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ON GENERAL TAUBERIAN REMAINDER THEOREMS

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Introduction.

In this paper we will consider two general Tauberian remainder theorems of the following kind:

Let K belong to some class of integrable functions and φ be a bounded measurable function satisfying a Tauberian condition. Furthermore suppose H is a non-increasing function such that $\lim_{x\to\infty} H(x)=0$. *Then*

$$K*\varphi(x) = \int_{-\infty}^{\infty} K(x-t)\,\varphi(t)\,dt = O(H(x)), \qquad x\to\infty ,$$

implies an estimation of φ,

$$\varphi(x) = O(H_1(x)), \qquad x\to\infty ,$$

for some non-increasing function H_1, with $\lim_{x\to\infty} H_1(x)=0$.

As a consequence of Wiener's general Tauberian theorem [8, p. 25] a necessary condition for the existence of such a function H_1 is that K fulfills the Wiener condition, that is the Fourier transform

$$\hat{K}(t) = \int_{-\infty}^{\infty} \exp(-ixt)\,K(x)\,dx$$

must be different from zero for all real t. Here the class of functions will be further restricted by conditions on the Fourier transform of K.

Problems of this kind were treated first by Beurling [1, p. 22] and later by Lyttkens [6], [7] and Ganelius [2], [3]. The results in this paper will generalize those of Beurling and also generalize and even sharpen some of the results of Lyttkens. It also includes most of Ganelius' results in [2].

I will use non-decreasing, submultiplicative functions p defined for positive arguments, which I for convenience suppose equal to one for negative arguments. Thus

$$p(x) \geqq p(0) = 1, \qquad p(x+y) \leqq p(x)p(y), \qquad p(\omega x) \geqq p(x) \quad \text{if } \omega \geqq 1 .$$

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Such functions are sometimes called weight-functions, cf. [1, p. 9]. Some interesting examples are p_1, p_2 and p_3 defined for positive arguments by

$$\begin{aligned} p_1(x) &= (\log(e+x))^q, & q &\geqq 0\,, \\ p_2(x) &= (1+x)^q, & q &\geqq 0\,, \\ p_3(x) &= \exp(qx), & q &\geqq 0\,, \end{aligned}$$

and naturally also their products. I will also use the fact that for each such function there exists an m such that

$$p(x) = O(\exp(mx)), \quad x \to \infty\,.$$

In the two theorems I first use a Tauberian condition which is strong and easy to work with and such that it doesn't hide the idea of the proofs. Later in section 4 this condition will be weakened.

We only attack Tauberian problems for slowly decreasing remainders, since otherwise we must use more specific properties of the kernels than those considered here, to get best possible results in the interesting special cases. For some results of this kind see e.g. Ganelius [3], [4].

1. A lemma.

The proofs of the theorems are based on a lemma, which is a small modification of an inequality used by Ganelius in proving Tauberian theorems. (Cf. [2, p. 9] or [3, p. 214]. A proof of this inequality in the periodic case is published in [5].) Since the modifications are essential for my proofs, I include a proof of the lemma.

LEMMA. *If u is an integrable function, then there exists a numerical constant C such that*

$$\sup_x |u(x)| \leqq C\cdot\left\{\inf_{x\leqq t\leqq x+V^{-1}} -(u(t)-u(x)) + \sup_\tau \left|\int_{-V}^{V} \exp(i\xi\tau)(1-|\xi|V^{-1})\hat{u}(\xi)\,d\xi\right|\right\}$$

for all positive V.

PROOF. We will use the fact that

$$G(x) = \int_{-\infty}^{\infty} u(x-t)\,\delta_V(t)\,dt = \int_{-\infty}^{\infty} \exp(ixt)\,\Delta_V(t)\,\hat{u}(t)\,dt$$

with

$$\delta_V(t) = (2\pi V)^{-1}(\tfrac{1}{2}t)^{-2}(\sin\tfrac{1}{2}tV)^2$$

and

$$\Delta_V(t) = \begin{cases} (1-|t|V^{-1})(2\pi)^{-1} & \text{when } |t|<V\,, \\ 0 & \text{when } |t|\geqq V\,, \end{cases}$$

which is true since $\hat{\delta}_V = 2\pi\Delta_V$.

Suppose that $\sup_t |u(t)| = S = \sup_t u(t)$ (The case $S = -\sup_t u(t)$ can be treated in an analogous way). Then for an arbitrary $\varepsilon > 0$ there exists an R such that $u(R) > S - \varepsilon$.

Write

$$G(R+r) = \int_{-r}^{r} \delta_V(t)\, u(R+r-t)\, dt + \int_{|t|\geqq r} \delta_V(t)\, u(R+r-t)\, dt$$

with $r = 16(\pi V)^{-1}$. Then it is easy to see that

$$G(R+r) \geqq \left\{S - \varepsilon + \inf_{-r\leqq t\leqq r} \big(u(R+r-t) - u(R)\big)\right\}\tfrac{3}{4} - \tfrac{1}{4}S\,,$$

if not

$$S - \varepsilon + \inf_{-r\leqq t\leqq r} \big(u(R+r-t) - u(R)\big) \leqq 0\,,$$

When the last inequality is true, then

$$S \leqq \varepsilon - \inf_{-r\leqq t\leqq r} \big(u(R+r-t) - u(R)\big)\,,$$

and otherwise

$$S \leqq \tfrac{3}{2}\varepsilon - \tfrac{3}{2}\cdot \inf_{-r\leqq t\leqq r} \big(u(R+r-t) - u(R)\big) + 2G(R+r)\,.$$

Thus in both cases

$$S \leqq \tfrac{3}{2}\varepsilon - \tfrac{3}{2}\cdot \inf_{x\leqq t\leqq x+2r} \big(u(t) - u(x)\big) + \sup_{\tau} \left| \int_{-V}^{V} \exp(i\xi\tau)\,(1 - |\xi|V^{-1})\, \hat{u}(\xi)\, d\xi \right|,$$

and since ε is arbitrary the lemma is true if, for example, $C = 18$.

2. The first theorem.

Let us start by introducing two classes of functions T and $E(P_0, \alpha, \beta)$, defined as follows.

T: all bounded an measurable functions φ, for which there exists a constant C such that the function defined by $\varphi(x) + Cx$ is non-decreasing.

$E(P_0, a, \beta)$: all integrable functions K, which fulfills the Wiener condition, and for which the function g, such that

$$g(t) = \hat{K}(t)^{-1}\,,$$

can be analytically continued in a strip $-\alpha < \operatorname{Im} t < \beta$, in the complex t-plane including the real axis, and where

$$(2.1) \qquad |g(t)| \leqq \mathrm{const}\cdot P_0(|t|)$$

for some weight-function P_0 and all t in the strip.

Let R_1 be the inverse function of that given by $x^{\frac{3}{2}}P_0(x)$ for all positive values of x. If now p is a weight-function with

$$\overline{\lim_{x\to\infty}}\, x^{-1}\log p(x) = \gamma\,,$$

we have the following theorem.

THEOREM 1. *If* $K \in E(P_0,\alpha,\beta)$ *with* $\beta>\gamma$ *and if* $\varphi\in T$ *then*

$$K*\varphi(x) = O\big(p(x)^{-1}\big), \quad x\to\infty\,, \tag{2.2}$$

implies

$$\varphi(x) = O\big(R_1(p(x))^{-1}\big), \quad x\to\infty\,.$$

PROOF. Let y be an arbitrary positive number, and apply the lemma to the function u given by $u(x)=\exp\big(-\tfrac{1}{2}(x-y)^2\big)\,\varphi(x)$. For convenience let C stand for positive numerical constants independent of y. Since $|u(y)|\leqq \sup_x|u(x)|$ we have by the lemma

$$|\varphi(y)| \leqq C\cdot\left\{-\inf_{x\leqq t\leqq x+V^{-1}}\big(u(t)-u(x)\big)+\sup_\tau\left|\int_{-V}^{V}\exp(i\xi\tau)(1-|\xi|V^{-1})\hat{u}(\xi)\,d\xi\right|\right\}$$

for all positive values of V. Writing

$$u(t)-u(x) = \exp\big(-\tfrac{1}{2}(t-y)^2\big)\,\{\varphi(t)-\varphi(x)\} + \\ +\ \varphi(x)\{\exp\big(-\tfrac{1}{2}(t-y)^2\big)-\exp\big(-\tfrac{1}{2}(x-y)^2\big)\}$$

we see that for $t>x$

$$u(t)-u(x) \geqq -C(t-x) - C(t-x)\,\sup_t|t\exp(-\tfrac{1}{2}t^2)|$$

and hence

$$|\varphi(y)| \leqq C\left\{V^{-1}+\sup_\tau\left|\int_{-V}^{V}\exp(i\xi\tau)(1-|\xi|V^{-1})\hat{u}(\xi)\,d\xi\right|\right\}. \tag{2.3}$$

To get an expression for $\hat{u}$ depending on g we observe the following: If both h and Q are integrable, where

$$Q(x) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(ixv)\,\hat{h}(v)\,g(v)\,dv \tag{2.4}$$

then

$$h*\varphi(x) = \psi*Q(x) \qquad\text{with}\qquad \psi(x) = K*\varphi(x)\,, \tag{2.5}$$

cf. [2, p. 7]. Now

$$\hat{u}(\xi) = \exp(-i\xi y)\int_{-\infty}^{\infty}\exp\big(i\xi(y-x) - \tfrac{1}{2}(y-x)^2\big)\,\varphi(x)\,dx$$

and hence we can use (2.4) with

$$h(\cdot) = \exp(-\tfrac{1}{2}(\cdot)^2 + i\xi(\cdot)),$$

since in this case h and Q are integrable. This can be seen if we translate the line of integration in (2.4) using Cauchy's integral theorem. Hence

$$\exp(i\xi y)\,\hat{u}(\xi) = \int_{-\infty}^{\infty} \psi(y-x)\,Q(x)\,dx \tag{2.6}$$

with

$$Q(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\bigl(ixv - \tfrac{1}{2}(v-\xi)^2\bigr)\,g(v)\,dv\,. \tag{2.7}$$

Let r and s be positive numbers, $0<r<\alpha$ and $\gamma<s<\beta$. Now we use Cauchy's integral theorem and translate the line of integration in (2.7) by a distance s upwards when x is positive and by a distance r downwards when x is negative. Thus we make the substitutions $v=t+\xi+is$ when x is positive and $v=t+\xi-ir$ when x is negative. Hence

$$Q(x) = (2\pi)^{-\frac{1}{2}} \exp(-sx+i\xi x) \int_{-\infty}^{\infty} \exp\bigl(ixt - \tfrac{1}{2}(t+is)^2\bigr)\,g(t+\xi+is)\,dt$$

for $x>0$, and

$$Q(x) = (2\pi)^{-\frac{1}{2}} \exp(rx+i\xi x) \int_{-\infty}^{\infty} \exp\bigl(ixt - \tfrac{1}{2}(t-ir)^2\bigr)\,g(t+\xi-ir)\,dt$$

for $x<0$. By aid of this formula, (2.6) and by Fubini's theorem the last term in (2.3) can be transformed. Thus we have

$$\begin{aligned}(2\pi)^{\frac{1}{2}} \int_{-V}^{V} \exp(i\xi\tau)(1-|\xi|\,V^{-1})\hat{u}(\xi)\,d\xi \\ = \int_{-\infty}^{\infty} \exp\bigl(-\tfrac{1}{2}(t-ir)^2\bigr)\,I_1(t)\,dt + \int_{-\infty}^{\infty} \exp\bigl(-\tfrac{1}{2}(t+is)^2\bigr)\,I_2(t)\,dt\,,\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}I_1(t) = \int_{-\infty}^{0} \psi(y-x)\exp(rx+ixt)\,\cdot \\ \cdot\left(\int_{-V}^{V} \exp\bigl(i\xi(x-y+\tau)\bigr)\,(1-|\xi|\,V^{-1})g(t+\xi-ir)\,d\xi\right)dx\,,\end{aligned}$$

$$\begin{aligned}I_2(t) = \int_{0}^{\infty} \psi(y-x)\exp(-sx+ixt)\,\cdot \\ \cdot\left(\int_{-V}^{V} \exp\bigl(i\xi(x-y+\tau)\bigr)\,(1-|\xi|\,V^{-1})g(t+\xi+is)\,d\xi\right)dx\,.\end{aligned}$$

By Schwarz's inequality

$$|I_1| \leqq \left(\int_{-\infty}^{0} |\psi(y-x)\exp(rx)|^2\,dx\right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\int_{-\infty}^{0} \left| \int_{-V}^{V} \exp(i\xi(x-y+\tau))(1-|\xi|V^{-1})g(t+\xi-ir)\,d\xi\right|^2 dx\right)^{\frac{1}{2}}.$$

Since φ is bounded in (2.2) we see that $|\psi(x)| \leqq Cp(x)^{-1}$ for all x, and then by the property of weight-functions we have

$$|\psi(y-x)| \leqq Cp(x)p(y)^{-1},$$

that is

$$\left(\int_{-\infty}^{0} |\psi(y-x)\exp(rx)|^2 dx\right)^{\frac{1}{2}} \leqq Cp(y)^{-1}.$$

Furthermore

$$\left(\int_{-\infty}^{0} \left| \int_{-V}^{V} \exp(i\xi(x-y+\tau))(1-|\xi|V^{-1})\,g(t+\xi-ir)\,d\xi\right|^2 dx\right)^{\frac{1}{2}}$$

$$\leqq \left(\int_{-\infty}^{\infty} \left| \int_{-V}^{V} \exp(i\xi x)(1-|\xi|V^{-1})\,g(t+\xi-ir)\,d\xi\right|^2 dx\right)^{\frac{1}{2}}$$

and using Parseval's relation we find that this is equal to

$$\left(2\pi \int_{-V}^{V} |(1-|\xi|V^{-1})\,g(t+\xi-ir)|^2\,d\xi\right)^{\frac{1}{2}}.$$

By (2.1) and the weight-function property this is less than or equal to

$$CP_0(r)\,P_0(|t|)\left(4\pi\int_{0}^{V} P_0(\xi)^2\,d\xi\right)^{\frac{1}{2}}.$$

Estimates of the same kind hold for I_2. Hence

$$\left|\int_{-V}^{V} \exp(i\xi\tau)(1-|\xi|V^{-1})\,\hat{u}(\xi)\,d\xi\right|$$

$$\leqq Cp(y)^{-1}\cdot\left(\int_{0}^{V} P_0(\xi)^2\,d\xi\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} P_0(|t|)\exp(-\tfrac{1}{2}t^2)\,dt,$$

which we put into (2.3), and then

$$|\varphi(y)| \leqq C\left\{V^{-1}+p(y)^{-1}\left(\int\limits_0^V P_0(\xi)^2\,d\xi\right)^{\frac{1}{2}}\right\}.$$

Thus we have

$$|\varphi(y)| \leqq C\{V^{-1}+p(y)^{-1}\,V^{\frac{1}{2}}\,P_0(V)\}\,.$$

Using the function R_1 to put $V=R_1(p(y))$ this results in

$$|\varphi(y)| \leqq CR_1(p(y))^{-1}\,,$$

and the proof of Theorem 1 is complete.

Remark. This case was treated by Beurling [1] and Lyttkens [6], [7] with

$$P_0(t) = (1+|t|)^q, \qquad q \geqq 0\,,$$

where Beurling had $p(x)=\exp(\gamma x)$ and Lyttkens in [6] had $p(x)=\exp(\gamma x)$ while in [7] she used a general weight-function. Theorem 1 even sharpens some of the results of Lyttkens and it also covers the case

$$P_0(t) = \exp(mt), \qquad m>0\,,$$

treated by Ganelius in [2].

3. The second theorem.

Here we will deal with the classes of functions T, as in section 2, and $E(P_1,P_2.\alpha.\beta)$ where

$E(P_1,P_2,\alpha,\beta)$: all integrable functions K, which fulfills the Wiener condition, and for which the function g, such that

$$g(t) = \hat{K}(t)^{-1}\,,$$

can be analytically continued in a strip $-\alpha<\mathrm{Imt}<\beta$, in the complex t-plane including the real axis, where both

$$|g(t)| \leqq \mathrm{const.}P_1(|t|) \tag{3.1}$$

and

$$|g'(t)| \leqq \mathrm{const.}P_2(|t|) \tag{3.2}$$

for some weight-functions P_1 and P_2 and all t in the strip.

Let R_2 be the inverse function of that given by $x^{\frac{3}{2}}(P_1(x)P_2(x))^{\frac{1}{2}}$ for all positive values of x, and let p be a weight-function with

$$\varlimsup_{x\to\infty} x^{-1}\log p(x) = \gamma\,,$$

then we have the following theorem.

THEOREM 2. *If* $K \in E(P_1, P_2, \alpha, \beta)$ *with* $\beta > \gamma$ *and if* $\varphi \in T$, *then*

$$K*\varphi(x) = O(p(x)^{-1}), \quad x \to \infty, \tag{3.3}$$

implies

$$\varphi(x) = O\big(R_2(p(x))^{-1}\big), \quad x \to \infty.$$

PROOF. The proof is the same as that of theorem 1 up to the estimation of (2.8). This time we make a direct estimation of I_1:

$$|I_1| \leqq Cp(y)^{-1} \sup_{x \leqq 0} |p(x) \exp(rx)| \int_{-\infty}^{\infty} \left| \int_{-V}^{V} \exp(i\xi x)(1-|\xi| V^{-1}) g(t+\xi-ir)\, d\xi \right| dx.$$

Now we use the following property: if f is integrable and if the right hand member exists then

$$\int_{-\infty}^{\infty} |f(x)|\, dx \leqq \left\{ \int_{-\infty}^{\infty} |\hat{f}(x)|^2\, dx \int_{-\infty}^{\infty} |\hat{f}'(x)|^2\, dx \right\}^{\frac{1}{2}}, \tag{3.4}$$

(cf. [1, p. 5].) By putting

$$f(x) = \int_{-V}^{0} \exp(i\xi x)(1+\xi V^{-1})\, g(t+\xi-ir)\, d\xi + \int_{0}^{V} \exp(i\xi x)(1-\xi V^{-1})\, g(t+\xi-ir)\, d\xi$$

we see after two partial integrations that f is integrable, hence we can apply (3.4), and

$$|I_1| \leqq Cp(y)^{-1} \int_{-\infty}^{\infty} \left| \int_{-V}^{V} \exp(i\xi x)(1-|\xi| V^{-1}) g(t+\xi-ir)\, d\xi \right| dx$$

$$\leqq Cp(y)^{-1} \left\{ \int_{-V}^{V} |(1-|\xi| V^{-1}) g(t+\xi-ir)|^2\, d\xi \cdot \int_{-V}^{V} \big(|(1-|\xi| V^{-1}) g'(t+\xi-ir)| + V^{-1} |g(t+\xi-ir)|\big)^2\, d\xi \right\}^{\frac{1}{2}}.$$

By (3.1) and (3.2) and since obviously

$$|g(t+\xi-ir)| \leqq C(1+|t|+|\xi|) P_2(|t|+|\xi|+r)$$

we have

$$|I_1| \leqq Cp(y)^{-1} \left\{ \int_0^V P_1(\xi)^2\, d\xi \int_0^V P_2(\xi)^2\, d\xi \right\}^{\frac{1}{2}} \{P_1(|t|)\, P_2(|t|)(1+|t|)\}^{\frac{1}{2}}.$$

Estimations of the same kind also hold for I_2. This time

$$\left|\int_{-V}^{V} \exp(i\xi\tau)(1-|\xi|V^{-1})\hat{u}(\xi)\,d\xi\right|$$

$$\leq Cp(y)^{-1}\left\{\int_0^V P_1(\xi)^2\,d\xi \int_0^V P_2(\xi)^2\,d\xi\right\}^{\frac{1}{4}} \int_{-\infty}^{\infty} \{(1+|t|)P_1(|t|)\,P_2(|t|)\}^{\frac{1}{2}} \exp(-\tfrac{1}{2}t^2)\,dt.$$

If we put this into (2.3) we obtain

$$|\varphi(y)| \leq C\left\{V^{-1}+p(y)^{-1}\left(\int_0^V P_1(\xi)^2\,d\xi \int_0^V P_2(\xi)^2\,d\xi\right)^{\frac{1}{4}}\right\},$$

which gives us

$$|\varphi(y)| \leq C\{V^{-1}+p(y)^{-1}\,V^{\frac{1}{2}}\,(P_1(V)\,P_2(V))^{\frac{1}{2}}\},$$

and if we put $V=R_2(p(y))$, then

$$|\varphi(y)| \leq CR_2(p(y))^{-1}.$$

This completes the proof of Theorem 2.

REMARK. This theorem generalizes the case where Beurling [1], Lyttkens [6], [7] and Ganelius [2] only suppose that

$$|g'(t)| \leq C(1+|t|)^{q-1}, \qquad \text{if} \quad q \geq 1,$$

since then

$$|g(t)| \leq C(1+|t|)^q,$$

and thus by theorem 2 it is true that

$$\varphi(y) = O\big(p(y)^{-1/(q+1)}\big), \quad y \to \infty.$$

4. Generalizations and comments.

The Tauberian condition can be weakened in the two theorems. For example in theorem 1 we can suppose that instead of $\varphi \in T$ we only know that φ is bounded and measurable, and that there exists an x_0 such that

$$\text{(4.1)} \qquad \varphi(t)-\varphi(x) \geq -CR_1(p(x))^{-1} \quad \text{if} \quad x_0 \leq x \leq t \leq x+R_1(p(x))^{-1}$$

To prove this we observe that if we put $V=R_1(p(y))$ from the beginning of the proof of theorem 1, the only difference this weaker Tauberian condition will make in the proof, is in the estimation of $u(t)-u(x)$. Now it remains to prove that for large values of y

$$A = -\inf_{x \leqq t \leqq x+V^{-1}} \exp\left(-\tfrac{1}{2}(t-y)^2\right)\left(\varphi(t)-\varphi(x)\right) \leqq C R_1(p(y))^{-1}.$$

We will use the fact that

$$R_1(mx) \leqq m R_1(x), \qquad m \geqq 1, \tag{4.2}$$

which follows from the definition of R_1. If y is sufficiently large then

$$\begin{aligned} A &= -\inf_{x \leqq t \leqq x+V^{-1}} \exp(-\tfrac{1}{2}t^2)\left(\varphi(t+y)-\varphi(x+y)\right) \\ &\leqq C\exp(-\varepsilon y^2) - \inf_{-\frac{1}{2}y \leqq x \leqq t \leqq x+V^{-1}} \exp(-\tfrac{1}{2}t^2)\left(\varphi(t+y)-\varphi(x+y)\right), \end{aligned}$$

for some $\varepsilon > 0$, and by (4.1)

$$\begin{aligned} A \leqq C\exp(-\varepsilon y^2) + C &\sup_{-\frac{1}{2}y \leqq x \leqq t \leqq x+V^{-1} \leqq V^{-1}} \exp(-\tfrac{1}{2}t^2) R_1(p(x+y))^{-1} - \\ &- \inf_{0 \leqq x \leqq t \leqq x+V^{-1}} \exp(-\tfrac{1}{2}t^2)\left(\varphi(t+y)-\varphi(x+y)\right), \end{aligned}$$

Using the weight function property and (4.2) we have

$$R_1(p(x+y))^{-1} \leqq p(-x)\, R_1(p(y))^{-1} \qquad \text{for} \quad -y \leqq x \leqq 0$$

and since by (4.2) we also know that $R_1(x) \leqq Cx$ and hence

$$\exp(-\varepsilon y^2) \leqq Cp(y)^{-1} \leqq CR_1(p(y))^{-1},$$

it only remains us to estimate the last term. This cannot be done directly from (4.1), since the interval $x \leqq t \leqq x + V^{-1}$ is larger than that allowed in (4.1) if $x > 0$. But since by (4.2)

$$p(t)\, R_1(p(t+y))^{-1} \geqq R_1(p(y))^{-1} \qquad \text{if} \quad t \geqq 0,$$

we may choose an n such that

$$p(x+V^{-1}) < n < 2p(x+V^{-1}),$$

split the interval $x \leqq t \leqq x+V^{-1}$, $x \geqq 0$, in n equal parts by $x = t_0 < t_1 \ldots < t_n = x + V^{-1}$ and put

$$B = \varphi(t+y) - \varphi(x+y) = \sum_{k=1}^{n}\left(\varphi(t_k+y) - \varphi(t_{k-1}+y)\right).$$

Now by (4.1)

$$B \geqq -C\sum_{k=1}^{n} R_1(p(y+t_{k-1}))^{-1} \geqq -C\, n\, R_1(p(y))^{-1} \geqq -Cp(x)\, R_1(p(y))^{-1},$$

and we see that

$$A \leqq CR_1(p(y))^{-1} \qquad \text{for large values of } y.$$

Hence we are through. The same reasoning applies in theorem 2.

For certain combinations of P_0 and p in theorem 1 the condition $\beta > \gamma$ is superfluous. A sufficient condition is that

$$R_1(p(nx)) \leqq CnR_1(p(x)), \qquad n \geqq 1 . \tag{4.3}$$

In the interesting case $P_0(x) = \exp(mx)$, which covers the situation when K is the convolution kernel associated with the Laplace or Stieltjes transform, this is true for all weight functions p. In the other cases it is, for example, true for all p such that

$$p(nx) \leqq Cnp(x), \qquad n \geqq 1 ,$$

since then (4.3) follows from (4.2). To prove this let us consider K_n and φ_n where

$$K_n(x) = K(nx) \qquad \text{and} \qquad \varphi_n(x) = \varphi(nx)$$

with n so large that $n\beta > \gamma$. Since

$$\hat{K}_n(t) = n^{-1} \hat{K}(tn^{-1}) ,$$

it is easy to see that

$$K \in E(P_0, \alpha, \beta) \qquad \text{implies} \qquad K_n \in E(P_0, n\alpha, n\beta)$$

and that $\varphi_n \in T$. If now

$$K * \varphi(x) = O(p(x)^{-1}), \quad x \to \infty ,$$

then

$$K_n * \varphi_n(x) = O(p(x)^{-1}), \quad x \to \infty ,$$

and by theorem 1

$$\varphi(nx) = O\big(R_1(p(x))^{-1}\big), \quad x \to \infty ,$$

and if (4.3) is fulfilled

$$\varphi(x) = O\big(R_1(p(x))^{-1}\big), \quad x \to \infty .$$

Even in this case the same holds for theorem 2. As might be seen from above, this is also true with the weaker Tauberian condition (4.1).

For certain p it also possible to weaken the conditions in another way. If for example $p(x) = \exp(\gamma x)$ we only need to suppose analyticity above the real axis in the definition of the class E (cf. [2, p. 5], [7, p. 317]). In theorem 1 we only translate the line of integration in (2.7) a distance s upwards when x is positive, where $\gamma < s < \beta$, and do nothing when x is negative. Similar estimations as in section 2 hold for (2.8).

In theorem 2 the same weakening of the conditions could be made for all p, with the same arguments as above, (cf. [7, p. 321]). Naturally this even holds with the other generalisations discussed in this section.

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