(\beta-\beta')/(\beta+\beta')$. Suppose that $f(x)$ is the characteristic function of a stable distribution function bounded to the left. Then $0<\gamma<2$ since $\gamma=2$ corresponds to the normal distribution. From Theorem 3 it follows that $\beta=0$, thus $c=-1$, and $0<\gamma<1$. Hence

$$\log f(x) = i\alpha x - b\,|x|^{\gamma}\,(1-i \operatorname{tg} \tfrac{1}{2}\pi\gamma \operatorname{sign} x), \qquad 0<\gamma<1 . \tag{3.10}$$

Conversely it is easily found that if $f(x)$ has the representation (3.10) the conditions of Theorem 3 are satisfied. Thus we have proved that in order for a stable distribution function to be bounded to the left it is necessary and sufficient that its characteristic function have the representation (3.10). The characteristic function (3.1) is an example of this kind.

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