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INFINITE KUMMER EXTENSIONS

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Let n be a positive integer and let F be a field which contains n distinct n^{th} roots of unity. Denote by F^* the multiplicative group of non-zero elements of F and denote by $(F^*)^n$ the subgroup of n^{th} powers of elements in F^*. Suppose C is an algebraic closure of F. The classical theory of Kummer extensions establishes a 1-1 correspondence between the subfields K of C which are finite abelian extensions of F of exponent dividing n and the subgroups Q of F^* containing $(F^*)^n$ and such that $Q/(F^*)^n$ is finite. The correspondence is such that if K corresponds to Q then $\mathrm{G}(K/F)$, the Galois group of K/F, is isomorphic to $Q/(F^*)^n$. For an exposition of these results see [1].

The object of this paper is to extend these results to arbitrary (not necessarily finite) abelian extensions K of F whose Galois group is of bounded order n, i.e. if $\sigma \in \mathrm{G}(K/F)$ then $\sigma^n = 1$.

Thus let K be a subfield of C containing F such that K/F is Galois and $\mathrm{G} = \mathrm{G}(K/F)$ is abelian of bounded order n. Define

$$S = S(K) = \{\alpha \in K^* : \alpha^n \in F^*\} .$$

For $\alpha \in S$ define a function χ_α on G with values in K^* by the rule

$$\chi_\alpha(\sigma) = \alpha/\sigma(\alpha) .$$

Just as in the ordinary theory of Kummer extensions it is easy to check that $\chi_\alpha(\sigma) \in Z$ and χ_α is a homomorphism of G into Z where Z denotes the group of n^{th} roots of unity in F^*. Further, the map $\alpha \to \chi_\alpha$ is a homomorphism of S into X, the group of characters on G. Note that since G is of bounded order any character on G is automatically continuous in the Galois topology on G.

Now let χ be any character on G. Since G is of bounded order n, one may assume that χ takes its values in Z. The map $\sigma \to \chi(\sigma)$ is continuous and satisfies

$$\sigma\tau \to \chi(\sigma)\,\chi(\tau) = \chi(\sigma)\,\sigma(\chi(\tau)) .$$

Thus $\sigma \to \chi(\sigma)$ is a continuous cocycle for the system (G,K^*). Since $H^1(G;K^*) = 1$ (see [2]) there exists $\beta \in K^*$ such that

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$$\chi(\sigma) = \beta/\sigma(\beta) .$$

Since $\chi(\sigma)^n = 1$ for all $\sigma \in \mathrm{G}$ one has $\sigma(\beta^n) = \beta^n$ for all $\sigma \in \mathrm{G}$. Thus $\beta \in S$. This shows that the map $\alpha \to \chi_\alpha$ of S into X is surjective. The kernel of this map is F^* and therefore $S/F^* \approx X$.

Now set

$$Q = Q(K) = \{\alpha^n : \alpha \in S\} .$$

Then $Q/(F^*)^n$ is isomorphic to S/F^*. Thus given an abelian extension K of F of bounded order n one can associate the group $Q/(F^*)^n$. According to the Pontriagin duality theorem then G is topologically isomorphic to the character group of $Q/(F^*)^n$.

Note that Q also has the property that K is generated over F by

$$S = Q^{1/n} = \{\alpha \in K^* : \alpha^n \in Q\} .$$

For let $K' = F(S)$ and let $\sigma \in \mathrm{G}(K/K')$. Then for any $\chi_\alpha \in X$,

$$\chi_\alpha(\sigma) = \alpha/\sigma(\alpha) = 1 .$$

The duality theorem gives that $\sigma = 1$. Therefore $K' = K$.

It remains to show that any subgroup Q of F^* containing $(F^*)^n$ is a $Q(K)$ for some field K and that the correspondence $K \to Q(K)$ is 1-1. Given such a Q set

$$Q^{1/n} = \{\alpha \in C : \alpha^n \in Q\}$$

and take $K = F(Q^{1/n})$. Then K is the splitting field over F of the set of polynomials $\{X^n - \beta : \beta \in Q\}$. Since each of these polynomials is separable K is a normal separable extension of F. On the other hand $K = \lim\uparrow E$ where E is the splitting field over F of some finite set of polynomials

$$\{X^n - \beta_i : \beta_i \in Q, i = 1,2,\ldots,s\} .$$

Therefore $\mathrm{G}(K/F) = \lim\downarrow \mathrm{G}(E/F)$. By the theory of ordinary Kummer extensions each $\mathrm{G}(E/F)$ is finite abelian of exponent dividing n. Therefore $\mathrm{G}(K/F)$ is abelian of bounded order n. Moreover, the field K just constructed has the property that

$$Q(K) = \{\alpha^n : \alpha \in S(K)\} = Q;$$

for let R be a subgroup of F^* such that

$$Q \supset R \supset (F^*)^n$$

and $R/(F^*)^n$ is finite. Then $Q = \lim\uparrow R$. If E is the subfield of C obtained by adjoining the set $\{\alpha \in C : \alpha^n \in R\}$ to F, then the theory of finite Kummer extensions gives that $Q(E) = R$. Therefore $Q(K) = \lim\uparrow Q(E) = \lim\uparrow R = Q$.

Finally, since any abelian extension K of bounded order n can be described by $K = F(S(K))$ it follows that the correspondence $K \to Q$ is 1-1. These considerations give the following

THEOREM. *There is a* 1-1 *correspondence between the abelian extensions* K *of* F *of bounded order* n *and the subgroups* Q *of* F^* *containing* $(F^*)^n$. *This correspondence has the property that if* K *corresponds to* Q *then* $\mathrm{G}(K/F)$ *is topologically isomorphic to the character group of* $Q/(F^*)^n$.

REFERENCES

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