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PROXIMITY AND HEIGHT

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1. Introduction.

Alfsen and Njåstad [3] have given an example of a p-equivalence class lacking a finest uniform structure. Other examples appear in [5] and [7, p. 27]. These examples, and certain corresponding facts about functions have led Fenstad (in [5]) to question the usefulness of the concept of proximity. In particular, the concept of generalized uniform structures has been introduced in [3] and developed in [8].

Motivated by these considerations, the author has formulated a new concept, *height*, to help clarify the order structure of p-classes of uniformities. Several unforeseen consequences and conjectures concerning the relationship between height and proximity have resulted. Among these are 1) the construction of an example of two uniformities in the same proximity class whose least upper bound is not in the proximity class, and differing from previous examples by the fact that the two uniformities in question are related in height; 2) the fact that height and proximity are, in a certain sense, dual; 3) the conjecture that a height class and a proximity class can have at most one element in common. Since only a partial result is known (Theorem 4.1), the latter is an open question. The purpose of this paper is to introduce the concept of height and develop some of the initial consequences referred to above.

In Section 2 we introduce a quasi-ordering on the family of all uniformities on a set X, label the resulting equivalence classes height classes, and prove some preliminary theorems.

Then, in Section 3, we review those facts about p-equivalence which we need, and relate proximity to height. The dual nature of proximity and height is discussed at the end of Section 3.

Section 4 contains some conjectures, examples, and counterexamples.

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2. Height.

Following common usage, we say that a subset U of $X \times X$ is *totally bounded* iff X is the union of a finite number of sets B such that $B \times B \subset U$. A uniformity $\mathscr{U}$ is said to be totally bounded iff each of its entourages is totally bounded.

Now we can define an ordering of uniformities by means of relative total boundedness. We say that a uniformity $\mathscr{U}$ on X is *less than or equal in height* ($\leqq^h$) to a uniformity $\mathscr{V}$ on X iff for each $U \in \mathscr{U}$ there exists $V \in \mathscr{V}$ such that $U \cup V^c$ is totally bounded. (V^c denotes the complement of V.)

The following results are immediate:

THEOREM 2.1. *The relation "less than or equal in height" is a quasi-ordering (reflexive and transitive).*

It is well known that a quasi-ordering leads to a family of equivalence classes (which in this case we shall call *height classes*) and a resulting partial ordering ($\leqq$) on the family of equivalence classes. If two uniformities belong to the same height class, we shall say that they are *equal in height* ($=^h$). We will denote the height class to which a given uniformity $\mathscr{U}$ belongs by $H(\mathscr{U})$.

THEOREM 2.2. *If $\mathscr{U} \leqq \mathscr{V}$ (every entourage of $\mathscr{U}$ is also a member of $\mathscr{V}$), then $\mathscr{U} \leqq^h \mathscr{V}$.*

THEOREM 2.3. *The smallest height class is the class of all totally bounded uniformities on X.*

It is well known that the set of all uniformities on X form a complete lattice under the ordering $\leqq$. We shall establish some initial results concerning $\leqq^h$. It will be convenient to use the common notation $\vee$ for both least upper bounds of uniformities and least upper bounds of height classes. (We must, of course, first establish that pairs of height classes have least upper bounds.)

THEOREM 2.4. *If $\mathscr{U} \leqq^h \mathscr{V}$ and $\mathscr{W} \leqq^h \mathscr{V}$ then $(\mathscr{U} \vee \mathscr{W}) \leqq^h \mathscr{V}$.*

PROOF. If $T \in (\mathscr{U} \vee \mathscr{W})$ is of the form $U \cap W$ with $U \in \mathscr{U}$ and $W \in \mathscr{W}$ and V_U and V_W are members of $\mathscr{V}$ and A_i, $i = 1, 2, \ldots, n$, and B_j, $j = 1, 2, \ldots, m$ are finite covers of X such that

$$(A_i \times A_i) \subset (U \cup (V_U)^c \quad \text{and} \quad (B_j \times B_j) \subset \big(W \cup (V_W)^c\big),$$

then

$$(A_i \cap B_j) \times (A_i \cap B_j) \subset \big(T \cup (V_U \cap V_W)^c\big).$$

COROLLARY 2.1. *If* $\mathscr{U} \leqq^h \mathscr{V}$, *then* $(\mathscr{U} \vee \mathscr{V}) =^h \mathscr{V}$.

PROOF. By Theorem 2.2., $\mathscr{V} \leqq^h \mathscr{V} \leqq^h (\mathscr{U} \vee \mathscr{V})$; and by Theorem 2.4, $(\mathscr{U} \vee \mathscr{V}) \leqq^h \mathscr{V}$.

COROLLARY 2.2. *If* $\mathscr{U}$ *is totally bounded and* $\mathscr{V}$ *is arbitrary then* $(\mathscr{U} \vee \mathscr{V}) =^h \mathscr{V}$.

PROOF. Theorem 2.3 and Corollary 2.1.

3. Proximity and height.

We shall review only what we need of the facts concerning proximity. For a more complete treatment, see [1], [2], and [4].

If $\mathscr{U}$ is a uniformity on X, then $\mathscr{U}$ induces a relation $\Subset$ on the power set of X in the following way. $A \Subset B$ iff there exists $U \in \mathscr{U}$ such that $U(A) \subset B$, where

$$U(A) = \{y\colon \text{there exists } x \in A \text{ such that } (x,y) \in U\}\,.$$

We say that $\mathscr{U}$ and $\mathscr{V}$ *are in the same proximity class* or that *they are p-equivalent* iff $\mathscr{U}$ and $\mathscr{V}$ induce the same $\Subset$ relation on the power set of X. If is well known (see [1]) that every proximity class contains a unique, smallest member $\mathscr{U}_\omega$ which may be constructed in the following way:

We say that a finite cover A_i, $i = 1,2,\ldots,n$, of X is a *p-cover* of X iff there exists a finite cover B_i, $i = 1,2,\ldots,n$, of X such that $B_i \Subset A_i$.

It has been shown in [1] that the sets $\bigcup_{i=1}^{n}(A_i \times A_i)$, where the A_i's are a p-cover of X, constitute a base for a totally bounded uniformity which is the smallest uniformity inducing $\Subset$. Following [1], we shall use $\mathscr{U}_\omega$, $\mathscr{V}_\omega$, etc. to denote the smallest member of the proximity class of $\mathscr{U}, \mathscr{V}, \ldots$ respectively.

It will be convenient to say that $\mathscr{U}$ is *less than or equal in proximity* ($\leqq^p$) to $\mathscr{V}$ if the relation $\Subset$ induced by $\mathscr{U}$ is a subclass of the relation $\Subset$ induced by $\mathscr{V}$. (This corresponds to the order given in [2].)

It is well known that the above construction of $\mathscr{U}_\omega$ gives a one to one correspondence between proximity classes and totally bounded uniformities which is an order-isomorphism when the ordering of proximity classes is defined as above.

In particular, we shall make use of the following Lemma proved in [3].

LEMMA. *Let* $\mathscr{U}_\omega$ *be the* (*totally bounded*) *coarsest uniform structure of a p-equivalence class* $\mathscr{P}$ *of uniform structures on a set* S. *If* V *is a subset of* $S \times S$ *for which*

$$A \Subset V(A)(\mathscr{P}) \quad \textit{for all} \quad A \subset S\,,$$

then we also have

$$A \Subset [V \cap U](A)(\mathscr{P}) \quad \textit{for all} \quad U \in \mathscr{U}_\omega,\ A \subset S\,.$$

COROLLARY 3.1. *If* $\mathscr{U} \leqq^p \mathscr{V}$, *then* $(\mathscr{U} \vee \mathscr{V}_\omega) =^p \mathscr{V}$.

We now prove the main theorem relating height and proximity.

THEOREM 3.1. *If* $\mathscr{U} \leqq^h \mathscr{V}$, *then* $\mathscr{U} \leqq [\mathscr{V} \vee (\mathscr{U} \vee \mathscr{V})_\omega]$.

PROOF. Let all of the entourages picked below be symmetric. Let U and $U^{\frac{1}{3}}$ be members of $\mathscr{U}$ such that $U^{\frac{1}{3}} \circ U^{\frac{1}{3}} \circ U^{\frac{1}{3}} \subset U$. Since $\mathscr{U} \leqq^h \mathscr{V}$ there exist $V \in \mathscr{V}$ and A_i, $i = 1, 2, \ldots, n$, such that the A_i cover X and

$$(U^{\frac{1}{3}} \cup V^c) \supset \bigcup_{i=1}^{n} (A_i \times A_i)\,.$$

Let $U^{\frac{1}{9}}$ be a member of $\mathscr{U}$ such that $U^{\frac{1}{9}} \circ U^{\frac{1}{9}} \circ U^{\frac{1}{9}} \subset U^{\frac{1}{3}}$ and let $V^{\frac{1}{3}} \in \mathscr{V}$ be such that $V^{\frac{1}{3}} \circ V^{\frac{1}{3}} \circ V^{\frac{1}{3}} \subset V$. Then let

$$B_i = (U^{\frac{1}{9}} \cap V^{\frac{1}{3}})(A_i)\,.$$

Clearly the B_i constitute a p-cover of X with respect to the p-equivalence class of $\mathscr{U} \vee \mathscr{V}$ and thus

$$U_\omega = \bigcup_{i=1}^{n} (B_i \times B_i) \in (\mathscr{U} \vee \mathscr{V})_\omega\,.$$

To complete the proof, it will suffice to show that

$$(U_\omega \cap V^{\frac{1}{3}}) \subset U\,.$$

Suppose $(t,u) \in (U_\omega \cap V^{\frac{1}{3}})$, then $(t,u) \in (B_i \times B_i)$ for some i and there exist x and $y \in A_i$ such that (x,t) and $(y,u) \in (U^{\frac{1}{9}} \cap V^{\frac{1}{3}})$. Therefore, $(x,y) \in V$. However, since $(A_i \times A_i) \subset (U^{\frac{1}{3}} \cup V^c)$, we have $(x,y) \in U^{\frac{1}{3}}$. Consequently, since (x,t) and $(y,u) \in U^{\frac{1}{9}} \subset U^{\frac{1}{3}}$ and all the entourages under consideration are symmetric, $(t,u) \in U$.

COROLLARY 3.2. *If a p-equivalence class $\mathscr{P}$ has a least upper bound, then the relation $\mathscr{U} \leqq^h \mathscr{V}$ between two members of $\mathscr{P}$ implies $\mathscr{U} \leqq \mathscr{V}$.*

PROOF. If $\mathscr{W}$ is the least upper bound of $\mathscr{P}$, then $\mathscr{W}_\omega = (\mathscr{U} \vee \mathscr{V})_\omega \subset \mathscr{V}$. So $\mathscr{V} = [\mathscr{V} \vee (\mathscr{U} \vee \mathscr{V})_\omega]$.

COROLLARY 3.3. *If H_1 and H_2 are height classes, then $H_1 \leqq H_2$ iff there exists $\mathscr{U}_1 \in H_1$ and $\mathscr{U}_2 \in H_2$ such that $\mathscr{U}_1 \leqq \mathscr{U}_2$.*

PROOF. If $\mathscr{U}_i \in H_i$ are such that $\mathscr{U}_1 \leqq \mathscr{U}_2$, then, by Theorem 2.2,

$\mathscr{U}_1 \leqq^h \mathscr{U}_2$. Conversely, if $H_1 \leqq H_2$ and $\mathscr{U}_1 \in H_1$ and $\mathscr{V} \in H_2$ so that $\mathscr{U}_1 \leqq^h \mathscr{V}$, then, by Theorem 2.3 and Corollary 2.1,

$$\mathscr{U}_2 = [\mathscr{V} \vee (\mathscr{U}_1 \vee \mathscr{V})_\omega] \in H_2$$

and, by Theorem 3.1, $\mathscr{U}_1 \leqq \mathscr{U}_2$.

THEOREM 3.2. *Every height class H has a unique, largest member $\mathscr{U}_h$. If $\mathscr{U} \leqq^h \mathscr{U}_h$, then $\mathscr{U} \leqq \mathscr{U}_h$.*

PROOF. It is well known that there exists a largest uniformity $\mathscr{U}_\Delta$ (generated by $\Delta =$ the diagonal of $X \times X$) on X. Clearly, every finite cover of X is a p-cover with respect to the p-equivalence class of $\mathscr{U}_\Delta$. If $\mathscr{U} \in H$, let $\mathscr{U}_h = (\mathscr{U} \vee (\mathscr{U}_\Delta)_\omega)$. Clearly, $\mathscr{U}_h \in H$. If $\mathscr{U}' \leqq^h \mathscr{U}_h$, then, by Theorem 3.1

$$\mathscr{U}' \leqq [\mathscr{U}_h \vee (\mathscr{U}' \vee \mathscr{U}_h)_\omega] = \mathscr{U}_h .$$

Since $\mathscr{U}' \in H$ implies $\mathscr{U}' \leqq^h \mathscr{U}_h$, $\mathscr{U}_h$ is the largest member of H.

COROLLARY 3.4. *The family of all height classes of uniformities on a set X forms a complete lattice.*

PROOF. Since the totally bounded uniformities form a height class which is the smallest one (Theorem 2.3), it will suffice to show that every subfamily of height classes has a least upper bound. Let H_a, $a \in \mathscr{A}$ be a subfamily of the family of all height classes on X and, by Theorem 3.2, let $\mathscr{U}_a \in H_a$ be the largest member of H_a for each a. If

$$\mathscr{U} = \bigvee_{a \in \mathscr{A}} \mathscr{U}_a ,$$

then $H(\mathscr{U})$ is certainly an upper bound for $\{H_a : a \in \mathscr{A}\}$. On the other hand, suppose H' is an upper bound for $\{H_a : a \in \mathscr{A}\}$, and $\mathscr{U}_h'$ is the largest member of H'. Then, by Theorem 3.2, $\mathscr{U}_a \leqq \mathscr{U}_h'$ for all $a \in \mathscr{A}$. Consequently $\mathscr{U} \leqq \mathscr{U}_h'$ which implies $H(\mathscr{U}) \leqq H$ by Corollary 3.3.

COROLLARY 3.5. *If $\{\mathscr{U}_a : a \in \mathscr{A}\}$ is a class of uniformities on X, then*

$$\bigvee_{a \in \mathscr{A}} H(\mathscr{U}_a) = H\Big(\bigvee_{a \in \mathscr{A}} \mathscr{U}_a\Big).$$

PROOF. By Corollary 3.3,

$$\bigvee_{a \in \mathscr{A}} H(\mathscr{U}_a) \leqq H\Big(\bigvee_{a \in \mathscr{A}} \mathscr{U}_a\Big).$$

On the other hand, if $\mathscr{U}_a'$ is the largest member of $H(\mathscr{U}_a)$, then, by Corollary 3.3,

$$H\Big(\bigvee_{a\in\mathscr{A}}\mathscr{U}_a\Big) \leqq H\Big(\bigvee_{a\in\mathscr{A}}\mathscr{U}_a'\Big),$$

and by the proof of Corollary 3.4,

$$H\Big(\bigvee_{a\in\mathscr{A}}\mathscr{U}_a'\Big) = \bigvee_{a\in\mathscr{A}}(H(\mathscr{U}_a')) = \bigvee_{a\in\mathscr{A}}(H(\mathscr{U}_a)).$$

We pause to mention here, that it is not necessarily true that

$$\bigwedge_{a\in\mathscr{A}} H(\mathscr{U}_a) = H\Big(\bigwedge_{a\in\mathscr{A}}\mathscr{U}_a\Big).$$

A counter example will be given in Section 4.

In certain respects, height and proximity appear to be dual concepts. In particular, the statements "every p-equivalence class has a unique smallest member" and "every height class has a unique largest member" appear to be dual. The duality is further suggested by the facts that the unique smallest members of p-equivalence classes are exactly the members of the smallest height class, and the unique largest members of height classes are exactly the members of the largest p-equivalence class. Furthermore, we have the facts that 1) the ordering of the p-equivalence classes is isomorphic to the ordering of the smallest height class (well known), and 2) the ordering of the height classes is isomorphic to the ordering of the largest p-equivalence class (Corollary 3.2 and Theorem 3.2).

4. Thron's conjecture and examples.

In considering the concepts of proximity and height, the question naturally arises: To what extent does the knowledge of the p-equivalence class and the height class of a uniformity determine the uniformity?

CONJECTURE 4.1. (THRON'S CONJECTURE). *There exists at most one uniformity in the intersection of a height class and a p-equivalence class.*

All attempts by the author to prove this conjecture have failed. On the other hand, the construction of a counter-example appears to be very difficult. We wish to find two distinct uniformities A and B such that $A =^h B$ and $A =^p B$. Since Theorem 3.1 implies that the p-equivalence class of A and B cannot contain $A \vee B$, one is led to look at existing examples of uniformities A and B such that

$$A =^p B \neq^p A \vee B\,.$$

The first such example was constructed by Alfsen and Njåstad [3].

For convenience, we describe it in a form slightly different from the original.

EXAMPLE 4.1. Let X be a set whose elements are indexed by the product of the integers with themselves, i.e.,

$$X = \{x_{i,j}\colon i=1,2,\ldots,\ j=1,2,\ldots\}.$$

Let $A_i=\{x_{i,j}\colon j=1,2,\ldots\}$, $B_j=\{x_{i,j}\colon i=1,2,\ldots\}$, and let

$$A = \bigcup_{i=1}^{\infty}(A_i\times A_i), \qquad B = \bigcup_{i=1}^{\infty}(B_j\times B_j).$$

By abuse of the language we will let A and B stand for the uniformities generated by A and B respectively. Then Alfsen and Njåstad have shown

$$[A \vee (A_\omega \vee B_\omega)] \neq [B \vee (A_\omega \vee B_\omega)],$$

$$[A \vee (A_\omega \vee B_\omega)] =^p [B \vee (A_\omega \vee B_\omega)] \neq^p [A \vee (A_\omega \vee B_\omega)] \vee [B \vee (A_\omega \vee B_\omega)] = A \vee B.$$

By Corollary 2.2, we have $A =^h [A\vee(A_\omega\vee B_\omega)]$ and similarly for B. Thus, if $A =^h B$, we would have the example for which we are looking. However, suppose $A \leqq^h B$, then there would exist a finite cover, $C_1, C_2, \ldots, C_n$, of X such that

$$B\cap(C_i\times C_i)\subset A.$$

It is easy to see that, if $x_{j,1}\in C_i$, then $x_{k,1}\notin C_i$ for $k\neq j$. Consequently, the n sets $C_1, C_2,\ldots, C_n$ cannot cover X. Thus A and B are not comparable in height.

We shall say that $\mathscr{U}$ is a *partition uniformity* if there exists a partition $\{A_a\colon a\in\mathscr{A}\}$ of X such that the single entourage $\bigcup_{a\in\mathscr{A}}(A_a\times A_a)$ is a base for $\mathscr{U}$.

By using Example 4.1 as a model we have been able to construct the following example of two uniformities A and B such $A\leqq^h B$, $A=^p B$, and $A \not\leqq B$.

EXAMPLE 4.2. Let A be as in Example 4.1. Let j be any positive integer. Let $n=\frac{1}{2}(j+1)$ if j is odd, and $\frac{1}{2}j$ if j is even. Let k be any positive integer or zero. Given j, let $i=1,2,\ldots,n$. If j is odd, let $l=2nk+i$. If j is even, let $l=(2k-1)n+i$. In either case let $m=l+n$. If j is odd, or if j is even and $1\leqq k$, let $B_{i,j,k}=\{x_{l,j}, x_{m,j}\}$. If j is even and $k=0$, let $B_{i,j,k}=\{x_{m,j}\}$. The $B_{i,j,k}$'s constitute a partition of X. Let

$$B = \bigcup(B_{i,j,k}\times B_{i,j,k}).$$

Let C_1 be the set of all $x_{l,j}$'s, and let C_2 be the set of all $x_{m,j}$'s. Clearly, $C_1 \cup C_2 = X$, and if

$$C = (C_1 \times C_1) \cup (C_2 \times C_2) ,$$

then $B \cap C = \Delta \subset A$ so that (if B is also used to denote the uniformity generated by B) $A \leqq^h B$.

Now A_ω has a base of elements of the form $U = \bigcup_{j=1}^n (F_j \times F_j)$ where each F_j is a union of A_i's, and the F_j's constitute a finite cover of X. Consequently, some F_j contains at least two A_i's. Suppose F_{j_1} contains both A_{i_1} and A_{i_2} such that $i_1 < i_2$ and $0 < n = i_2 - i_1$. Let $i_1 = s + rn$, where $0 < s \leqq n$. If r is even, let $t = 2n-1$ and $k = \frac{1}{2} r$. If r is odd, let $t = 2n$ and $k = \frac{1}{2}(r+1)$. Then $B_{s,t,k} = \{x_{i_1,t}, x_{i_2,t}\}$. Clearly

$$(B_{s,t,k} \times B_{s,t,k}) \subset (B \cap U) \not\subset A .$$

We conclude that

$$\mathscr{U} = A \vee (A_\omega \vee B_\omega) = A \vee B_\omega \not\leqq B \vee A_\omega = B \vee (A_\omega \vee B_\omega) = \mathscr{V} .$$

Thus, we have constructed an example of two uniformities in a p-equivalence class which are comparable in height but not comparable in the usual ordering.

We note that $\mathscr{U} \vee \mathscr{V}$ is not in the p-equivalence class of $\mathscr{U}$ and $\mathscr{V}$. (This follows either from Theorem 3.1 or by observing that $A \cap B = \Delta$.) However, any extension of the approach used in Example 4.2 must fail to give us a counter example to Thron's conjecture because of:

THEOREM 4.1. *If $\mathscr{U}$ and $\mathscr{V}$ are partition uniformities such that $\mathscr{U} =^h \mathscr{V}$, then $[\mathscr{U} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)] = [\mathscr{V} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)]$.*

PROOF. Suppose $A = \bigcup_{a \in \mathscr{A}} (A_a \times A_a)$ is a base for $\mathscr{U}$ and $B = \bigcup_{b \in \mathscr{B}} (B_b \times B_b)$ is a base for $\mathscr{V}$. Since $\mathscr{U} =^h \mathscr{V}$, there exists a finite cover C_i, $i = 1, 2, \ldots, n$, of X such that $B \cap (C_i \times C_i) \subset A$, and a finite cover D_j, $j = 1, 2, \ldots, m$ of X such that $A \cap (D_j \times D_j) \subset B$. Clearly for $b \in \mathscr{B}$, $B_b \cap C_i$ can intersect only one A_a. Since the C_i's cover X, B_b can intersect at most n A_a's. Similarly each A_a can intersect at most m B_b's. Let E_1 be a maximal family of B_b's having the property P: For $a \in \mathscr{A}$, no more than one member of E_1 can intersect A_a.

Such an E_1 exists by Zorn's lemma. Having constructed $E_1, E_2, \ldots, E_k$ we may construct E_{k+1} maximal such that E_{k+1} does not intersect $E_1, E_2, \ldots, E_k$ and satisfying property P. Now $E_1, E_2, \ldots, E_{nm+1}$ must exhaust all the B_b's since if, for some B_b,

$$B_b \notin E_l, \qquad l = 1, 2, \ldots, nm+1 ,$$

then there exists A_{a_l} and B_{b_l} such that $B_{b_l} \in E_l$ and both B_b and B_{b_l} intersect A_{a_l}. Since B_b can intersect at most n distinct A_a's there can occur at most n distinct A_a's among the A_{a_l}'s. Since each A_a can intersect at most m B_b's, there can occur at most $m \cdot n$ distinct B_b's among the B_{b_l}'s. But this is a contradiction since the E's are disjoint. Let

$$F_l = \bigcup_{B_b \in E_l} B_b, \qquad l = 1, 2, \ldots, nm+1 .$$

Clearly, $B(F_l) = F_l$, so the F_l's constitute a p-cover of X with respect to the p-equivalence class of B. Therefore,

$$F = \bigcup_{l=1}^{nm+1} (F_l \times F_l) \in \mathscr{V}_\omega \subset (\mathscr{U}_\omega \vee \mathscr{V}_\omega) .$$

Now, if $(x, y) \in A \cap F$ both x and y are members of the same A_a and thus, by the method of construction of F, both x and y are members of the same B_b so that $(x, y) \in B$. Thus, we see finally that

$$A \cap F \subset B, \qquad (A \cap F) \in [\mathscr{U} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)]$$

which implies $B \in [\mathscr{U} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)]$ which in turn implies $\mathscr{V} \leqq [\mathscr{U} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)]$ so that

$$[\mathscr{V} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)] \leqq [\mathscr{U} \vee (\mathscr{U}_\omega \vee \mathscr{V}_\omega)] .$$

Similarly, we can prove the reverse inequality.

The conjecture suggested by dualizing Corollary 3.1 is:

CONJECTURE 4.2. *If* $\mathscr{U} \leqq^h \mathscr{V}$, *then* $(\mathscr{U}_h \wedge \mathscr{V}) =^h \mathscr{U}$.

EXAMPLE 4.3. Let A and B be the two partition uniformities constructed in Example 4.1 above. Consider $\mathscr{U} = A_h \wedge B_h$. Let

$$C_1 = \{x_{i,i} \colon i = 1, 2, \ldots\} ,$$

let $C_2 = C_1{}^c$, and let $C = \bigcup_{i=1}^2 (C_i \times C_i)$. Then $(C \cap A) \in A_h$ and $(C \cap B) \in B_h$. If

$$D = \{(x_{i,i}, x_{i,i}) \colon i = 1, 2, \ldots\} \cup \{(x_{i,j}, x_{k,l}) \colon i \neq j \text{ and } k \neq l\} ,$$

then D generates a partition uniformity $\mathscr{U}$, and

$$(C \cap A) \cup (C \cap B) \subset D .$$

Consequently, $\mathscr{U} \leqq (A_h \wedge B_h)$, so that $\mathscr{U} \leqq^h A$ by Corollary 3.3. Clearly $\mathscr{U}$ is not totally bounded. Thus, if we show that $A \wedge \mathscr{U}_h$ is totally bounded, we will have a counter-example to Conjecture 4.2 by applying Theorem 2.3.

Consider $\mathscr{W} = (A \wedge B_h)$; clearly any entourage $W \in \mathscr{W}$ must contain an entourage $W^{\frac{1}{3}} \in \mathscr{W}$ such that $W^{\frac{1}{3}} \circ W^{\frac{1}{3}} \circ W^{\frac{1}{3}} \subset W$. But since $W^{\frac{1}{3}} \in \mathscr{W}$, the entourage $W^{\frac{1}{3}}$ must contain a subset of the form

$$A \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{n} (B_j \cap C_k) \times (B_j \cap C_k) \right),$$

where C_k, $k = 1, 2, \ldots, n$, is a finite cover of X. Suppose $(x_{i,1}, x_{j,1}) \in C_k \times C_k$, then for arbitrary l and m, $(x_{i,l}, x_{i,1})$ and $(x_{j,1}, x_{j,m})$ are members of A, thus

$$(x_{i,l}, x_{j,m}) \in W^{\frac{1}{3}} \circ W^{\frac{1}{3}} \circ W^{\frac{1}{3}} \subset W,$$

and since the C_k's constitute a finite cover of X, W is totally bounded, and thus $\mathscr{W}$ is totally bounded.

Since $(A \wedge \mathscr{U}_h) \leqq \mathscr{W}$, $(A \wedge \mathscr{U}_h)$ is totally bounded, and this completes the proof.

We conclude with a theorem showing that height determines the Cauchy ultrafilters, and an example showing that height distinguishes between more uniformities than do the Cauchy ultrafilters.

Theorem 4.2. *If $\mathscr{U}$ and $\mathscr{V}$ are two uniformities such that $\mathscr{U} = {}^h \mathscr{V}$, then $\mathscr{U}$ and $\mathscr{V}$ have the same Cauchy ultrafilters.*

Proof. It will suffice to show that, if $\mathscr{U} \leqq^h \mathscr{V}$, then every ultrafilter which is Cauchy with respect to $\mathscr{V}$ is also Cauchy with respect to $\mathscr{U}$.

Suppose $\mathscr{F}$ is an ultrafilter which is Cauchy with respect to $\mathscr{V}$. Pick $U \in \mathscr{U}$. Let V be a member of $\mathscr{V}$ and let $A_1, A_2, \ldots, A_n$ be a finite cover of X such that

$$A = \bigcup_{i=1}^{n} (A_i \times A_i) \subset U \cup V^c.$$

Since $\mathscr{F}$ is Cauchy with respect to $\mathscr{V}$ there exists $F \in \mathscr{F}$ such that $F \times F \subset V$. Let $B_i = A_i \cap F$. Since $\bigcup_{i=1}^{n} B_i = F$, some B_i is a member of $\mathscr{F}$. For x and y members of this B_i we have $x, y \in F \cap A_i$ which implies

$$(x,y) \in \big(V \cap (A_i \times A_i)\big) \subset U.$$

Thus, $B_i \times B_i \subset U$ and we have shown $\mathscr{F}$ Cauchy with respect to $\mathscr{U}$.

Example 4.4 Let R be the set of real numbers. Let d_1 be the usual metric on the real numbers, and let $d_2(x,y) = |x^2 - y^2|$. Let $\mathscr{U}$ be the uniformity generated by d_1. Let $\mathscr{V}$ be the uniformity generated by d_1 and d_2. It is well known that $(X, \mathscr{U})$ is complete, i.e., the only Cauchy ultrafilters are the "principal" ultrafilters generated by singletons.

Since $\mathscr{U} \leqq \mathscr{V}$, every ultrafilter which is Cauchy with respect to $\mathscr{V}$ is also Cauchy with respect to $\mathscr{U}$. On the other hand, every principal ultrafilter must be Cauchy with respect to $\mathscr{V}$. Thus $\mathscr{U}$ and $\mathscr{V}$ have the same Cauchy ultrafilters.

In order to see that $\mathscr{U} \neq^h \mathscr{V}$, it will suffice to show that $\mathscr{V} \nleqq^h \mathscr{U}$. In turn, for this, it will suffice to show that, given $\varepsilon > 0$,

$$W = \{(x,y)\colon\ |x^2-y^2| < 1 \text{ or } |x-y| \geqq \varepsilon\}$$

is not totally bounded. Let $A_1, A_2, \ldots, A_n$ be any finite number of subsets of R such that $(A_i \times A_i) \subset W$. Let m be a positive integer such that $1/m < \varepsilon$. Consider the points

$$x_i = m^2n^2 + \frac{i}{mn}, \qquad i = 1,2,\ldots,n,n+1 \ .$$

For any pair x_{i_1} and x_{i_2} we have $|x_{i_1} - x_{i_2}| < 1/m < \varepsilon$, and

$$|x_{i_1}^2 - x_{i_2}^2| = |x_{i_1} - x_{i_2}| \cdot |x_{i_1} + x_{i_2}| \geqq \frac{1}{mn} \cdot 2mn = 2 \ .$$

Consequently, no more than one x_i can be a member of any A_j, $j = 1,2, \ldots, n$. This implies that the A_j's do not cover R.

Thus, $\mathscr{U}$ and $\mathscr{V}$ are not equal in height, but they have the same Cauchy ultrafilters.

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