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ON THE DECOMPOSITION OF A CHOQUET SIMPLEX INTO A DIRECT CONVEX SUM OF COMPLEMENTARY FACES

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The purpose of this note is to study the problem mentioned in the title. We shall give a necessary and sufficient condition that a face F shall induce such a decomposition (Theorem 1). This condition is satisfied if F is closed; which in the metrizable case implies that the complementary face F' is an $F_{\sigma\delta}$-set (Theorem 2). The proofs are mainly combinatorial, and the presentation involves a number of preliminary propositions which may be of some independent interest.

There is a close relationship between the material of the first part of the present note and O. Hustad's investigations on supplementary subcones [5]. In particular one may obtain the decomposition of a simplex into a direct convex sum of a closed face F and its complement F' (Corollary of Theorem 1) by application of the Corollary 2 of Proposition 10 of [5].

Throughout the paper K is assumed to be a compact convex subset of a locally convex Hausdorff space E over the reals, and all occurring functions are assumed to be real valued. The concept of a *face* is defined e.g. in [1, p. 99], and we recall that the face generated by a point x of K, can be expressed as follows:

$$\text{face}(x) = \bigcup_{n=1}^{\infty} D(x,n)\,, \tag{1}$$

where

$$D(x,\alpha) = (\alpha x-(\alpha-1)K) \cap K, \qquad \alpha \geqq 1\,. \tag{2}$$

We shall use the symbol F' to denote the union of all faces not meeting a given face F. Thus by definition

$$x \in F' \iff \text{face}(x) \cap F = \emptyset\,. \tag{3}$$

For later references we state the following fundamental property of simplexes, which is obtained by the lattice-characterization of simplexes

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(cf. e.g. [4, p. 145]) and by the Decomposition Lemma for vector lattices [3, p. 19].

PROPOSITION 1. *If K is a simplex, and if $x=\sum_{i=1}^{n}\mu_i y_i$, $x=\sum_{j=1}^{m}\nu_j z_j$ are two proper convex combinations on K, then there exists a third convex combination $x=\sum_{ij}\varrho_{ij}w_{ij}$ on K such that y_i, z_j can be expressed by the following convex combinations*:

$$y_i = \sum_{j=1}^{m}\varrho_{ij}\mu_i^{-1}w_{ij}, \qquad i=1,\ldots,n, \tag{4}$$

$$z_j = \sum_{i=1}^{n}\varrho_{ij}\nu_j^{-1}w_{ij}, \qquad j=1,\ldots,m. \tag{5}$$

Note that the investigations up to Theorem 1 only depend on the conclusion of Proposition 1, and so they are independent of the topological properties of K and E.

PROPOSITION 2. *If F is a face of a simplex K, then F' is also a face of K.*

PROOF. Clearly it suffices to prove that F' is convex. To this end consider a proper convex combination

$$x = \mu_1 y_1 + \mu_2 y_2, \tag{6}$$

where $y_1, y_2 \in F'$.

If $x \notin F'$, then there exists a point $z_1 \in \mathrm{face}(x)\cap F$. By the definition of a face, there exists a convex combination

$$x = \nu_1 z_1 + \nu_2 z_2, \tag{7}$$

where $z_1 \in F$, $z_2 \in K$, and $\nu_1 \neq 0$. In this case also $\nu_2 \neq 0$, for otherwise $x=z_1 \in F$ and so $y_i \in \mathrm{face}(x) \subset F$ contrary to the assumption $y_i \in F'$ for $i=1,2$. Thus (6) and (7) are both proper convex combinations, and so there exists a convex combination $x=\sum_{i,j=1}^{2}\varrho_{ij}w_{ij}$ on K satisfying (4) and (5).

Assume first $\varrho_{11} \neq 0$. Then the expression (5) for z_1 implies $w_{11} \in \mathrm{face}(z_1) \subset F$, and the expression (4) for y_1 implies $w_{11} \in \mathrm{face}(y_1)$, which gives a contradiction since $y_1 \in F'$.

Assume next $\varrho_{11}=0$. Then the expression (5) for z_1 implies $z_1=w_{21}$ and $\varrho_{21}=\nu_1 \neq 0$, and now the expression (4) for y_2 implies $z_1=w_{21} \in \mathrm{face}(y_2)$, which is a contradiction since $z_1 \in F$ and $y_2 \in F'$.

These two contradictions complete the proof.

If F is a face of a simplex K, then the set F' will be termed the *complementary face* of F. This terminology, however, is only partly justified by the properties of F'. Clearly F' is the largest face disjoint from F, but it is by no means certain that K itself is the smallest face containing both F and F'. In fact we shall establish a necessary and sufficient condition that $K = \text{face}(F \cup F')$, and we first observe that this is equivalent to $K = \text{conv}(F \cup F')$ by virtue of the following

PROPOSITION 3. *If F and G are faces of a simplex K, then*

$$\text{face}(F \cup G) = \text{conv}(F \cup G) \tag{8}$$

PROOF. Let y_1 be an arbitrary element of $\text{face}(F \cup G)$. By the explicite expression for the face generated by a convex set [1, p. 99], there exists a point $x \in \text{conv}(F \cup G)$ and a convex combination

$$x = \mu_1 y_1 + \mu_2 y_1 , \tag{9}$$

where $y_2 \in K$ and $\mu_1 \neq 0$. In this case we may also assume $\mu_2 \neq 0$, for otherwise $y_1 = x \in \text{conv}(F \cup G)$ and there is nothing more to prove.

Since $x \in \text{conv}(F \cup G)$, there exists a convex combination

$$x = \nu_1 z_1 + \nu_2 z_2 , \tag{10}$$

where $z_1 \in F$ and $z_2 \in G$.

If $\nu_1 = 0$, then $x = z_2 \in G$, and by (9) $y_1 \in \text{face}(x) \subset G$. Similarly $\nu_2 = 0$ implies $y_1 \in F$. In both cases we are through, and so we may assume $\nu_1 \neq 0$ and $\nu_2 \neq 0$ for the rest of the proof.

Now (9) and (10) are proper convex combinations, and so there exists a convex combination $x = \sum_{i,j=1}^{2} \varrho_{ij} w_{ij}$ on K satisfying (4) and (5). Here the proof splits up in a few simple cases:

1) Assume $\varrho_{11} = 0$. Then the expression (4) for y_1 implies $y_1 = w_{12}$ and $\varrho_{12} = \mu_1 \neq 0$. Now the expression (5) for z_2 implies $y_1 = w_{12} \in \text{face}(z_2) \subset G$.

2) Assume $\varrho_{12} = 0$ and apply a similar argument to yield $y_1 \in F$.

3) Assume $\varrho_{11} \neq 0$ and $\varrho_{12} \neq 0$. Then the expressions (5) imply $w_{11} \in \text{face}(z_1) \subset F$ and $w_{12} \in \text{face}(z_2) \subset G$. Hence by the expression (4) for y_1, $y_1 \in \text{conv}(F \cup G)$ and the proof is complete.

If F is a face of K and $x \in K \setminus F'$, then there exists an $\alpha \geqq 1$ such that $D(x,\alpha) \cap F \neq \emptyset$, and we shall write

$$\delta(x,F) = \inf\{\alpha \mid D(x,\alpha) \cap F \neq \emptyset\} . \tag{11}$$

One may term $\delta(x,F)$ the "relative distance" from x to F, and it is natural to write $\delta(x,F) = \infty$ if $x \in F'$, although that will not be needed in the sequel.

PROPOSITION 4. *Let F be a face of the convex set K and let x be a point of $K \setminus F \cup F'$. If*

$$y_1 \in D(x,\alpha_0) \cap F, \quad \textit{where} \quad \alpha_0 = \delta(x,F)\,,$$

then there is a convex combination

$$x = \mu y_1 + (1-\mu) y_2\,, \tag{12}$$

where $y_2 \in F'$ *and* $\mu = \alpha_0^{-1}$.

Proof. By the definition 2, there is a point $y_2 \in K$ such that

$$y_1 = \alpha_0 x - (\alpha_0 - 1) y_2\,.$$

Hence there is a convex combination

$$x = \mu y_1 + (1-\mu) y_2\,, \tag{13}$$

where $\mu = \alpha_0^{-1} \neq 0$. Also $\mu \neq 1$, for otherwise $x = y_1 \in F$ contrary to assumptions.

To verify that $y_2 \in F'$ we assume the converse, by which there exists a point $w_1 \in \mathrm{face}(y_2) \cap F$. By the definition of a face there must be a convex combination

$$y_2 = \varrho w_1 + (1-\varrho) w_2\,, \tag{14}$$

where $w_2 \in K$ and $\varrho \neq 0$. Also $\varrho \neq 1$, for otherwise $y_2 = w_1 \in F$, which by (13) would imply $x \in F$ contrary to assumptions. Now consider the point z defined by

$$z = \nu y_1 + (1-\nu) w_1\,, \tag{15}$$

where $\nu = \mu(\mu + \varrho - \mu\varrho)^{-1}$. It is easily verified that $0 < \nu < 1$. Hence (15) is a proper convex combination. In particular $z \in F$. By substitution of (14) and (15) into (13) one obtains

$$x = \nu^{-1}\mu z + (1 - \nu^{-1}\mu) w_2\,.$$

Solving for z and remembering that $\mu = \alpha_0^{-1}$, one obtains

$$z = \nu\alpha_0 x - (\nu\alpha_0 - 1) w_2 \in D(x, \nu\alpha_0) \cap F\,.$$

This contradicts the definition of α_0 since $\nu < 1$, and so the proof is complete.

Proposition 5. *Let F be a face of a simplex K, and consider a proper convex combination*

$$x = \mu y_1 + (1-\mu) y_2\,, \tag{16}$$

where $y_1 \in F$ *and* $y_2 \in F'$. *If a convex combination*

$$x = \nu z_1 + (1-\nu) z_2\,, \tag{17}$$

is distinct from (16) *and if* $z_1 \in F$, *then* $z_2 \notin F'$ *and* $\nu < \mu$.

PROOF. Note first that $x \notin F$ for otherwise (16) would imply $y_2 \in \text{face}(x) \subset F$ contrary to the assumption $y_2 \in F'$. Also $x \notin F'$ since $y_1 \in \text{face}(x) \cap F$. It follows that $\nu \neq 1$, since $\nu = 1$ implies $x = z_1 \in F$. Also we may assume $\nu \neq 0$ for otherwise $z_2 = x \notin F'$ and $\nu < \mu$; hence there is nothing more to prove.

Now (16) and (17) are proper convex combinations, and so there exists a convex combination

$$x = \sum_{i,j}^{2} \varrho_{ij} w_{ij}$$

satisfying (4) and (5) with $\mu_1 = \mu$, $\mu_2 = 1 - \mu$, $\nu_1 = \nu$, $\nu_2 = 1 - \nu$.

We first observe that $\varrho_{21} = 0$. In fact, if $\varrho_{21} \neq 0$, then the formulas (4) and (5) for y_2 and z_1 would yield $w_{21} \in \text{face}(y_2)$ and $w_{21} \in \text{face}(z_1) \subset F$, which is a contradiction since $y_2 \in F'$.

Since $\varrho_{21} = 0$, it follows from the formulas (4) and (5) for y_2 and z_1 that $\varrho_{22} = \varrho_{21} + \varrho_{22} = \mu_2 = 1 - \mu$ and $\varrho_{11} = \varrho_{11} + \varrho_{21} = \nu_1 = \nu$, and furthermore, that $y_2 = w_{22}$ and $z_1 = w_{11}$.

Next, we observe that $\varrho_{12} \neq 0$. In fact, if $\varrho_{12} = 0$, then it would follow from the relation $\varrho_{11} + \varrho_{12} + \varrho_{21} + \varrho_{22} = 1$ that $\mu = \nu$, and from the equation

$$x = \varrho_{11} w_{11} + \varrho_{22} w_{22} = \nu z_1 + (1 - \mu) y_2$$

together with (16) and (17) that these were identical in contradiction with the assumptions.

Now, since

$$1 = \varrho_{11} + \varrho_{12} + \varrho_{22} = \nu + \varrho_{12} + 1 - \mu ,$$

we obtain $\mu = \nu + \varrho_{12} > \nu$. Furthermore, since $\varrho_{12} \neq 0$, the formulas (4) and (5) for y_1 and z_2 yield $w_{12} \in \text{face}(y_1) \subset F$ and $w_{12} \in \text{face}(z_2)$. Hence $\text{face}(z_2) \cap F \neq \emptyset$, and so $z_2 \notin F'$.

This completes the proof.

THEOREM 1. *Let F be a face of a simplex K and let F' be the complementary face. For a given $x \in K \setminus F \cup F'$ there is at most one convex combination*

$$x = \mu_1 y_1 + \mu_2 y_2 , \tag{18}$$

with $y_1 \in F$ and $y_2 \in F'$.

Such a convex combination exists if and only if the "relative distance" $\alpha_0 = \delta(x, F)$ is attained in F, that is, if

$$F \cap D(x, \alpha_0) \neq \emptyset;$$

in which case the point y_1 of (18) *is the unique member of this intersection and $\mu_1 = \alpha_0^{-1}$.*

PROOF. 1) We first prove that a convex combination (18) with $y_1 \in F$ and $y_2 \in F'$, must satisfy the requirements $y_1 \in D(x, \alpha_0)$ and $\mu_1 = \alpha_0^{-1}$.

To this end consider an arbitrary number $\alpha > \alpha_0$. Then there is an element $z_1 \in F \cap D(x, \alpha)$, and by the definition (2) there exists a $z_2 \in K$ such that

$$z_1 = \alpha x - (\alpha - 1) z_2 . \tag{19}$$

Observe that $\alpha > 1$, since $\alpha = 1$ would imply $x = z_1 \in F$ contrary to assumption.

Writing $\nu_1 = \alpha^{-1}$ and $\nu_2 = 1 - \nu_1$, we may convert (19) into the proper convex combination

$$x = \nu_1 z_1 + \nu_2 z_2 . \tag{20}$$

Since $z_1 \in F$, we may apply Proposition 5 to obtain $\nu_1 \leqq \mu_1$, or equivalently $\alpha \geqq \mu_1^{-1}$. Since $\alpha > \alpha_0$ was arbitrary, we must have $\alpha_0 \geqq \mu_1^{-1}$.

Solving (18) for y_1, one obtains $y_1 \in D(x, \mu_1^{-1})$. Hence $\mu_1^{-1} \geqq \alpha_0$, and so we must have $\mu_1^{-1} = \alpha_0$ and $y_1 \in D(x, \mu_1^{-1})$.

2) Next we prove that $D(x, \alpha_0) \cap F$ can not have more than one element, which by the first part of the proof will establish the uniqueness of a convex combination (18) with $y_1 \in F$ and $y_2 \in F'$.

To this end we assume that y_1 and z_1 are two members of $D(x_1, \alpha_0) \cap F$. Applying the definition (2) and solving for y_1 and z_1, we observe that y_1 and z_1 occur in convex combinations like (18) and (20) with $\mu_1 = \nu_1 = \alpha_0^{-1}$. By Proposition 5, this entails $y_1 = z_1$.

3) Finally assume $D(x, \alpha_0) \cap F \neq \emptyset$. By Proposition 4, there is a convex combination (18) with $y_1 \in F$ and $y_2 \in F'$, and the proof is complete.

COROLLARY. *If F is a closed face of a (compact) simplex, then every $x \in K \setminus F \cup F'$ can be decomposed uniquely into a convex combination*

$$x = \mu_1 y_1 + \mu_2 y_2 \tag{21}$$

with $y_1 \in F$ and $y_2 \in F'$.

PROOF. Let $\alpha_0 = \delta(x, F)$. By compactness

$$F \cap D(x, \alpha_0) = \bigcap_{\alpha > \alpha_0} F \cap D(x, \alpha) \neq \emptyset ,$$

and the conclusion follows from Theorem 1.

Following the terminology of [2], we shall say that a function on K is of class $\mathscr{G}$ if it is affine and l.s.c., and it is of class $\mathscr{G}_\delta$ if it is the pointwise limit of a descending sequence from $\mathscr{G}$. Also we shall use the symbol μ_x to denote the unique positive normalized boundary measure [1, p. 98] with barycenter x in a simplex K.

PROPOSITION 6. *If K is a metrizable simplex and $f \in C(K)$, then the function $x \rightsquigarrow \int f d\mu_x$ is of class $\mathcal{G}_\delta$.*

PROOF. The u.s.c. upper envelope $\bar{f}$ is pointwise limit of the (downward-) directed set of all continuous and concave proper majorants of f (cf. e.g. [4, p. 140]). By a standard argument (based on the existence of a countable base for the compact metrizable space K) there is a descending *sequence* $\{g_n\}$ of continuous concave functions on K which converge pointwise to $\bar{f}$.

By a known result (cf. e.g. [4, p. 145]), the l.s.c. lower envelopes $\underline{g}_n$ are affine for $n=1,2,\ldots$. Hence $\underline{g}_n$ is a descending sequence from $\bar{\mathcal{G}}$, and the limit $k=\inf_n \underline{g}_n$ is of class $\mathcal{G}_\delta$.

It is known (cf. e.g. [4, p. 145]) that

$$\underline{g}_n(x) = \int g_n \, d\mu_x ,$$

for all $x \in K$, $n=1,2,\ldots$. By the Monotone Convergence Theorem and by the definition of boundary measure [1, p. 98],

$$\begin{aligned} k(x) &= \inf_n \int \underline{g}_n \, d\mu_x = \inf_n \int g_n \, d\mu_x \\ &= \int \bar{f} \, d\mu_x = \int f \, d\mu_x . \end{aligned}$$

This completes the proof since $k \in \mathcal{G}_\delta$.

THEOREM 2. *If F is a closed face of a metrizable simplex K, then F' is an $F_{\sigma\delta}$-set.*

PROOF. Let K be metrizable, and define a function k by

$$k(x) = \mu_x(F), \qquad x \in K . \tag{22}$$

The indicator function χ_F is u.s.c., and so there exists a descending sequence $\{f_n\}$ from $C(K)$ which converges pointwise to χ_F. By the Monotone Convergence Theorem

$$k(x) = \int \chi_F \, d\mu_x = \lim_{n\to\infty} g_n(x) ,$$

where $g_n(x) = \int f_n d\mu_x$ for $n=1,2,\ldots$. By Proposition 6, $g_n \in \mathcal{G}_\delta$ for $n=1,2,\ldots$. It follows in particular that k is affine.

For every natural number n, let $(g_{n,m})_{m=1,2}\ldots$ be a descending sequence from $\mathcal{G}$ converging pointwise to g_n, and define

$$k_n = \inf\{g_{i,j} \mid i,j=1,\ldots,n\} .$$

Now $\{k_n\}$ is a descending sequence of l.s.c. functions which converges pointwise to k.

If $x \in F$, then $\mathrm{Spt}(\mu_x) \subset F$ (cf. e.g. [1, p. 100]), and so $k(x) = 1$.

We claim that if $x \in F'$, then $k(x) = 0$. To verify this assertion, we assume the converse, that is $\mu_x(F) \neq 0$. We first observe that $\mu_x(F) \neq 1$, for otherwise $\mathrm{Spt}(\mu_x) \subset F$ and so $x \in F$, contrary to the assumption $x \in F'$. Now we write $\mu_x(F) = \lambda$, and we define two positive normalized measures π_1 and π_2 as follows

$$\pi_1 = \lambda^{-1}(\mu_x)_F, \qquad \pi_2 = (1-\lambda)^{-1}(\mu_x)_{\complement F}.$$

Now

$$\mu_x = \lambda \pi_1 + (1-\lambda)\pi_2 .$$

Writing y_1 and y_2 for the barycenters of π_1 and π_2, we obtain a proper convex combination

$$x = \lambda y_1 + (1-\lambda) y_2 .$$

Hence $y_1 \in \mathrm{face}(x)$. On the other hand $\mathrm{Spt}(\pi_1) \subset F$, and so $y_1 \in F$. This gives the desired contradiction since $x \in F'$. Thus we have completed the proof that $k(x) = 0$ for $x \in F'$.

Applying the decomposition of the first part of the theorem together with the fact that k is an affine function, we obtain

$$k(x) = 0 \iff x \in F' . \tag{23}$$

Now define

$$E_{m,n} = \{x \mid k_n(x) \leq 1/m\}, \qquad m,n = 1,2,\ldots .$$

By the lower semi-continuity of k_n, $E_{m,n}$ is closed for all m,n. By virtue of (23) and the fact that $k_n \searrow k$, we shall have

$$F' = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_{m,n} .$$

Hence F' is an $F_{\sigma\delta}$-set, and the proof is complete.

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