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# CONVEX SETS AND CHEBYSHEV SETS II

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## Introduction.

In a previous paper [1], the author proved the following result:

(V) *For every integer* $n \geqq 3$ *there exists an n-dimensional non-smooth Banach space E with the property that every Chebyshev set in E is convex.*

(A subset $M$ of a normed linear space $E$ is called a Chebyshev set if each point in $E$ has a unique nearest point in $M$. We shall take all spaces as real linear spaces; i.e. every complex space is identified with its underlying real space.)

Theorem (V) is a supplement to the following list of well-known results:

(I) *A finite dimensional Banach space E is rotund if and only if every non-empty closed convex set in E is a Chebyshev set.*

(II) *A finite dimensional Banach space E is rotund and smooth if and only if the Chebyshev sets in E are identical with the non-empty closed convex sets in E.*

(III) *A 2-dimensional Banach space E is smooth if and only if every Chebyshev set in E is convex.*

(IV) *If E is a finite dimensional smooth Banach space, then every Chebyshev set in E is convex.*

This collection of theorems naturally suggests the problem of characterizing those finite dimensional Banach spaces in which every Chebyshev set is convex,—in terms of the geometrical properties of the unit ball. In the present paper we shall give such a characterization for spaces of dimension 3. The proof is based on a lemma which also yields the basis for a new proof of (V).

One might also ask whether it is possible to characterize those finite dimensional spaces which are smooth,—in terms of the Chebyshev sets. In the final section we shall give a solution of this (as it comes out) not very interesting problem.

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For references to the theorems (I)–(IV), see [1]. Section 2 (Terminology) and section 3 (Preliminaries) of [1] may also be useful to the reader.

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**Preliminaries.**

We shall briefly review some notation introduced in [1]. Let $E$ be a normed linear space with unit ball $K$. For a fixed Chebyshev set $M$ in $E$ the metric projection of $E$ onto $M$ (i.e. the mapping which to each point in $E$ assigns the unique nearest point in $M$) is denoted by $\pi$. If $\pi(z) = \pi(x)$ for every $x \in E\setminus M$ and every $z$ on the halfline emanating from $\pi(x)$ and passing through $x$, then $M$ is called a sun. For $x \in E\setminus N$ we put

$$K_x = x + \|\pi(x) - x\| K ,$$

and we let $T_x$ be the supporting cone of $K_x$ at the point $\pi(x)$; that is, $T_x$ is the intersection of all closed halfspaces containing $K_x$ and bounded by supporting hyperplanes of $K_x$ at $\pi(x)$. A point $x \in E\setminus M$ is said to be of type 1 if $\pi(x)$ is a smooth point of $K_x$. If $\pi(x)$ is a non-smooth point, then $x$ is said to be of type 2. No type is ascribed to points in $M$.

If $M$ is a Chebyshev set in a finite dimensional space, then $M$ is a sun, and the metric projection onto $M$ is continuous (cf. section 3 of [1]).

**Basic Lemma.**

In the proof of the lemma we shall use the following two propositions. Proposition 1 has been proved in [1].

Proposition 1. *Let $E$ be a normed linear space, and let $M$ be a Chebyshev set in $E$ which is a sun. Then* $(\operatorname{int} T_x) \cap M = \emptyset$ *for every* $x \in E\setminus M$.

Proposition 2. *Let $E$ be a normed linear space, and let $M$ be a Chebyshev set in $E$ which is a sun and has a continuous metric projection $\pi$. If for every $x \in E\setminus M$ there exists a $y \in E\setminus M$ such that $x \in \operatorname{int} T_y$ and $y$ is the limit of some sequence $\{y_n\}$ of points of type* 1, *then $M$ is convex.*

Proof. Assuming the condition fulfilled, we shall prove the convexity of $M$ by proving that every point $x \in E\setminus M$ can be separated from $M$ by a closed hyperplane; (i.e. there exists a closed hyperplane such that $x$ is contained in one of the open halfspaces determined by the hyperplane and $M$ is contained in the complement of this open halfspace). If $x \in E\setminus M$ is of type 1, then by proposition 1 the supporting hyperplane of $K_x$ at $\pi(x)$ separates $x$ from $M$. So, consider a point $x \in E\setminus M$ of type

2, and let $y$ and $\{y_n\}$ be as stated in the condition. Let $H_n$ be the (unique) supporting hyperplane of $K_{y_n}$ at $\pi(y_n)$. By proposition 1, $H_n$ separates $K_{y_n}$ and $M$. The hyperplanes $H_n$ are of the form

$$H_n = \{z \in E:\ \langle \xi_n, z\rangle = \langle \xi_n, \pi(y_n)\rangle\} ,$$

where $\xi_n$ is a non-zero element of $E^*$, the dual of $E$. If we require

$$\|\xi_n\| = 1 ,$$

and

$$\begin{aligned} \langle \xi_n, \pi(y_n)\rangle &= \sup\{\langle \xi_n, z\rangle:\ z \in K_{y_n}\} \\ &= \inf\{\langle \xi_n, z\rangle:\ z \in M\} , \end{aligned} \tag{1}$$

then $\xi_n$ is uniquely determined, and

$$\langle \xi_n, \pi(y_n)\rangle = \langle \xi_n, y_n\rangle + \|\pi(y_n) - y_n\| . \tag{2}$$

Since the sequence $\{\xi_n\}$ is contained in the unit ball of $E^*$, and any such ball is known to be weakly compact (i.e. $w^*$-compact), it follows that $\{\xi_n\}$ contains a weakly convergent subsequence. Hence, we may as well assume that $\{\xi_n\}$ is weakly convergent. Let $\xi$ be the limit point. Let

$$H = \{z \in E:\ \langle \xi, z\rangle = \langle \xi, \pi(y)\rangle\} .$$

It follows from (2) that

$$\langle \xi, \pi(y)\rangle = \langle \xi, y\rangle + \|\pi(y) - y\| .$$

Hence, $\xi$ is not identically zero, and therefore $H$ is a closed hyperplane. Furthermore, using (1) and the fact that every $z \in K_y$ is the limit of a sequence $\{z_n\}$, where $z_n \in K_{y_n}$, it is easy to verify that

$$\begin{aligned} \langle \xi, \pi(y)\rangle &= \sup\{\langle \xi, z\rangle:\ z \in K_y\} \\ &= \inf\{\langle \xi, z\rangle:\ z \in M\} . \end{aligned}$$

Consequently, the hyperplane $H$ separates $K_y$ and $M$. But then $H$ also separates $T_y$ and $M$. And since $x \in \operatorname{int} T_y$, it follows that $x$ is separated from $M$ by $H$. This completes the proof of proposition 2.

Let $K$ be the unit ball of a normed linear space $E$ (or more generally, a convex body in a locally convex topological linear space). One may define a face of $K$ to be a non-empty subset of $\operatorname{bd} K$ which is the intersection of $K$ and some collection of supporting hyperplanes of $K$. Clearly, every face is closed and convex, and every non-empty intersection of faces is a face. Among the faces containing a fixed $x \in \operatorname{bd} K$ there is a minimal one, namely the intersection of $K$ and all supporting hyperplanes of $K$ at $x$; we shall denote this minimal face by $N_x$. Note that a face $N_x$ may have faces of $K$ as proper subsets. It is well-known that if $E$ is finite dimensional, then every face of $K$ is of the form $N_x$, and every face may be obtained as the intersection of $K$ and just one supporting hyperplane.

LEMMA. *Let $E$ be a normed linear space with unit ball $K$, and let $\mathscr{F}$ be the set of those faces $N_x$ of $K$ which are determined by non-smooth boundary points $x$ of $K$. Assume that the following three conditions are fulfilled:*

(a) *Every $N_x \in \mathscr{F}$ contains more than one point.*

(b) *If $N_x, N_y \in \mathscr{F}$, and $N_x \cap N_y \neq \emptyset$, then $N_x = N_y$.*

(c) *The set $\mathscr{F}$ is at most countable.*

*Then every Chebyshev set in $E$ which is a sun and has a continuous metric projection is convex.*

REMARKS. Consider the following conditions:

(a′) Every exposed point of $K$ is a smooth point of $K$.

(b′) If $N_x \in \mathscr{F}$, and $H$ is a supporting hyperplane of $K$ such that $H \cap N_x \neq \emptyset$, then $N_x \subset H$.

(b″) If $N_x \in \mathscr{F}$, then $K$ has the same supporting cone at every point in $N_x$.

(c′) Every $N_x \in \mathscr{F}$ is contained in an open set $O_x$ such that $N_y \in \mathscr{F}$ and $N_y \cap O_x \neq \emptyset$ implies $N_x = N_y$.

Condition (a′) is an obvious consequence of (a). If $E$ is finite dimensional, then (a) and (a′) are equivalent, according to a remark above. It is easy to verify that each of the conditions (b′) and (b″) is equivalent to (b). The proof below will show that condition (c) may be replaced by condition (c′). However, if $E$ is separable, and in particular if $E$ is finite dimensional, condition (c′) implies (c). Condition (c′) always implies (b).

PROOF OF THE LEMMA. Let $M$ be a Chebyshev set in $E$ which is a sun and has a continuous metric projection $\pi$. Let $x \in E \setminus M$ be a point of type 2. To prove the convexity of $M$ it suffices, by proposition 2, to prove the existence of a point $y \in E \setminus M$ such that $x \in \operatorname{int} T_y$ and $y$ is the limit of a sequence of points of type 1.

Clearly, we may assume that $x = o$ and $\|\pi(o)\| = 1$, whence $K_o = K$. Let $N = N_{\pi(o)}$, and let $L$ be the flat spanned by $N$. Note that the flat spanned by $\pi(o)-N$ is the subspace $-\pi(o)+L$, and that this subspace has dimension $\geqq 1$ by condition (a), whence

$$\bigl(-\pi(o)+L\bigr)\setminus\bigl(\pi(o)-N\bigr) \neq \emptyset .$$

First, let us prove that

$$(3) \qquad z+K \subset K_z \quad \text{for every} \quad z \in -\pi(o)+L .$$

Since $-\pi(o)+L$ is parallel to every supporting hyperplane of $K$ at $\pi(o)$, and $K \subset T_o$, it follows that

$$z+K \subset T_o$$

for every $z \in -\pi(o)+L$. Furthermore, by proposition 1,

$$(\operatorname{int} T_o) \cap M = \emptyset .$$

Hence, we have

$$\bigl(\operatorname{int}(z+K)\bigr) \cap M = \emptyset$$

for every $z \in -\pi(o)+L$ which proves (3).

Let the mapping $\tau\colon E\setminus M \to \operatorname{bd} K$ be defined by

$$\tau(z) = \frac{\pi(z)-z}{\|\pi(z)-z\|}.$$

Then, by condition (b), we have

$$(4) \qquad z \in -\pi(o)+L \text{ and } \tau(z) \in N \quad \text{implies} \quad -z+T_z = T_z .$$

Using (3) and (4) we shall prove the following two statements:

$$(5) \qquad z \in \pi(o)-N \quad \text{implies} \quad \tau(z) \in N \text{ and } o \in \operatorname{int} T_z .$$

$$(6) \qquad z \in \bigl(-\pi(o)+L\bigr)\setminus\bigl(\pi(o)-N\bigr) \quad \text{implies} \quad \tau(z) \notin N .$$

Let $z \in \pi(o)-N$. Then $\pi(o) \in z+N$ which by (3) implies

$$z+K = K_z$$

and

$$\pi(z) = \pi(o) .$$

But then it follows that $\tau(z) \in N$. By condition (b) this implies

$$-z+T_z = T_o .$$

Using (4) we get $T_z = T_o$, and therefore

$$o \in \operatorname{int} T_z .$$

This completes the proof of (5). To prove (6), let

$$z \in \left(-\pi(o)+L\right)\setminus\left(\pi(o)-N\right),$$

and suppose $\tau(z) \in N$. We know by (3) that $z+K \subset K_z$. Suppose that

$$z+K \subsetneqq K_z .$$

Then clearly

$$T_o \subset \operatorname{int}(-z+T_z),$$

whence, by (4) and proposition 1, we get

$$\pi(o) \notin M ,$$

a contradiction. Hence,

$$z+K = K_z .$$

This clearly implies $\pi(z) \in L$ and $\pi(z) \neq \pi(o)$. Since $N$ is convex, and the segment $[\pi(z),\pi(o)]$ is contained in the flat $L$ spanned by $N$, it follows that $N$ contains a segment which is parallel to $[\pi(z),\pi(o)]$. Let $u$ be the midpoint of such a segment in $N$. Then it is easy to verify, using condition (b), that for a sufficiently large positive real $\lambda$ we have

$$\tfrac{1}{2}\left(\pi(z)+\pi(o)\right)-\lambda u+\lambda K \subset T_o$$

and

$$\pi(z),\pi(o) \in \operatorname{bd}\left(\tfrac{1}{2}\left(\pi(z)+\pi(o)\right)-\lambda u+\lambda K\right) .$$

This, however, implies that the point

$$\tfrac{1}{2}\left(\pi(z)+\pi(o)\right)-\lambda u$$

has both $\pi(z)$ and $\pi(o)$ as nearest points in $M$, a contradiction. Hence, we have proved (6).

Now we choose $y$ to be any point in the relative boundary of $\pi(o)-N$. It follows from (5) that

$$o \in \operatorname{int} T_y .$$

It remains to prove that $y$ is the limit of a sequence of points of type 1. To prove this it suffices to prove that if

$$z \in \left(-\pi(o)+L\right)\setminus\left(\pi(o)-N\right),$$

and

$$[z,y[ \,\cap \left(\pi(o)-N\right) = \emptyset ,$$

then some point $v \in [z,y[$ is of type 1. So, let $z$ be as stated, and suppose that the subset

$$\tau([z,y])$$

of $\operatorname{bd} K$ does not contain any smooth points of $K$. (Clearly, $[z,y] \subset E\setminus M$.) Since $\pi$ is continuous, it follows that $\tau$ is continuous, whence $\tau([z,y])$ is a continuum. It is well-known (e.g. G. T. Whyburn [3, (10.3) p. 16]) that such a set is not the union of any finite or countable family of disjoint proper closed subsets. Since

$$\tau([z,y]) = \bigcup\{\tau([z,y]) \cap N_p:\ p \in \tau([z,y])\}$$

we therefore conclude by (b) and (c) that

$$\tau([z,y]) = \tau([z,y]) \cap N_p$$

for some $p \in \tau([z,y])$. Since, by (5), we have $\tau(y) \in N$, it follows from condition (b) that $N_p = N$, and therefore

$$\tau([z,y]) \subset N .$$

This, however, contradicts that $\tau(z) \notin N$ by (6). Hence, we conclude that there exists a point $v \in [z,y[$ such that $\tau(v)$ is a smooth point of $K$, i.e. the point $v$ is of type 1. This completes the proof of the lemma.

Remark. Clearly, the lemma is also true for spaces with non-symmetric unit balls (cf. remark 2 of [1]).

## A geometrical characterization of 3-dimensional Banach spaces in which every Chebyshev set is convex.

The characterization is as follows:

Theorem 1. *Let $E$ be a 3-dimensional Banach space with unit ball $K$. Then every Chebyshev set in $E$ is convex if and only if every exposed point of $K$ is a smooth point of $K$.*

The "only if" part of theorem 1 is contained in the following well-known result:

Proposition 3. *Let $E$ be a normed linear space with unit ball $K$. If every Chebyshev set in $E$ is convex, then every exposed point of $K$ is a smooth point of $K$.*

Proof. Suppose $K$ contains an exposed non-smooth point $x$. Then it is easy to see that there exist two different supporting hyperplanes $H_1$ and $H_2$ of $K$ with

$$H_1 \cap K = H_2 \cap K = \{x\} .$$

But then the union of the two closed halfspaces which are bounded by $H_1$ and $H_2$, and do not contain $K$ is a non-convex Chebyshev set. This proves the proposition.

PROOF OF THEOREM 1. Assume that every exposed point of $K$ is smooth. We shall verify that then the conditions (a)–(c) of the lemma are fulfilled. Since in a finite dimensional space every Chebyshev set is a sun and has a continuous metric projection, this will prove that every Chebyshev set in $E$ is convex and thus complete the proof of the theorem.

Our assumption immediately implies that no $N_x \in \mathscr{F}$ is one-pointed, whence (a) is fulfilled. (Compare the remarks following the statement of the lemma). Since trivially no $N_x \in \mathscr{F}$ has dimension $\geqq 2$, it follows that every $N_x \in \mathscr{F}$ is a closed segment. This again implies that (b) is fulfilled. Finally, that condition (c) is fulfilled follows from a theorem of M. Fujiwara [2], saying that a convex body in a 3-dimensional (euclidean) space has at most a countable number of edges, and edge being a non-degenerate segment which is the intersection of the body and two different supporting hyperplanes.

REMARKS. It is easy to verify that theorem 1 is also valid for spaces with a non-symmetric unit ball.

It is obvious that in a 2-dimensional space every non-smooth point of the unit ball is exposed. Hence, in such a space smoothness is equivalent to every exposed point be smooth. (This observation together with proposition 3 proves the "if" part of theorem (III) in the introduction.) Therefore, by theorem (III), the characterization in theorem 1 is also valid for 2-dimensional spaces. Whether it also holds for spaces of dimension $\geqq 4$ is unknown.

Let us call a face $N_x$ of a convex body $K$ in a finite dimensional space $E$ a perfect face if $N_x$ has the same dimension as the intersection of all supporting hyperplanes of $K$ at $x$. It is well-known that the number of 0-dimensional perfect faces ("corner-points") and $(n-1)$-dimensional perfect faces ("facets") of an $n$-dimensional convex body is at most countable. The theorem of Fujiwara quoted above states that a 3-dimensional convex body has at most a countable number of 1-dimensional perfect faces. The main idea in Fujiwara's proof is that of "slicing" the body by means of families of parallel hyperplanes, thus reducing the problem to a 2-dimensional problem. It is possible to extend this procedure to spaces of higher dimensions, and thereby obtain a proof (using induction on the dimension of the space) of the following result: The number of perfect faces of an $n$-dimensional convex body is at most countable. Using this, it follows immediately from the lemma that in a finite dimensional space $E$ every Chebyshev set is convex provided that every face in $\mathscr{F}$ is perfect, no face in $\mathscr{F}$ is 0-dimensional, and any two faces in $\mathscr{F}$ are disjoint. If $E$ is 3-dimensional, and no face in $\mathscr{F}$ is 0-dimensional, then (as we have seen above) all faces in $\mathscr{F}$ are perfect, and any two such faces are disjoint. However, for spaces of dimension 4 or more, this is no longer true.

The generalized theorem of Fujiwara is of independent interest; we shall mention a reformulation of it. Let $K$ be a convex body in an $n$-dimensional space $E$, with $o \in \operatorname{int} K$, and let $K^\circ$ be the polar body in the dual of $E$. For any face $N$ of $K$, let $N^*$ denote the dual face of $N$ in $K^\circ$. Then

$$\dim N + \dim N^* \leqq n-1 . \tag{1}$$

It is easy to verify that we have equality in (1) if and only if $N$ is a perfect face. Hence, an $n$-dimensional convex body $K$ with $o \in \operatorname{int} K$ has at most a countable number of faces $N$ such that equality holds in (1).

**A new proof of theorem (V).**

We shall prove the following theorem which has theorem (V) as an immediate corollary:

THEOREM 2. *Let $F$ be a reflexive Banach space of dimension $\geqq 3$ containing a smooth closed subspace $F_1$ of deficiency* 1. *Then $F$ can be renormed to a non-smooth Banach $E$ with the property that every Chebyshev set in $E$ which is a sun and has a continuous metric projection is convex.*

COROLLARY (Theorem (V)). *For every integer $n \geqq 3$ there exists an $n$-dimensional non-smooth Banach space $E$ with the property that every Chebyshev set in $E$ is convex.*

PROOF OF THEOREM 2. We may consider $F$ as a product $G \times R \times R$, where $G$ is a closed subspace of deficiency 2, such that $F_1 = G \times R \times \{0\}$. Then the dual of $F$ is $G^* \times R \times R$, where $G^*$ is the dual of $G$; the value of an element $(\alpha',\beta',\gamma')$ in $G^* \times R \times R$ at a point $(\alpha,\beta,\gamma)$ in $G \times R \times R$ is $\langle \alpha,\alpha'\rangle + \beta\beta' + \gamma\gamma'$, where $\langle \alpha,\alpha'\rangle$ is the value of $\alpha'$ at $\alpha$.

We shall denote the unit ball of $F$ by $C$, and let

$$\begin{aligned} F_2 &= G \times \{0\} \times \{0\} , \\ C_1 &= C \cap F_1 , \\ C_2 &= C \cap F_2 . \end{aligned}$$

The element $(o,0,1)$ will be denoted by $e$. With

$$\alpha(t) = (1-t^2)^{\frac{1}{2}} + 1 ,$$

and

$$\beta(t) = (\tfrac{4}{3}-t^2)^{\frac{1}{2}} - 3^{-\frac{1}{2}} ,$$

we then define

$$K = \bigcup \{te+\alpha(t)C_2+\beta(t)C_1 : \ t \in [-1,1]\} .$$

We claim the following to be true:

(i) The set $K$ is a symmetric convex body in $F$.

(ii) The set $(e+C_2)\cup(-e+C_2)$ is the set of points in $\operatorname{bd} K$ which are non-smooth points of $K$.

(iii) If $H$ is a supporting hyperplane of $K$, and $H\cap(e+C_2)\neq\emptyset$, then $H\cap K=e+C_2$. Similarly for $-e+C_2$.

For a proof of these three statements the reader is referred to the proofs of the statements (i), (iv), (v), and (vi) of proposition 2 in [1]. After a slight change in notation, the proofs in [1] carry over to the present situation. (In the proof of the closedness of $K$, let compactness and continuity be with respect to the weak topology on $F$).

Now, by statement (i), $K$ is the unit ball of a new norm on $F$, equivalent to the original one. Let $E$ be the Banach space thus obtained. By (ii), $E$ is a non-smooth space. To complete the proof of the theorem it suffices to verify that the conditions (a), (b), and (c) of the lemma are fulfilled. This follows easily from (ii) and (iii).

**Some remarks concerning characterizations of smooth spaces in terms of the Chebyshev sets.**

It is well-known that a normed linear space is smooth if and only if each of its (2-dimensional) subspaces is smooth. Let us say that a space has property $\mathscr{C}_1$ if every Chebyshev set in the space is convex. Then it follows easily from the theorems (III) and (IV) and the remark above that for a finite dimensional Banach space $E$ the following conditions are equivalent:

(i) $E$ is smooth.

(ii) Every 2-dimensional subspace of $E$ has property $\mathscr{C}_1$.

(iii) $E$ and every (2-dimensional) subspace of $E$ has property $\mathscr{C}_1$.

Let us say that a space has property $\mathscr{C}_2$ if every non-empty closed convex set in the space is a Chebyshev set. Since a space is rotund if and only if each of its (2-dimensional) subspaces is rotund, it follows immediately from the theorems (I) and (II) that if in the conditions above we replace "smooth" by "rotund" or "rotund and smooth", and replace "$\mathscr{C}_1$" by "$\mathscr{C}_2$" or "$\mathscr{C}_1$ and $\mathscr{C}_2$", respectively, then the conditions are still equivalent. Note that in all three cases the conditions (i) and (ii) are equivalent for arbitrary normed linear spaces.

The characterization above of smooth finite dimensional spaces in terms of the Chebyshev sets seems to be as good as one can hope for. For let $F$ be a finite dimensional space which fulfills the conditions of theorem 2, and let $K$ be as defined there. Further, let $K'$ be the symmetric smooth body obtained by replacing the function $\beta(t)$ in the definition of $K$ by the function $(1-t^2)^{\frac{1}{2}}$. Then for any pair $H,H'$ of parallel supporting hyperplanes of $K$ and $K'$, respectively, the sets $H \cap K$ and $H' \cap K'$ consist either of a single point, namely if $H$ and $H'$ are not parallel to $F_2$, or the two sets are both translates of homothets of $C_2$. Since both $K$ and $K'$ produce only convex Chebyshev sets, it follows therefore easily that $K$ and $K'$ actually produce the same Chebyshev sets.

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