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ON THE TOTAL REGULARITY OF FUNCTION-TO-FUNCTION TRANSFORMATIONS OF TRIANGULAR TYPE

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By a function-to-function transformation of triangular type we mean a transformation of the form

$$F_x(f) = \int_0^x f(t)\, d\alpha_x(t) , \tag{1}$$

where $d\alpha_x$ is a bounded measure on $(0,x)$, and α_x is supposed to be normalized so that $\alpha_x(t+)=\alpha_x(t)$, $0<t<x$. The function f is assumed to be Borel-measurable and bounded on every $(0,x)$. In particular, when $\alpha_x(t)=\beta(t/x)$, we have

$$H_x(f) = \int_0^1 f(xt)\, d\beta(t) . \tag{2}$$

The latter is analogous to the well-known Hausdorff sequence-to-sequence transformation.

The transformation F given by (1) is regular (convergence- and limit-preserving), if and only if (cf. [3, p. 16])

$$\int_0^x |d\alpha_x(t)| \leqq M , \tag{3}$$

for all $x>0$,

$$\int_0^x d\alpha_x(t) \to 1, \quad x \to \infty , \tag{4}$$

and

$$\int_E d\alpha_x(t) \to 0, \quad x \to \infty , \tag{5}$$

for every bounded and Borel-measurable set E.

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In this paper we shall give necessary and sufficient conditions for a transformation of this type to be totally regular, i.e. regular and infinite-limit-preserving. The result will correspond to a classical one by W. A. Hurwitz [1] concerning triangular sequence-to-sequence transformations.

THEOREM 1. *The transformation F given in* (1) *is totally regular if and only if* (4) *and* (5) *holds and there is a $t_0 \geqq 0$, so that α_x is non-decreasing in $[t_0,x]$ for every $x > t_0$.*

In the Hausdorff case theorem 1 takes the following form.

THEOREM 2. *The transformation H defined in* (2) *is totally regular if and only if β is a bounded positive measure with*

$$\beta(0+)=\beta(0) \qquad \textit{and} \qquad \beta(1)-\beta(0)=1\,.$$

In the proof of our theorems we shall apply the following definition and lemma (cf. Natanson [2, p. 207 and 266]).

DEFINITION. *A number λ (finite or infinite) is called a derived number of the function f at the point x_0, if there exists a sequence*

$$\{h_\nu\}_1^\infty, \quad h_\nu \neq 0, \quad h_\nu \to 0 \text{ as } \nu \to \infty\,,$$

so that

$$\lim_{\nu\to\infty} \frac{f(x_0+h_\nu)-f(x_0)}{h_\nu} = \lambda\,.$$

We write $Df(x_0)$ for the set of derived numbers.

LEMMA. *Let f be defined and finite on $[a,b]$. If at every point of $[a,b]$ all the derived numbers of f are non-negative, then f is a non-decreasing function.*

If we recall the necessary and sufficient conditions for regularity quoted at the beginning of this paper, the necessity of the conditions in theorem 1 is easily seen. We then conclude the sufficiency from the following result.

THEOREM 3. *If F is totally regular, there exists a $t_0 \geqq 0$, so that α_x is non-decreasing in $[t_0,x]$ for every $x > t_0$.*

PROOF. The function α_x defines the totally regular transformation F according to (1). Let us further assume that, given $t_0 \geqq 0$, there exist $x > t_0$ and τ with $t_0 \leqq \tau \leqq x$, so that for some $\lambda \in D\alpha_x(\tau)$, we have $\lambda < 0$. First let $t_0 = 0$. We get $x_1 > 0$, τ_1 with $0 \leqq \tau_1 \leqq x_1$ and $\lambda_1 \in D\alpha_{x_1}(\tau_1)$ with $\lambda_1 < 0$. Suppose that $\lambda_1 \neq -\infty$. (If $\lambda_1 = -\infty$ only small modifications are necessary.) Then there is a sequence $\{h_\nu^{(1)}\}_1^\infty$, $h_\nu^{(1)} \neq 0$, so that

$$\{\alpha_{x_1}(\tau_1+h_\nu^{(1)})-\alpha_{x_1}(\tau_1)\}/h_\nu^{(1)} \to \lambda_1, \quad \nu\to\infty,$$

that is

$$\{\alpha_{x_1}(\tau_1+h_\nu^{(1)})-\alpha_{x_1}(\tau_1)\}/(\lambda_1 h_\nu^{(1)}) \to 1, \quad v\to\infty.$$

Choose ν_1 so that

$$\{\alpha_{x_1}(\tau_1+h_{\nu_1}^{(1)})-\alpha_{x_1}(\tau_1)\}/(\lambda_1 h_{\nu_1}^{(1)}) > 0,$$

and suppose that $h_{\nu_1}^{(1)}>0$. We define a function f on $[0,x_1]$ by

$$f(t) = \begin{cases} -1/(\lambda_1 h_{\nu_1}^{(1)}) & \text{if } t\in(\tau_1,\tau_1+h_{\nu_1}^{(1)}], \\ 0 & \text{if } t\notin(\tau_1,\tau_1+h_{\nu_1}^{(1)}]. \end{cases}$$

(If $h_{\nu_1}^{(1)}<0$ we take $f(t)=1/(\lambda_1 h_{\nu_1}^{(1)})$ for t in $(\tau_1+h_{\nu_1}^{(1)},\tau_1]$ and $=0$ outside this interval.) Since F is regular we get

$$\int_0^{x_1} f(t)\,d\alpha_x(t) \to 0, \quad x\to\infty,$$

and

$$\int_{x_1+}^{x} d\alpha_x(t) \to 1, \quad x\to\infty.$$

Consequently there is an $\omega_1>\max(1,x_1)$ so that

$$\int_0^{x_1} f(t)\,d\alpha_x(t) < \tfrac{1}{2},$$

and

$$\int_{x_1+}^{x} d\alpha_x(t) < 1+\tfrac{1}{2} \quad \text{for } x>\omega_1.$$

Let $x_1,\ldots,x_{n-1},\tau_1,\ldots,\tau_{n-1},\omega_{n-1}>\max(n-1,x_{n-1})$ and f on $[0,x_{n-1}]$ be chosen so that

$$\int_0^{x_{n-1}} f(t)\,d\alpha_x(t) < 1/n,$$

and

$$\int_{x_{n-1}+}^{x} d\alpha_x(t) < 1+1/n \quad \text{for } x>\omega_{n-1}.$$

Then according to our assumptions there exist (with $t_0=\omega_{n-1}$) $x_n>\omega_{n-1}$ and τ_n with $\omega_{n-1}\leqq\tau_n\leqq x_n$ so that

$$\{\alpha_{x_n}(\tau_n+h_\nu^{(n)})-\alpha_{x_n}(\tau_n)\}/h_\nu^{(n)} \to \lambda_n < 0, \quad \nu\to\infty,$$

for some sequence $\{h_\nu^{(n)}\}_{\nu=1}^{\infty}$, $h_\nu^{(n)} \neq 0$, $h_\nu^{(n)} \to 0$, $\nu \to \infty$. Suppose as before that $\lambda_n \neq -\infty$ and take ν_n so that

$$\{\alpha_{x_n}(\tau_n + h_{\nu_n}^{(n)}) - \alpha_{x_n}(\tau_n)\} / \lambda_n h_{\nu_n}^{(n)} > 1 - 1/n .$$

Suppose that $h_{\nu_n}^{(n)} > 0$ and define f on $(x_{n-1}, x_n]$ by

$$f(t) = \begin{cases} -n/(\lambda_n h_{\nu_n}^{(n)}) + n - 1 & \text{if } t \in (\tau_n, \tau_n + h_{\nu_n}^{(n)}] , \\ n-1 & \text{if } t \notin (\tau_n, \tau_n + h_{\nu_n}^{(n)}] . \end{cases}$$

(If $h_{\nu_n}^{(n)} < 0$ we choose f as before, mutatis mutandis).

In this way the function f is inductively given, and evidently $f(t)$ tends to $+\infty$ with t. We get

$$\int_0^{x_n} f(t)\, d\alpha_{x_n}(t) = \int_0^{x_{n-1}} f(t)\, d\alpha_{x_n}(t) + \int_{x_{n-1}+}^{x_n} f(t)\, d\alpha_{x_n}(t)$$

$$< 1/n + (n-1) \int_{x_{n-1}+}^{x_n} d\alpha_{x_n}(t) - n\{\alpha_{x_n}[\tau_n + h_{\nu_n}^{(n)}] - \alpha_{x_n}[\tau_n +]\} / \lambda_n h_{\nu_n}^{(n)}$$

$$< 1/n + (n-1)(1+1/n) - n(1-1/n) = 1 .$$

Consequently, for all n,

$$F_{x_n}(f) \leqq 1 ,$$

and

$$\liminf_{x \to \infty} F_x(f) \leqq 1 ,$$

which contradicts the total regularity of F.

Therefore the assumption in the beginning of the proof must be false and accordingly there exists a $t_0 \geqq 0$ so that all the derived numbers of the function α_x in $[t_0, x]$ are non-negative for $x > t_0$ and, by the lemma α_x is non-decreasing in $[t_0, x]$ for every $x > t_0$. The proof of theorem 3 is hence complete.

REFERENCES

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