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A MEASURE THEORETIC CHARACTERIZATION OF CHOQUET SIMPLEXES

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A point set A in R^n is affinely independent if and only if the convex hulls of B and C are disjoint, hence separable by a hyperplane, for any partition $\{B,C\}$ of A. (This is related to a classical theorem of Radon, cf. e.g. [4].) In the present paper this result is generalized to a theorem on a compact convex set K in a locally convex space, according to which K is a simplex if and only if every boundary measure on K admits a Hahndecomposition by halfspaces (determined by an affine Borel function of class $\mathscr{A}$, cf. definition below).

We are indebted to R. Phelps for valuable discussions on the subject and also for making available to us the manuscript of his forthcoming book [6].

1. Definitions and basic properties.

The setting of the present note is similar to that of [1], and we shall use the concepts of that paper rather freely. Thus K shall be a compact convex subset of a locally convex Hausdorff vector space E over R, $\mathscr{H}$ shall be the class of all K-restrictions of continuous, affine functionals, and $\mathscr{S}$ shall be the class of all continuous and convex, real valued functions on K. The *lower envelope* $\underline{f}$ of a real valued function f, bounded below on K, is the greatest l.s.c. convex minorant of f. It can be expressed as follows

$$(1.1)\qquad \underline{f}(x) = \sup\{g(x) \mid g \in \mathscr{S},\ g(y)<f(y) \text{ for all } y \in K\}$$
$$= \sup\{h(x) \mid h \in \mathscr{H},\ h(y)<f(y) \text{ for all } y \in K\}\,.$$

The upper envelope $\bar{f}$ is defined dually and admits the dual characterizations.

In the sequel we shall use the word *measure* to denote a regular Borel measure on K, and vector-valued integrals are taken in the weak sense. Thus $\int t\,d\mu(t)$ denotes the *resultant* of μ, and it denotes the *barycenter* of μ if μ is positive and normalized (probability measure).

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We shall make repeated use of the following elementary fact: If $\{f_\alpha\}$ is an ascending net of l.s.c. functions such that $\sup_\alpha \int f_\alpha d\mu < \infty$ for some measure μ, then the l.s.c. function $f = \sup_\alpha f_\alpha$ is μ-integrable and

$$\int f d\mu = \sup_\alpha \int f_\alpha \, d\mu \,. \tag{1.2}$$

A measure μ is said to be a *boundary measure* if

$$\int (\bar{f} - f) \, d|\mu| = 0 \tag{1.3}$$

for every $f \in \mathscr{C}(K)$.

In the sequel $\mathscr{F}$ and $\mathscr{G}$ shall be the classes of all real valued functions on K which are pointwise limits of descending and ascending nets from $\mathscr{H}$, respectively. Symbols such as $\mathscr{F}_{\sigma\delta}$, $\mathscr{G}_{\delta\sigma}$ etc., will be used in the customary meaning, δ, σ denoting pointwise limits of descending and ascending sequences. The smallest class of functions containing $\mathscr{F}$ and $\mathscr{G}$ and being closed under pointwise limits of monotone sequences, will be denoted by $\mathscr{A}$.

Clearly every function in $\mathscr{F}$ is u.s.c. and affine, and every function in $\mathscr{G}$ is l.s.c. and affine. The converse statements are also valid by virtue of the following:

PROPOSITION 1. *If f is an u.s.c. affine function on K, then the set of all $h \in \mathscr{H}$ such that $f(x) < h(x)$ for all $x \in K$, is directed downward. Consequently $\mathscr{F}$ comprises all u.s.c. affine functions. Similarly $\mathscr{G}$ comprises all l.s.c. affine functions.*

PROOF. Let $h_i \in \mathscr{H}$ and $f(x) < h_i(x)$ for all $x \in K$ and $i = 1, 2$. Let α, β be two real numbers bounding f, h_1, h_2 below and above, respectively, and define the following "ordinate sets" in $E \times \mathsf{R}$

$$L = \{(x, \eta) \mid x \in K, \ \alpha \leqq \eta \leqq f(x)\} \,,$$
$$U_i = \{(x, \eta) \mid x \in K, \ h_i(x) \leqq \eta \leqq \beta\}, \quad i = 1, 2 \,.$$

Clearly L, U_1, U_2 are convex and compact, and by an elementary theorem, $U = \mathrm{conv}(U_1, U_2)$ is also compact.

The sets L and U are disjoint. In fact if $(x, \eta) \in U$, then there is a convex combination

$$x = \lambda y + (1 - \lambda) z, \qquad 0 \leqq \lambda \leqq 1, \quad y, z \in K \,,$$

such that

$$\eta \geqq \lambda h_1(y) + (1 - \lambda) h_2(z) > \lambda f(y) + (1 - \lambda) f(z) = f(x) \,,$$

and hence $(x, \eta) \notin L$.

By a well known separation property (based on the Hahn–Banach Theorem), the sets L and U may be separated by a hyperplane H in $E \times R$. Now H is seen to be the graph of a continuous affine functional whose K-restriction has the desired property

$$(1.4) \qquad f(x) < h(x) < h_i(x)$$

for all $x \in K$ and $i = 1, 2$.

Now the last part of the proposition is an immediate consequence of (1.1).

Clearly $\mathscr{A}$ is contained in the class $\mathscr{B}_a$ of affine Borel functions, but the two classes are not identical in general. By an example of G. Choquet [3] (cf. also [6]) there exists an affine Borel function (of second Baire class) which does not enjoy the property (1.5) of our next proposition. The relationship between $\mathscr{A}$ and $\mathscr{B}_a$ is similar to the relationship between the monotone class generated by convex closed sets and the class of convex Borel sets. The latter two classes have been proved to coalesce in $\mathbf{R}^2$ by V. Klee [5], but to the best of our knowledge the problem is open even for $\mathbf{R}^3$.

PROPOSITION 2. *If μ is a positive normalized measure with barycenter x and if f is a function of class $\mathscr{A}$, then f is μ-integrable and*

$$(1.5) \qquad f(x) = \int f d\mu .$$

PROOF. Let $\mathscr{H}$ be the class of all μ-integrable functions of class $\mathscr{A}$ for which (1.5) holds. If $g \in \mathscr{G}$, then there is a net $\{h_\alpha\}$ from $\mathscr{H}$ such that $h_\alpha \nearrow g$. Now

$$\sup_\alpha \int h_\alpha d\mu = \sup_\alpha h_\alpha(x) = g(x) < \infty ,$$

and by (1.2) g is integrable and

$$\int g d\mu = \sup_\alpha \int h_\alpha d\mu = g(x) .$$

Hence $\mathscr{G} \subset \mathscr{H}$. Similarly one may prove $\mathscr{F} \subset \mathscr{H}$.

Next consider an increasing sequence $\{f_n\}$ from $\mathscr{H}$ which converges pointwise to a real valued function f. Then

$$\sup_n \int f_n d\mu = \sup_n f_n(x) = f(x) < \infty ,$$

and by the Monotone Convergence Theorem, f is integrable and

$$\int f d\mu = \sup_n \int f_n d\mu = f(x) .$$

Hence $f \in \mathscr{K}$. Similarly one may prove that $\mathscr{K}$ is closed under pointwise limits of descending sequences. It follows that $\mathscr{K} = \mathscr{A}$, and the proof is accomplished.

A non-zero signed boundary measure with total mass zero and resultant in the origin is said to be an *affine dependence* on $\partial_e K$, and K is said to be a *simplex* if there is no affine dependence on $\partial_e K$ (cf. [1]). By a theorem of G. Choquet and P. A. Meyer [2, p. 145], K is a simplex if and only if $\bar{f}$ is an u.s.c. *affine* function for every $f \in \mathscr{S}$, or equivalently if and only if $\underline{f}$ is a l.s.c. *affine* function for every $f \in -\mathscr{S}$. Hence it follows from Proposition 1, that *K is a simplex if and only if $\bar{f} \in \mathscr{F}$ for every $f \in \mathscr{S}$, or equivalently if and only if $\underline{f} \in \mathscr{G}$ for every $f \in -\mathscr{S}$.*

2. Hahn-decomposition by half-spaces.

We first prove that any two mutually singular boundary measures on a simplex can be "separated up to ε" by a function from $\mathscr{H}$.

PROPOSITION 3. *If μ and ν are mutually singular, positive boundary measures on a simplex K, then for every $\varepsilon > 0$ there exists an $h \in \mathscr{H}$ such that $0 \leqq h \leqq 1$ and*

$$\int h\, d\nu \leqq \varepsilon, \quad \int (1-h)\, d\mu \leqq \varepsilon . \tag{2.1}$$

PROOF. By the mutual singularity of μ and ν there exists a continuous function f on K such that $0 \leqq f \leqq 1$ and

$$\int f\, d\nu \leqq \tfrac{1}{2}\varepsilon, \quad \int (1-f)\, d\mu \leqq \tfrac{1}{2}\varepsilon . \tag{2.2}$$

By (1.1) $\underline{f}$ is the supremum of the set of all $g \in \mathscr{S}$ such that $g(x) < f(x)$ for all $x \in K$. This set is closed under finite suprema ("réticulé supérieurement"). In particular it is directed upward, and by (1.2) it has a member g such that

$$\int g\, d\mu \geqq \int \underline{f}\, d\mu - \tfrac{1}{2}\varepsilon .$$

This inequality subsists with g^+ in the place of g, and clearly $g^+ \in \mathscr{S}$, $0 \leqq g^+ \leqq f$ and $g^+(x) < 1$ for all $x \in K$. Hence by (2.2) and by the characteristic property (1.3) of boundary measures

$$\int g^+ d\nu \leqq \int f d\nu \leqq \tfrac{1}{2}\varepsilon , \tag{2.3}$$

and

$$\int g^+ d\mu \geqq \int f d\mu - \tfrac{1}{2}\varepsilon \geqq \mu(K) - \varepsilon . \tag{2.4}$$

Since K is a simplex and $g^+ \in \mathscr{S}$, the function $\overline{g^+}$ is u.s.c. and *affine*. By (1.1) $\overline{g^+}$ is the infimum of the set of all $h \in \mathscr{H}$ such that $h(x) > g^+(x)$ for all $x \in K$. By Proposition 1 this set is directed downward and by (1.2) it has a member h such that

$$\int h\,d\nu \leqq \int \overline{g^+}\,d\nu + \tfrac{1}{2}\varepsilon .$$

We may assume $h \leqq 1$ since $g^+(x) < 1$ for all $x \in K$. Hence $0 \leqq h \leqq 1$, and by (2.3), (2.4) and by use of (1.3) once more

$$\int h\,d\nu \leqq \int g^+ d\nu + \tfrac{1}{2}\varepsilon \leqq \varepsilon ,$$

and

$$\int h\,d\mu \geqq \int g^+ d\mu \geqq \mu(K) - \varepsilon .$$

These relations complete the proof.

PROPOSITION 4. *Let μ and ν be mutually singular, positive boundary measures on a simplex K. For every $\varepsilon > 0$ there exists an (affine) function g of class $\mathscr{G}_\delta$ such that $0 \leqq g \leqq 1$, and*

$$\int g\,d\nu = 0, \qquad \int (1-g)\,d\mu < \varepsilon . \tag{2.5}$$

Moreover, there exsists an (affine) function f of class $\mathscr{G}_{\delta\sigma}$ such that $0 \leqq f \leqq 1$, and

$$\int f\,d\nu = \int (1-f)\,d\mu = 0 . \tag{2.6}$$

PROOF. By Proposition 3 there exist functions $h_n \in \mathscr{H}$ such that $0 \leqq h_n \leqq 1$ and

$$\int h_n d\nu \leqq 2^{-n}, \qquad \int (1-h_n) d\mu \leqq 2^{-n}, \tag{2.7}$$

for $n = 1, 2, \ldots$. Define

$$g_{n,p} = \underline{h_{n+1} \wedge \ldots \wedge h_{n+p}}, \qquad n, p = 1, 2, \ldots .$$

The functions $g_{n,p}$ are l.s.c. and *affine* since K is a simplex. By Proposition 1,

$$g_{n,p} \in \mathscr{G}, \qquad n, p = 1, 2, \ldots .$$

Now define

$$g_n = \inf_p g_{n,p}, \qquad n = 1, 2, \ldots .$$

Clearly $g_n \in \mathscr{G}_\delta$, and

$$\int g_n d\nu \leqq \int h_{n+p} d\nu \leqq 2^{-n-p}, \qquad n,p=1,2,\ldots .$$

Hence

(2.8) $$\int g_n d\nu = 0, \qquad n=1,2,\ldots .$$

By the characteristic property (1.3) of a boundary measure,

$$\int (1-g_{n,p})d\mu = \int (1-h_{n+1} \wedge \ldots \wedge h_{n+p})d\mu$$
$$\leqq \sum_{k=n+1}^{n+p} \int (1-h_k)d\mu \leqq 2^{-n}(1-2^{-p}), \quad n,p=1,2,\ldots .$$

Hence by the Monotone Convergence Theorem

(2.9) $$\int (1-g_n)d\mu = \sup_p \int (1-g_{n,p})d\mu \leqq 2^{-n}, \qquad n=1,2,\ldots .$$

By (2.8) and (2.9) the requirement (2.5) is satisfied with $g=g_n$ when $2^{-n} \leqq \varepsilon$.

Next define $f=\sup_n g_n$. Clearly $f \in \mathscr{G}_{\delta\sigma}$. By the Monotone Convergence Theorem and by (2.8)

$$\int f\, d\nu = 0 .$$

Clearly $1-f \leqq 1-g_n$ for $n=1,2,\ldots$. Hence by (2.9)

$$\int (1-f)d\mu = 0 .$$

Thus, f has the desired property (2.6).

THEOREM 1. *A convex compact set K is a simplex if and only if every (signed) boundary measure μ admits an affine function f of class $\mathscr{A}$ such that*

(2.10) $$\mu^-(\{x \mid f(x) \geqq 0\}) = \mu^+(\{x \mid f(x) \leqq 0\}) = 0 .$$

PROOF. 1. Assume K to be a simplex. By Proposition 5 there exists an affine function g of class $\mathscr{G}_{\delta\sigma}$ such that $0 \leqq g \leqq 1$ and

(2.11) $$\int g\, d\mu^- = \int (1-g)d\mu^+ = 0 .$$

Let $f=g-\frac{1}{2}$, and define $A=\{x \mid f(x) \geqq 0\}$, $B=\{x \mid f(x) \leqq 0\}$. Clearly $\frac{1}{2}\chi_A \leqq g$, $\frac{1}{2}\chi_B \leqq 1-g$, and by (2.11)

$$\mu^-(A) = \mu^+(B) = 0 .$$

Thus $f \in \mathscr{G}_{\delta\sigma} \subset \mathscr{A}$, and (2.10) is satisfied.

2. Assume K to be a non-simplex. By the definition of a simplex there exists an affine dependence μ on $\partial_e K$. We assume the positive and negative parts of μ to be normalized, and we denote the common barycenter of μ^+ and μ^- by x. Thus we have

$$\mu^+(K) = \mu^-(K) = 1 \tag{2.12}$$

$$\int t\,d\mu^+(t) = \int t\,d\mu^-(t) = x \tag{2.13}$$

We claim that such a measure μ cannot admit any function f of class $\mathscr{A}$ for which (2.10) is valid. In fact, assume $f \in \mathscr{A}$ and

$$\mu^-(A) = \mu^+(B) = 0\,, \tag{2.14}$$

where $A = \{x \mid f(x) \geqq 0\}$, $B = \{x \mid f(x) \leqq 0\}$. By (2.12) and (2.14),

$$\mu^+(\{x \mid f(x) > 0\}) = \mu^-(\{x \mid f(x) < 0\}) = 1\,.$$

Hence there is an $\alpha > 0$ such that

$$\mu^+(A_\alpha) \geqq \tfrac{1}{2}, \qquad \mu^-(B_\alpha) \geqq \tfrac{1}{2}\,, \tag{2.15}$$

where $A_\alpha = \{x \mid f(x) \geqq \alpha\}$, $B_\alpha = \{x \mid f(x) \leqq -\alpha\}$. By virtue of (2.13), (2.15) and by Proposition 2

$$\tfrac{1}{2}\alpha \leqq \int_{A_\alpha} f\,d\mu^+ \leqq \int f\,d\mu^+ = f(x)\,,$$

$$-\tfrac{1}{2}\alpha \geqq \int_{B_\alpha} f\,d\mu^- \geqq \int f\,d\mu^- = f(x)\,.$$

This contradiction completes the proof.

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