MATH. SCAND. 18 (1966), 35–44

SPECIAL TRIGONOMETRIC SERIES AND THE RIEMANN HYPOTHESIS

L. A. RUBEL and E. G. STRAUS

By a special trigonometric series, we mean a series of the form

$$(1) \qquad \sum_{n=0}^{\infty} \frac{\sin \lambda_n x}{\lambda_n}, \qquad 0 < \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots, \qquad \lambda_n \to \infty ,$$

which we abbreviate as $\sum \lambda^{-1} \sin \lambda x$. We call the numbers λ_n the frequencies of the series. Throughout this paper, we restrict our attention to the range $x > 0$. Let ζ denote the Riemann zeta function, and let

$$\varrho = \beta + i\gamma$$

denote the non-trivial zeros of ζ.

Rademacher [2] has shown that if the Riemann hypothesis is true, then the series

$$(2) \qquad \sum_{\gamma > 0} \frac{\sin \gamma x}{\gamma}$$

has certain jump discontinuities. It converges uniformly on closed intervals of $x > 0$ that exclude the logarithms of the prime powers, to a function that jumps $(-\frac{1}{2})p^{-\frac{1}{2}k} \log p$ at each $k \log p$, $k = 1, 2, \ldots$. He suggested that such unusual behaviour of this special trigonometric series might constitute evidence against the Riemann hypothesis.

At the same time, he began a program of synthesizing a special trigonometric series that has these jumps, in the hope that the distribution of the synthetic frequencies would illuminate the distribution of the actual frequencies γ. We begin this paper by completing his program; indeed, we construct in Theorem 1 a special trigonometric series having any preassigned jumps, subject only to the mild restriction that the places where the jumps are to occur have no finite limit point. There is also considerable latitude in the density of the frequencies of such a series. Our methods are elementary and use no number theory or properties of ζ.

Received October 9, 1965.

Next, extending Rademacher's original argument, we show that the series (2) has the described jumps if the hypothesis

$$\sum_{\beta>\frac{1}{2},\gamma>0} \frac{(\beta-\frac{1}{2})^2}{\gamma} < \infty \tag{3}$$

holds. Now (3) is considerably weaker than the Riemann hypothesis, and seems to be only slightly out of reach of what is now known about the distribution of the zeros ϱ. We end the paper by showing that if the exponent 2 in (3) were increased to $2+\varepsilon$, for any positive ε, then the corresponding series would converge.

In summary, it seems doubtful that considerations of this kind can shed much light on the distribution of the zeros of the Riemann zeta function.

THEOREM 1. *Given any sequence $\{\sigma_n\}$ of real numbers indexed by the integers n, $1 \leqq n < N$, $N \leqq \infty$, and any identically indexed sequence $\{s_n\}$ of positive real numbers, that has no limit point, then there exists a special trigonometric series that converges uniformly on each closed subinterval of $x > 0$ that contains no s_n, and whose sum $H(x)$ jumps by σ_n at s_n for each n in the index set.*

In addition to the assertions of the theorem, it is possible to make the frequencies λ distinct. Furthermore, given any positive continuous function $u(t)$, $0 \leqq t < \infty$, such that $u(t) \to \infty$ as $t \to \infty$, it is possible to make the number of frequencies λ that do not exceed t smaller than $tu(t)$. We have not looked at the converse problem of thickening the sequence λ. Neither have we considered the problem of what jump discontinuities are possible for a series of the form $\sum \varphi(\lambda) \sin \lambda x$ for any function other than $\varphi(\lambda) = 1/\lambda$.

PROOF. We arrange the s_n in increasing order. We shall suppose that $s_1 = \pi$. This involves no loss of generality, since for each $a > 0$,

$$\sum \frac{\sin (a\lambda)x}{(a\lambda)} = \frac{1}{a} \sum \frac{\sin \lambda(ax)}{\lambda}.$$

It is an elementary fact that the series

$$S(x) = \sum_{k=1}^{\infty} \frac{\sin (2k-1)x}{2k-1}$$

is 2π-periodic, converges uniformly on closed sub-intervals of $(-\pi, 0)$ to $-\pi/4$, and converges uniformly on closed sub-intervals of $(0, \pi)$ to

$+\pi/4$. Now it is easily verified that if δ is restricted by $0<\delta<\frac{1}{2}$, then the two series

$$\sum \frac{\sin(2k-1+\delta)x}{2k-1+\delta} \quad \text{and} \quad \sum \frac{\sin(2k-1+\delta)x}{2k-1}$$

are uniformly equiconvergent; that is, the difference of the n-th partial sums converges uniformly on $x>0$. We then see that

$$\sum \left(\frac{\sin(2k-1-\delta)x}{2k-1-\delta}+\frac{\sin(2k-1+\delta)x}{2k-1+\delta}\right) \tag{4}$$

converges suitably to a function with the same jumps as $2S(x)\cos\delta x$. The same is true for the series $\sum_0$ obtained by rewriting (4) in the form (1); that is,

$$\sum\nolimits_0 = \frac{\sin(1-\delta)x}{1-\delta}+\frac{\sin(1+\delta)x}{1+\delta}+\frac{\sin(3-\delta)x}{3-\delta}+\frac{\sin(3+\delta)x}{3+\delta}+\dots. \tag{5}$$

Therefore, on choosing δ properly, we can make the jumps of $\sum_0$ at π any number we please in the interval $(-\pi/2,\pi/2)$. By choosing finitely many numbers $\delta_1,\dots,\delta_k$, adding the corresponding series of the form (5), and rearranging the sum by non-decreasing frequencies, we can make the jump at π any number we please. By choosing the numbers δ_i to be distinct, we may make the frequencies distinct.

We have, then, produced a series $\sum_1$ of the form (1) that jumps first at s_1 by the amount σ_1. The series $\sum_1$ will also jump on some subset of $\{ns_1\}$, $n=2,3,4,\dots$, and nowhere else. Suppose that the first of these is n_0s_1. Now if $s_2<n_0s_1$, we construct by the above procedure a series $\sum_2$ that jumps first at s_2 by the amount σ_2. If $s_2>n_0s_1$, we instead make the first jump of $\sum_2$ cancel the jump of $\sum_1$ at n_0s_1. A similar provision is made in case $s_2=n_0s_1$. We provide, in any event, that the frequencies that occur in $\sum_2$ are not only distinct, but are different also from the frequencies that occur in $\sum_1$. This is possible since the frequencies that occur in $\sum_1$ lie in a finite union of arithmetic progressions.

We now proceed in order, producing series $\sum_n$ that have the required jumps, or that cancel (perhaps only partially) unnecessary jumps introduced in the preceding steps. Because the s_n have no finite limit point, and because the series $\sum_j$ each have jumps spaced further apart than s_1, we arrive by addition at a formal series

$$\sum = \sum\nolimits_1+\sum\nolimits_2+\dots \tag{6}$$

that has formal jumps σ_n at the s_n, and no other jumps. If we let $\sum_n'$ denote $\sum_n$ with finitely many initial terms deleted, then $\sum_n'$ has exactly the same jumps as $\sum_n$. We shall specify later which terms should be deleted, but to begin with, we demand at least that the frequencies that occur in $\sum_n'$ should all exceed n. We now let

$$\sum' = \sum_1' + \sum_2' + \dots, \tag{7}$$

where $\sum'$ denotes the rearrangement of the formal series by increasing frequencies λ. Such a rearrangement is possible, and the rearranged λ approach infinity, since for each positive number N, there are at most finitely many frequencies that do not exceed N.

We denote the sequence of all the places where the jumps of the accumulated $\sum_i$ occur by $s_1', s_2', \dots$, arranged in increasing order — we have seen that $s_n' \to \infty$ as $n \to \infty$. Let $I_1, I_2, \dots$ be closed intervals whose union is the complement of $\{s_j'\}$, $j = 1, 2, \dots$, and such that each closed subinterval of this complement is contained in a finite union of the I_m. We denote by $\sum_n'[M,N]$ the sum of the terms of $\sum_n'$ with frequencies λ lying in the interval $M \leqq \lambda \leqq N$, with a corresponding meaning for $\sum'[M,N]$. Our next deletion requirement on the $\sum_n'$ is that for each n, M, N, we have

$$|\textstyle\sum_n'[M,N]| \leqq 2^{-n} \quad \text{on} \quad I_1 \cup \dots \cup I_n .$$

This is possible by the uniform convergence of each of the $\sum_n$ on closed intervals excluding its jumps. By the Cauchy convergence criterion, it follows that $\sum'$ converges uniformly on each interval I_m, for if we write, for $n \geqq m$,

$$|\textstyle\sum'[M,N]| \leqq |\sum_1'[M,N]| + \dots + |\sum_n'[M,N]| + \\ + \{|\sum_{n+1}'[M,N]| + |\sum_{n+2}'[M,N]| + \dots\}$$

so that on I_m, if $n \geqq m$,

$$|\textstyle\sum'[M,N]| \leqq \{|\sum_1'[M,N]| + \dots + |\sum_n'[M,N]|\} + 2^{-n} .$$

Given $\varepsilon > 0$, we need only choose $n \geqq m$ so that $2^{-n} < \varepsilon/2$, and then choose M_0 so that for $M, N \geqq M_0$,

$$|\textstyle\sum_j'[M,N]| \leqq \varepsilon/2n \qquad \text{for} \quad j = 1, 2, \dots, n .$$

It follows that $|\sum'[M,N]| < \varepsilon$ on I_m whenever $M, N \geqq M_0$, and we have proved that $\sum'$ converges uniformly on I_m. Consequently, $\sum'$ converges uniformly on each closed interval that contains no s_n'.

A more delicate study must be made in the neighborhood of a point s_j' where the jumps of the series $\sum_n$ occur. To each such s_j' we assign a closed interval T_j that contains s_j' in its interior and that excludes all other places of jumps, s_i'. It remains to be proved that $\sum'$ has the proper jumps. But first, we must delete some more initial terms from the $\sum_n$; this will not affect the preceding arguments.

For each positive integer n, consider those intervals T_m for $m \leqq n$ in which $\sum_n'$ has no jump. We delete enough initial terms from $\sum_n'$ so that on each such T_m,

$$|\textstyle\sum_n'[M,N]| < 2^{-n} \qquad \text{for all } M,N\ .$$

Now we write

$$\textstyle\sum_n'[M,N] = \sum^{(m)}[M,N] + \sum_1^m[M,N] + \sum_2^m[M,N] + \dots,$$

where $\sum_j^m = \sum_j'$ if $\sum_j'$ does not jump in T_m, and where $\sum_j^m = 0$ (the empty sum) otherwise.

By an earlier argument, we may, given $\varepsilon > 0$, choose M_0 so that for $M,N \geqq M_0$ we have

$$|\textstyle\sum_1^m[M,N]| + |\sum_2^m[M,N]| + \dots < \varepsilon\ .$$

In other words, the jump of $\sum'$ in T_m is the same as the jump of $\sum^{(m)}$ in T_m. It remains to prove that if $\sum_{n_1},\dots,\sum_{n_k}$ are those $\sum_j$ that jump in T_m, and if

$$\textstyle\sum^{[m]} = \sum_{n_1} + \sum_{n_2} + \dots + \sum_{n_k}$$

rearranged by increasing frequencies, then the jump of $\sum^{[m]}$ at s_m' is the sum of the jumps of the $\sum_{n_j}$ there.

The case where only one of the $\sum_j$ jumps at s_m' is trivial. We consider here only the special case where $\sum_1$ has an unnecessary jump at 2π because of a necessary jump at π, and $\sum_2$ cancels this jump. The general case is similar. Working modulo uniformly convergent series of continuous functions near $x = 2\pi$, the n-th partial sum of $\sum_1$ is

$$2(\textstyle\sum \cos \delta_i x) \displaystyle\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$$

and the n-th partial sum of $\sum_2$ is

$$4\left(\textstyle\sum \cos \delta_j' \dfrac{x}{2}\right) \displaystyle\sum_{k=1}^{n} \frac{\sin(2k-1)x/2}{2k-1}.$$

Now $\sum_1$ jumps by the amount $\pi \sum \cos 2\pi\delta_i$ at $x = 2\pi$, and $\sum_2$ jumps by the amount $-2\pi \sum \cos \pi\delta_j'$ at $x = 2\pi$, so that in order to have the correct cancellation, we must have

$$\textstyle\sum \cos 2\pi\delta_i - 2\sum \cos \pi\delta_j' = 0\ .$$

Now, by a familiar formula,

$$\sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1} = \frac{1}{2}\int_0^x \frac{\sin 2nt}{\sin t}\,dt\ .$$

Writing $\sum = \sum_1 + \sum_2$, where $\sum$ is arranged in order of increasing frequencies, we see that for each sufficiently large real number t, there are twice as many frequencies of $\sum_2$ that do not exceed t as there are frequencies of $\sum_1$ that do not exceed t, give or take one or two terms. We must prove then that the following sequence $u_n(x)$ converges uniformly in a neighborhood of $x = 2\pi$:

$$u_n(x) = (\sum \cos\delta_i x) \int_0^x \frac{\sin 2nt}{\sin t}\, dt + 2\left(\sum \cos\delta_j' \frac{x}{2}\right) \int_0^{x/2} \frac{\sin 4nt}{\sin t}\, dt \,.$$

Let us prove the result for $2\pi \leqq x \leqq 2\pi + 1$, say. The range $x < 2\pi$ is treated similarly. We write

$$x = 2\pi + y$$

and observe that

$$u_n(x) = (\sum \cos\delta_i x) \int_0^y \frac{\sin 2nt}{\sin t}\, dt - 2\left(\sum \cos\delta_j' \frac{x}{2}\right) \int_0^{\frac{1}{2}y} \frac{\sin 4nt}{\sin t}\, dt \,.$$

Now

$$\lim_{n\to\infty} \int_0^y \sin 2nt \left(\frac{1}{\sin t} - \frac{1}{t}\right) dt = 0$$

uniformly for $0 \leqq y \leqq 1$ by the Riemann–Lebesgue Lemma, in the form

$$\left| \int_{-\infty}^{\infty} f(x) e^{itx}\, dx \right| \leqq \tfrac{1}{2} \int_{-\infty}^{\infty} |f(x + \pi/t) - f(x)|\, dx \,,$$

applied to the function $f(x) = (1/\sin x) - (1/x)$ for $0 < x < y$, $f(x) = 0$ otherwise. Since the cosine sums are continuous, it follows that we need only prove that the sequence $u_n'(x)$ converges uniformly for $2\pi \leqq x \leqq 2\pi + 1$:

$$u_n'(x) = (\sum \cos\delta_i x) \int_0^y \frac{\sin 2nt}{t}\, dt + 2(\sum \cos\delta_j' x) \int_0^{\frac{1}{2}y} \frac{\sin 4nt}{t}\, dt \,.$$

Changing variables, we write

$$u_n'(x) = \left\{\sum \cos\delta_i x - 2 \sum \cos\delta_j' \frac{x}{2}\right\} \int_0^{2ny} \frac{\sin t}{t}\, dt \,,$$

so that

$$u_n'(x) - u_m'(x) = \left\{\sum \cos\delta_i x - 2 \sum \cos\delta_j' \frac{x}{2}\right\} \int_{2my}^{2ny} \frac{\sin t}{t}\, dt \,.$$

Since $\int_a^b t^{-1} \sin t\,dt$ is a uniformly bounded function of a and b, and since the cosine sums are continuous, we may, given $\varepsilon > 0$, choose a number $\delta > 0$ so that for $2\pi \leqq x \leqq 2\pi + \delta$,

$$|u_n'(x) - u_m'(x)| < \varepsilon \qquad \text{for all pairs } m, n .$$

But uniformly for $\delta \leqq y \leqq 1$,

$$\lim_{m,n\to\infty} \int_{2my}^{2ny} \frac{\sin t}{t} dt = 0 ,$$

and the result follows.

Next, modifying Rademacher's method, we exhibit the connection between the series (2) and the zeros of ζ.

THEOREM 2. *If the hypothesis* (3) *holds, then the series* (2) *converges uniformly in each closed interval that excludes all numbers of the form* $k \log p$, $k = 1, 2, 3, \ldots, p$ *prime, to a function that jumps*

$$-\tfrac{1}{2} p^{-\frac{1}{2}k} \log p$$

at each $k \log p$.

PROOF. We begin with the Riemann- von Mangoldt Formula [1, p. 36], which asserts that for $t > 1$, $t \neq p^k$,

$$\sum_{n \leqq t} \Lambda(n) = t - \sum_{\varrho} \varrho^{-1} t^{\varrho} - \tfrac{1}{2} \log(1 - t^{-2}) - \log 2\pi ,$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \text{ prime},\ k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

and where we arrange the ϱ in order of non-decreasing $|\gamma|$. It follows that the series

$$\Sigma \varrho^{-1} t^{\varrho}$$

converges to a function $f(t)$ that jumps $-\log p$ at each positive integral power p^k of each prime p, uniformly in each closed subinterval of $t > 1$ that contains no prime power. Now write $t = \exp x$, and $F(x) = f(t)$, so that

$$F(x) = \sum \frac{\exp(\beta + i\gamma)x}{\beta + i\gamma} .$$

Using the fact that the zeros ϱ occur in conjugate pairs, we have

$$F(x) = \sum_{\gamma > 0} \frac{\exp \beta x}{\beta^2 + \gamma^2} (2\beta \cos \gamma x + 2\gamma \sin \gamma x) .$$

Writing $c(x)$ as notation for a generic continuous function, and using the facts that the β are bounded and that $\sum \gamma^{-2} < \infty$ [3, p. 181], we have

$$F(x) = c(x) + 2 \sum_{\gamma>0} \frac{\sin \gamma x}{\gamma} \exp \beta x . \tag{8}$$

Since with each β we also have its mirror image in the line $\gamma = \frac{1}{2}$, we may rewrite (8) as

$$F(x) = c(x) + \sum_{\gamma>0} \{\exp \beta x + \exp (1-\beta)x\} \frac{\sin \gamma x}{\gamma} .$$

Let us now write, formally,

$$\begin{aligned} F(x) - c(x) - 2(\exp \tfrac{1}{2}x) \sum_{\gamma>0} \frac{\sin \gamma x}{\gamma} \\ = \sum_{\gamma>0} \{\exp \beta x + \exp (1-\beta)x - 2 \exp \tfrac{1}{2}x\} \frac{\sin \gamma x}{\gamma} . \end{aligned} \tag{9}$$

Now for x in any fixed interval I,

$$\exp \beta x + \exp (1-\beta)x - 2 \exp \tfrac{1}{2}x = O\big((\beta - \tfrac{1}{2})^2\big) ,$$

uniformly in I. Consequently, if the hypothesis (3) holds, then the second series in (9) converges uniformly in each closed interval, and the series $\sum \gamma^{-1} \sin \gamma x$ will therefore have the expected behavior.

The next result shows that the hypothesis (3) is "almost" within reach.

THEOREM. *For any number* $q > 2$,

$$\sum_{\beta > \frac{1}{2}} (\beta - \tfrac{1}{2})^q / \gamma < \infty .$$

PROOF. Let $N(\sigma, T)$ represent the number of zeros $\varrho = \beta + i\gamma$ of the ζ function that satisfy $\sigma \leqq \beta$ and $0 \leqq \gamma \leqq T$. According to a theorem of A. Selberg [3, p. 204], we have

$$N(\sigma, T) = O(T^r \log T) \tag{10}$$

uniformly for $\frac{1}{2} \leqq \sigma \leqq 1$, where

$$r = 1 - \tfrac{1}{4}(\sigma - \tfrac{1}{2}) .$$

Let

$$S(\sigma, T) = \sum_{\beta \geqq \sigma,\, 0 \leqq \gamma \leqq T} \frac{1}{\gamma} = \sum_{n=1}^{N(\sigma, T)} \frac{1}{\gamma_n} . \tag{11}$$

According to (10) we have, for suitable constants c_i, that

$$n < c_0 \gamma_n^{\,r} \log \gamma_n$$

and hence

(12) $$\gamma_n > c_1 (n/\log n)^s ,$$

where

$$s = \frac{1}{r} = \frac{1}{1 - \frac{1}{4}(\sigma - \frac{1}{2})}.$$

Substituting (12) in (11), we obtain

$$S(\sigma, T) < c_2 \int_1^{T \log T} \left(\frac{\log t}{t}\right)^s dt .$$

Now consider, for $q > 0$,

$$\int_{\frac{1}{2}}^{1} (\sigma - \tfrac{1}{2})^{q-1} S(\sigma, T)\, d\sigma = \sum_{\beta > \frac{1}{2},\, 0 \leqq \gamma \leqq T} \int_{\frac{1}{2}}^{\beta} \frac{(\sigma - \frac{1}{2})^{q-1}}{\gamma} d\sigma = \frac{1}{q} J(T) ,$$

where

$$J(T) = \sum_{\beta > \frac{1}{2},\, 0 \leqq \gamma \leqq T} \frac{(\beta - \frac{1}{2})^q}{\gamma}.$$

We have

$$J(T) = q \int_{\frac{1}{2}}^{1} (\sigma - \tfrac{1}{2})^{q-1} S(\sigma, T)\, d\sigma < c_3 \int_{\sigma = \frac{1}{2}}^{1} (\sigma - \tfrac{1}{2})^{q-1} \int_{t=1}^{T} \left(\frac{\log t}{t}\right)^s dt\, d\sigma ,$$

from which it follows that

$$J(T) < c_3 \int_{t=1}^{T \log T} \frac{\log t}{t} \int_{\sigma = \frac{1}{2}}^{1} (\sigma - \tfrac{1}{2})^{q-1} \left(\frac{\log t}{t}\right)^a d\sigma\, dt ,$$

where

$$a = \tfrac{1}{4} \frac{\sigma - \frac{1}{2}}{1 - \frac{1}{4}(\sigma - \frac{1}{2})},$$

and therefore that

(13) $$J(T) < c_3 \int_{t=1}^{T \log T} \frac{\log t}{t} \int_{\sigma = \frac{1}{2}}^{1} (\sigma - \tfrac{1}{2})^{q-1} \left(\frac{\log t}{t}\right)^b d\sigma\, dt ,$$

where

$$b = \tfrac{1}{4}(\sigma - \tfrac{1}{2}) .$$

But for any $\lambda > -1$ and u with $0 < u < 1$, we have

$$\int_0^{\frac{1}{2}} x^\lambda u^x\, dx = \frac{1}{(-\log u)^{\lambda+1}} \int_0^{-\frac{1}{2}\log u} y^\lambda e^{-y}\, dy < \frac{1}{(-\log u)^{\lambda+1}} \Gamma(\lambda + 1) .$$

If we use this estimate in (13) with $x=\sigma-\frac{1}{2}$, $\lambda=q-1$, and $u=(t^{-1}\log t)^{\frac{1}{2}}$, we obtain

$$\sum_{\beta>\frac{1}{2},\,0\leqq\gamma\leqq T}\frac{(\beta-\frac{1}{2})^q}{\gamma}<c_4\int_1^{T\log T}\frac{\log t}{t}\frac{1}{(\log t)^q}\,dt\,.$$

Since the right side is bounded whenever $q>2$, we have the desired result. We remark that a slight modification of the proof actually establishes that

$$\sum\frac{(\beta-\frac{1}{2})^q}{\gamma(\log\gamma)^p}<\infty\qquad\text{whenever}\quad q\geqq 0\ \text{and}\ p+q>2\,.$$

REFERENCES

1. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig ,1909.
2. H. Rademacher, *Remarks concerning the Riemann–von Mangoldt formula*, Report of the Institute in the Theory of Numbers, University of Colorado, 1959, pp. 31–37.
3. E. C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford, 1951.

UNIVERSITY OF ILLINOIS, URBANA, U.S.A.
AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES, U.S.A.