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A NEW PROOF FOR FØLNER'S CONDITION FOR MAXIMALLY ALMOST PERIODIC GROUPS

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Introduction.

A topological group is called maximally almost periodic (m.a.p.) if the continuous characters separate points of the group. An important part of Pontrjagin's duality theorem states that any Abelian, Hausdorff locally compact group is m.a.p. A standard proof makes use of the Haar integral and the associated L^1-algebra; after the construction of a positive definite function the Krein–Milman theorem is applied to obtain a suitable character.

A general condition for m.a.p. of an Abelian, topological group has been given by Følner ([2] and [3]; see also Cotlar and Ricabarra [1].). The proof makes use of the Banach mean value; it is observed that the convolution of two positive definite functions is almost periodic (Godement [4]), and an almost periodic function is the uniform limit of finite linear combinations of characters. Our alternative proof dispenses with considerations drawn from the theory of almost-periodic functions, but is modelled after the above-mentioned classical proof for locally compact groups. A Banach mean value will replace the Haar integral; we need a substitute for the ordinary group algebra. For this we will use a suitable subalgebra of the full generalized group algebra; thus our proof of Følner's result will show an application of the generalized group algebra to a practical problem.

In the first section we include a characterization of sets with positive upper Banach mean value.

1. Banach mean values and relatively accumulating subsets.

The concept of a relatively dense subset of an Abelian group is well known; in this section we are lead to the definition of a new concept, relatively accumulating subsets; and we include proofs for the simple

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connections between these and the lower and upper Banach mean values of the sets.

We will assume that G is an Abelian group. We recall from the theory of Banach mean values (see [2]): If f is a bounded real function on G, the upper mean value of f is defined by

$$\overline{M}(f) = \overline{M}_{(x)}(f(x)) = \inf_{\Omega} \sup_{x \in G} \sum_{n=1}^{k} \alpha_n f(x+a_n);$$

$$\Omega = \{\alpha_1,\ldots,\alpha_k; a_1,\ldots,a_k\}; \quad \alpha_n > 0, \ \sum_{n=1}^{k} \alpha_n = 1; \ a_n \in G \ \text{ for } \ n=1,2,\ldots,k .$$

Dually, the lower mean value of f is defined by

$$\underline{M}(f) = -\overline{M}(-f) = \sup_{\Omega} \inf_{x \in G} \sum_{n=1}^{k} \alpha_n f(x+a_n) .$$

The Banach mean value, M, is a translation-invariant, linear functional defined on the space of bounded real functions on G with the property

$$\underline{M}(f) \leqq M(f) \leqq \overline{M}(f) .$$

It is easily extended to the complex case. We have $M(1)=1$. If G is not locally precompact, it is easily seen by an argument due to A. D. Aleksandroff that M will not be strictly positive for continuous, positive functions (open sets).

We will need the following concepts to construct functions for which M will be positive.

Let A be a subset of G. A is relatively dense if a finite number of translates of A covers G. The minimum number of such translates will be called the *D-index* of A.

DEFINITION. A is *relatively accumulating* if the following condition holds: There exists a natural number n_0 such that for any natural number m, at least $m+1$ of any mn_0+1 translates of A have a common, non-empty intersection. The smallest number n_0 with this property is called the *A-index* of the set A.

PROPOSITION 1. *A relatively dense set A is relatively accumulating; and the A-index of A is not larger than the D-index.*

PROOF. Assume that $\{A+c_i\}$, $i=1,2,\ldots,n_0$, cover G, and let m be a natural number. Consider any mn_0+1 translates of A,

$$A+b_i, \qquad i=1,2,\ldots,mn_0+1 .$$

Choose a in A, and consider the mn_0+1 elements

$$f_i = a+b_1+b_2+\ldots+b_{i-1}+b_{i+1}+\ldots+b_{mn_0+1};\ i = 1,\ldots,mn_0+1\ .$$

Then at least $m+1$ of these elements must belong to one of the sets $A+c_i$, let

$$f_{i_1},\ldots,f_{i_{m+1}} \in A+c_k\ .$$

Then

$$a+b_1+\ldots+b_{mn_0+1}-c_k \in (A+b_{i_1}) \cap \ldots \cap (A+b_{i_{m+1}});$$

and this proves the proposition.

PROPOSITION 2. *Let A be a subset of G and n a number such that no $n+1$ translates of A are disjoint. Then $A-A$ is relatively dense, with D-index not larger than n.*

PROOF. Let $A+b_i$, $i=1,\ldots m$, be m disjoint translates of A $(m<n+1)$. If there are no $m+1$ disjoint translates of A, then the sets $A-A+b_i$, $i=1,\ldots,m$, cover G. For if $x\in G$, we have

$$(x+A)\cap(A+b_i) \neq \emptyset$$

for some i. Let $x+a_1=a_2+b_i$ with $a_1,a_2\in A$. Then

$$x = a_2-a_1+b_i \in A-A+b_i;$$

hence the sets $A-A+b_i$ cover G.

In particular, if A is a relatively accumulating set with A-index n, then $A-A$ is relatively dense with D-index not larger than n.

THEOREM 1. *Let G be an Abelian group and A a subset with characteristic function χ_A. We have:*

$\underline{M}(\chi_A)>0$ *if and only if A is relatively dense.*
$\overline{M}(\chi_A)>0$ *if and only if A is relatively accumulating.*

PROOF. The first part follows directly from the definitions. If A is not relatively accumulating, we can, for any natural number n_0, find an m and translates $\chi_{A+x_1},\ldots,\chi_{A+x_{mn_0+1}}$ such that no $m+1$ of these have a non-empty, common intersection. Choose $\alpha_i=1/(mn_0+1)$, then

$$\sup_{x\in G}\sum_{i=1}^{mn_0+1}\alpha_i\chi_{A+x_i}(x) \leqq \frac{m}{mn_0+1} < \frac{m}{mn_0} = \frac{1}{n_0}$$

and $\overline{M}(\chi_A)=0$. Now suppose that A is relatively accumulating with A-index n_0, and let $\sum_{i=1}^{n}\alpha_i\chi_{A+x_i}$ be any convex combination of translates of χ_A. We can always assume $\beta\leqq\alpha_i\leqq 2\beta$, $i=1,2,\ldots,n$, for some $\beta>0$. (Of course several of the translates may coincide.) Now determine the number m such that $mn_0<n\leqq(m+1)n_0$. Then

$$\frac{1}{2(m+1)n_0} \leqq \beta .$$

(since $\sum_{i=1}^{n}\alpha_i = 1$), hence all

$$\alpha_i \geqq \frac{1}{2(m+1)n_0}, \qquad i = 1,2,\ldots,n .$$

By hypothesis, at least $m+1$ of the sets $A+x_i$ have a non-empty, common intersection; for x in this intersection we have

$$\sum_{i=1}^{n} \alpha_i \chi_{A+x_i}(x) \geqq \frac{m+1}{2(m+1)n_0} = \frac{1}{2n_0}.$$

Hence

$$\sup_{x\in G} \sum_{i=1}^{n} \alpha_i \chi_{A+x_i}(x) \geqq \frac{1}{2n_0} \qquad \text{and} \qquad \overline{M}(\chi_A) \geqq \frac{1}{2n_0} > 0 .$$

We note that, since $\underline{M}(\chi_A) \leqq \overline{M}(\chi_A)$, the first part of Proposition 1 is a consequence of this theorem.

Corollary. *Let A be a relatively accumulating subset of G. There is then a Banach mean value on G, M, such that $M(\chi_A) > 0$.*

Proof. Relatively accumulating sets are exactly the sets with positive upper mean value. For the conclusion, see [3].

2. Generalized group algebras.

A brief account of the generalized group algebra for a locally compact group is given in [5, p. 275]. We note that local compactness is no necessary requirement for this construction. In fact, let G be an Abelian, Hausdorff topological group, let $\mathrm{UC}(G)$ be the Banach space of all uniformly continuous bounded complex functions on G (supremum norm). The dual space $\mathrm{UC}^*(G)$ is a Banach space, it is organized to a Banach algebra by the convolution product. Let $I, J \in \mathrm{UC}^*(G)$. The convolution of I and J is defined by

$$(I * J)(f) = I_{(y)}(J(f_{-y})) \qquad \text{for} \qquad f \in \mathrm{UC}(G);$$

that is, I is applied to the function $y \to J(f_{-y})$, where $f_y(x) = f(x-y)$. Translation is defined in $\mathrm{UC}^*(G)$ by

$$I_x(f) = I(f_{-x}) .$$

"Involution" is defined in $\mathrm{UC}^*(G)$ by

$$I^{\sim}(f) = \overline{I(\overline{f^{\star}})}, \qquad \text{where} \qquad f^{\star}(x) = f(-x) .$$

This latter operation will not in general satisfy the relation $(I*J)^{\sim} = J^{\sim}*I^{\sim}$; thus UC*$(G)$ will not always be a $^{\sim}$-algebra. However, we shall study a subalgebra where this relation does hold. The following relations are easily verified:

$$I_x*J = I*J_x = (I*J)_x, \qquad (I_x)^{\sim} = (I^{\sim})_{-x},$$
$$\|I_x\| = \|I^{\sim}\| = \|I\|, \qquad \|I*J\| \leqq \|I\|\cdot\|J\|.$$

In general, UC*(G) is a complex Banach algebra.

If M is a Banach mean value, $M'(g) = \frac{1}{2}(M(g)+M(g^\star))$ will define a Banach mean value with the property $M'(g) = M'(g^\star)$ for all $g \in \mathrm{UC}(G)$. Thus there will be no loss in assuming that M has this property. Furthermore, let f be a real function in UC(G) with $f = f^\star$. We define an element I of UC*(G) by

$$I(g) = M_{(x)}(f(x)g(x)).$$

The algebra $\mathscr{A}_f$ generated by all translates of I consists of all elements of the form $\sum_{i=1}^n \alpha_i (I^{n_i})_{x_i}$, $\alpha_i \in C$.

PROPOSITION 3. *For J_1 and $J_2 \in \mathscr{A}_f$, we have $(J_1*J_2)^{\sim} = J_2^{\sim}*J_1^{\sim}$.*

PROOF. Let $J_1 = \sum_{i=1}^m \alpha_i (I^{m_i})_{x_i}$ and $J_2 = \sum_{j=1}^n \beta_j (I^{n_j})_{y_j}$. Applying the above relations, we get

$$(J_1*J_2)^{\sim} = \sum_{i,j} \overline{\alpha_i\beta_j}\left((I^{m_i+n_j})^{\sim}\right)_{-x_i-y_j},$$
$$J_2^{\sim}*J_1^{\sim} = \sum_{i,j} \overline{\alpha_i\beta_j}\left((I^{n_j})^{\sim}*(I^{m_i})^{\sim}\right)_{-x_i-y_j}.$$

It is then sufficient to show that $(I^m)^{\sim} = I^m$, which can be done as follows:

$$\begin{aligned}
(I^m)^{\sim}(g) &= \overline{(I*I*\ldots*I)(\overline{g^\star})} \\
&= \overline{(I*\ldots*I)_{(x_2)}[I((\overline{g^\star})_{-x_2})]} \\
&= \overline{(I*\ldots*I)_{(x_2)}I_{(x_1)}(\overline{g^\star(x_1+x_2)})} \\
&= \overline{(I*\ldots*I)_{(x_2)}I_{(x_1)}(\overline{g(-x_1-x_2)})} \\
&= \overline{I_{(x_m)}[I_{(x_{m-1})}\{\ldots\{I_{(x_1)}(\overline{g(-x_1-x_2-\ldots-x_m)})\}\ldots\}} \\
&= \overline{M_{(x_m)}\{M_{(x_{m-1})}\{\ldots\{M_{(x_1)}(f(x_1)\ldots f(x_m)\overline{g(-x_1-\ldots-x_m)})\}\ldots\}} \\
&= M_{(x_m)}\{M_{(x_{m-1})}\ldots\{M_{(x_1)}(f(-x_1)\ldots f(-x_m)g(x_1+\ldots+x_m))\}\ldots\} \\
&= M_{(x_m)}\{\ldots\{M_{(x_1)}(f(x_1)\ldots f(x_m)g(x_1+\ldots+x_m))\}\} \\
&= I^m(g).
\end{aligned}$$

We can conclude that $\mathscr{A}_f$ is a commutative $^{\sim}$-algebra which is closed under translations.

We now gather the most important properties of the closure of $\mathscr{A}_f$ in a theorem (For Banach $\tilde{}$-algebras, see (5)).

THEOREM 2. *Let f be a real function in* $\mathrm{UC}(G)$ *with $f=f^\star$, and let the algebra $\mathscr{A}_f$ be given as above. The closure $\overline{\mathscr{A}_f}$ of $\mathscr{A}_f$ in* $\mathrm{UC}^*(G)$, *is a commutative, Banach $\tilde{}$-algebra closed under translations. For any element $J \in \overline{\mathscr{A}_f}$, the translation $x \to J_x$ is a continuous function from G into $\overline{\mathscr{A}_f}$.*

PROOF. From standard theory, the closure of a commutative subalgebra is commutative. Translations and involution are isometries, hence $\overline{\mathscr{A}_f}$ will be closed under these operations. The identity $(J_1 * J_2)^\sim = J_2^\sim * J_1^\sim$ is extended from $\mathscr{A}_f \times \mathscr{A}_f$ to $\overline{\mathscr{A}_f} \times \overline{\mathscr{A}_f}$ by continuity. For the last result, let first $J = I^m$. We have

$$\begin{aligned} I^m(g) &= M_{(x_m)} \ldots M_{(x_1)}\big(f(x_1) \ldots f(x_m) g(x_1 + \ldots + x_m)\big)\,, \\ (I^m)_x(g) &= M_{(x_m)} \ldots M_{(x_1)}\big(f(x_1) \ldots f(x_m) g(x_1 + \ldots + x_m + x)\big) \\ &= M_{(x_m)} \ldots M_{(x_1)}\big(f(x_1 - x) f(x_2) \ldots f(x_m) g(x_1 + \ldots + x_m)\big)\,, \\ |(I^m)_{y_1}(g) &- (I^m)_{y_2}(g)| \\ &= \big|M_{(x_m)} \ldots M_{(x_1)}\big(\big(f(x_1 - y_1) - f(x_1 - y_2)\big) f(x_2) \ldots f(x_m) g(x_1 + \ldots + x_m)\big)\big| \\ &\leqq \|f\|^{m-1} \|g\| M_{(x_m)} \ldots M_{(x_1)}\big(f(x_1 - y_1) - f(x_1 - y_2)\big)\,. \end{aligned}$$

Let $\delta > 0$, and choose a symmetric neighbourhood V of 0 in G such that for $z_1 - z_2 \in V$

$$|f(z_1) - f(z_2)| \leqq \delta / \|f\|^{m-1}\,.$$

Then, for $y_1 - y_2 \in V$,

$$|(I^m)_{y_1}(g) - (I^m)_{y_2}(g)| \leqq \delta \|g\|\,,$$

hence $\|(I^m)_{y_1} - (I^m)_{y_2}\| \leqq \delta$. It follows immediately that translation by x is a continuous function of x for any $J \in \mathscr{A}_f$.

Let $J \in \overline{\mathscr{A}_f}$; and $\delta > 0$. Choose $J_1 \in \mathscr{A}_f$ such that $\|J - J_1\| \leqq \delta/3$. Then

$$\|J_x - (J_1)_x\| = \|(J - J_1)_x\| = \|J - J_1\| \leqq \delta/3$$

for any $x \in G$. Choose a neighbourhood V of 0 in G such that for $x_1 - x_2 \in V$

$$\|(J_1)_{x_1} - (J_1)_{x_2}\| \leqq \delta/3\,.$$

Now let $y_1 - y_2 \in V$. We have

$$\|J_{y_1} - J_{y_2}\| \leqq \|J_{y_1} - (J_1)_{y_1}\| + \|(J_1)_{y_1} - (J_1)_{y_2}\| + \|(J_1)_{y_2} - J_{y_2}\| \leqq \delta\,.$$

THEOREM 3. *Let $\mathscr{B}$ be a subalgebra of* $\mathrm{UC}^*(G)$ *such that $\mathscr{B}$ is a complex Banach $\tilde{}$-algebra, and let V be a strongly continuous, unitary representation of G. There is then a unique $\tilde{}$-representation A of $\mathscr{B}$ such that*

$$\langle A_I\xi,\eta\rangle = I_{(x)}(\langle V_x\xi,\eta\rangle) \quad \textit{for} \quad I \in \mathscr{B};\ \xi,\eta \in H ,$$

where H is the Hilbert space for the representation, and $\langle ,\rangle$ the inner product.

PROOF. It is easily seen that $\langle V_x\xi,\eta\rangle$ is a function in UC(G), so the right side of the equality is well-defined. For constant ξ,

$$\eta \to I_{(x)}(\langle V_x\xi,\eta\rangle)$$

is a bounded, conjugate linear functional on H; we define $A_I\xi$ as the unique element $\xi' \in H$ such that

$$\langle \xi',\eta\rangle = I_{(x)}(\langle V_x\xi,\eta\rangle) \quad \text{for} \quad \eta \in H .$$

A_I is linear on H;

$$\|A_I\xi\|^2 = \langle \xi',\xi'\rangle = I_{(x)}(\langle V_x\xi,A_I\xi\rangle) \leqq \|I\|\,\|\xi\|\,\|A_I\xi\| ;$$

hence $\|A_I\xi\| \leqq \|I\|\,\|\xi\|$ and A_I is bounded with $\|A_I\| \leqq \|I\|$. The mapping $I \to A_I$ is easily seen to be a representation. We have

$$\langle A_{I*J}\xi,\eta\rangle = (I*J)_{(x)}(\langle V_x\xi,\eta\rangle) = I_{(y)}[J_{(x)}(\langle V_{x+y}\xi,\eta\rangle)]$$

and

$$\begin{aligned}\langle (A_I\circ A_J)\xi,\eta\rangle &= \langle A_I(A_J\xi),\eta\rangle \\ &= I_{(y)}(\langle V_y(A_J\xi),\eta\rangle) \\ &= I_{(y)}(\langle A_J\xi, V_{-y}\eta\rangle) \\ &= I_{(y)}[J_{(x)}(\langle V_x\xi, V_{-y}\eta\rangle) = I_{(y)}[J_{(x)}(\langle V_{x+y}\xi,\eta\rangle)] .\end{aligned}$$

Hence $A_{I*J} = A_I \circ A_J$.

We have

$$\begin{aligned}\langle \xi, A_{I^\sim}\eta\rangle = \overline{\langle A_{I^\sim}\eta,\xi\rangle} &= \overline{I^\sim_{(x)}(\langle V_x\eta,\xi\rangle)} \\ &= \overline{\overline{I_{(x)}(\overline{\langle V_{-x}\eta,\xi\rangle})}} \\ &= I_{(x)}(\langle \xi, V_{-x}\eta\rangle) \\ &= I_{(x)}(\langle V_x\xi,\eta\rangle) = \langle A_I\xi,\eta\rangle = \langle \xi,(A_I)^*\eta\rangle ,\end{aligned}$$

hence $(A_I)^* = A_{I^\sim}$, and A is a $^\sim$-representation.

3. Proof of Følner's condition.

We will prove the strong version of Følner's condition.

THEOREM 4. *Let G be an Abelian, Hausdorff topological group. Then G is m.a.p. if and only if the following condition holds: For any $x_0 \in G$, $x_0 \neq 0$, there exists a relatively accumulating subset E of G, such that $x_0 \notin \overline{E-E+E-E}$.*

That this condition is necessary, is trivial; we only have to prove the sufficiency.

Let x_0 and E be as above, and choose a symmetric neighbourhood W of 0 with

$$x_0 \notin E-E+E-E+W+W+W+W+W .$$

By the corollary to Theorem 1 we can choose a Banach mean value M with $M(\chi_E)>0$; again, as in section 2, we can assume $M(g)=M(g^\star)$ for all $g \in \mathrm{UC}(G)$. For the construction of the positive-definite function φ, we follow Følner [2] and [3]. Let h be a uniformly continuous function from G into $[0,1]$ with $h(0)=1$ and $h(x)=0$ for $x \notin W$. Then

$$j(x) \;=\; \sup_{y\in E} h(x-y)$$

is a uniformly continuous function from G into $[0,1]$ with $j(x)=1$ for $x \in E$ and $j(x)=0$ for $x \notin E+W$, and

$$\varphi(x) \;=\; M_{(t)}\big(j(x-t)j(-t)\big)$$

is a real, continuous positive-definite function on G, with $\varphi(0)>0$, $\varphi(x)=0$ for $x \notin E-E+W+W$, $M(\varphi)>0$. Let

$$f'(x) \;=\; \sup_{y\in E-E+W+W} h(x-y) ;$$

then $f'(x)$ is a uniformly continuous real function from G into $[0,1]$ with $f'(x)=1$ for $x \in E-E+W+W$; $f'(x)=0$ for $x \notin E-E+W+W+W$. Define

$$f(x) \;=\; \tfrac{1}{2}\big(f'(x)+f'(-x)\big) ,$$

then f has all the properties just stated for f', and $f=f^\star$.

As in section 2 we construct the functional I and the corresponding Banach $\tilde{\ }$-algebra. We have

$$I(\varphi) \;=\; M_{(x)}\big(f(x)\varphi(x)\big) \;=\; M_{(x)}\big(\varphi(x)\big) \;>\; 0 .$$

Assume $\varphi_{-x_0}(x) \neq 0$ for some $x \in E-E+W+W+W$, then $x+x_0 \in E-E+W+W$, that is,

$$x_0 \in E-E+E-E+W+W+W+W+W$$

which is a contradiction. Hence

$$I(\varphi_{-x_0}) \;=\; M_{(x)}\big(f(x)\varphi_{-x_0}(x)\big) \;=\; 0 .$$

It is well known that a continuous positive definite function φ on G corresponds to a strongly continuous unitary representation V of G (over a Hilbert space H') such that $\varphi(x)=\langle V_x\xi_0,\xi_0\rangle$. (See f. ex. Godement [4]; the construction of V does not depend on the assumption that G be locally compact). Let A be the corresponding $\tilde{}$-representation of $\overline{\mathscr{A}_f}$ (Theorem 3). We have

$$\langle A_{I_{x_0}}\xi_0,\xi_0\rangle = (I_{x_0})_{(x)}(\langle V_x\xi_0,\xi_0\rangle) = I_{(x)}(\langle V_{x+x_0}\xi_0,\xi_0\rangle) = I(\varphi_{-x_0}) = 0$$

and

$$\langle A_I\xi_0,\xi_0\rangle = I(\langle V_x\xi_0,\xi_0\rangle) = I(\varphi) > 0 .$$

Hence $A_{I_{x_0}} \neq A_I$; and $A_{I-I_{x_0}} \neq 0$. We now refer to Hewitt and Ross [5, Theorem 21.37, p. 330]. By this theorem, there must exist an irreducible $\tilde{}$-representation T of $\overline{\mathscr{A}_f}$ (over a Hilbert space H) such that $T_{I-I_{x_0}} \neq 0$. Since $\overline{\mathscr{A}_f}$ is commutative, any irreducible $\tilde{}$-representation is 1-dimensional. Now let

$$\xi \in H, \qquad \xi \neq 0, \qquad J \in \overline{\mathscr{A}_f} .$$

Define $U_x(T_J\xi) = T_{J_x}(\xi)$. This is well-defined, for if $T_J\xi = T_{J'}\xi$, we have $T_{J-J'}\xi = 0$ and

$$\begin{aligned}\langle T_{J_x-J'_x}\xi, T_{J_x-J'_x}\xi\rangle &= \langle (T_{J_x-J'_x})^* \circ T_{J_x-J'_x}\xi,\xi\rangle \\ &= \langle T_{((J-J')^\sim)_{-x}*(J-J')_x}\xi,\xi\rangle = \langle T_{J-J'}\xi, T_{J-J'}\xi\rangle = 0;\end{aligned}$$

that is, $T_{J_x}\xi = T_{J'_x}\xi$.

It is easily checked that U_x is a unitary operator on the 1-dimensional space H. We have

$$U_{x+y}(T_J\xi) = T_{J_{x+y}}\xi = T_{J_{y+x}}\xi = T_{(J_y)_x}\xi ,$$

$$U_xU_y(T_J\xi) = U_x(T_{J_y}\xi) = T_{(J_y)_x}\xi ;$$

hence $x \to U_x$ is a representation of G over H. We show that this representation is continuous: Let $\eta = T_J\xi$; any element in H can be written in this form. Consider the map $x \to U_x\eta = T_{J_x}\xi$. Since $J \in \overline{\mathscr{A}_f}$, the map $x \to J_x$ is continuous (Theorem 2), the map $J' \to T_{J'}$ is continuous, and finally, the map $B \to B\xi$ from the bounded operators on H into H is continuous, hence the result.

Since $T_{I_{x_0}} \neq T_I$, H is 1-dimensional, and $\xi \neq 0$, we have $T_I\xi \neq T_{I_{x_0}}\xi$; that is, $U_{x_0}(T_I\xi) \neq T_I\xi$. Hence, U_{x_0} is not the identity operator.

Let ξ_0 have unit length in H, and let $\gamma(x) = \langle U_x\xi_0,\xi_0\rangle$. By the above argument $U_{x_0}\xi_0 \neq \xi_0$, hence $\gamma(x_0) \neq 1$. Thus γ is a continuous character on G with $\gamma(x_0) \neq 1$; and the proof is complete.

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