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EQUIVALENCE OF TWO METHODS OF INTERPOLATION

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Introduction.

The purpose of this paper is to show the equivalence of two methods of introducing interpolation spaces, defined by Gagliardo (see for instance [2]) and by Peetre (see for instance [6]). (We remark that in view of the results in [5], we also get the equivalence with the interpolation spaces in [3], [4].)

Section 1 deals with Gagliardo's and Peetre's definitions of interpolation spaces. The short Section 2 contains some preliminaries to the proof of the equivalence, which is given in Section 3. For the main result see Theorem 3.1.

The problem treated in this paper has been suggested to me by professor Jaak Peetre. I thank him for valuable advice and for his great interest in my work.

1. The definition of the interpolation spaces.

Let A_0 and A_1 be two Banach-spaces, which are continuously embedded in a Banach-space $\mathscr{A}$. The corresponding norms are denoted by $\|a\|_{A_0}$ respectively $\|a\|_{A_1}$. (Often in the applications we have $A_0 \subset A_1$ or $A_1 \subset A_0$.)

a. Gagliardo's definition of interpolation spaces between A_0 and A_1. We introduce

$$(1.1) \qquad M(a) = \{(x,y) \mid \exists a_i \in A_i,\ i=0,1,\ \text{with}\ a=a_0+a_1,\ \|a_0\|_{A_0} \leqq x,\ \|a_1\|_{A_1} \leqq y\}\,.$$

$M(a)$ is defined for all $a \in A_0+A_1$. The pointset $M(a)$ is contained in the quadrant $x \geqq 0$, $y \geqq 0$ with the following properties:

$$(1.2) \qquad M(a) \text{ is convex}\,.$$

$$(1.3) \qquad \text{If } (x,y) \in M(a), \text{ then } (x+h,y+k) \in M(a),\ h,k \geqq 0\,.$$

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We also assume that

$$\inf_{(x,y)\in M(a)} x = 0 \tag{1.4}$$

$$\inf_{(x,y)\in M(a)} y = 0 . \tag{1.5}$$

(1.4) and (1.5) are fulfilled for instance when $A_0 \cap A_1$ is dense in A_0 or A_1, respectively. Denote by ∂M the boundary of M excluding those points belonging to the positive halfaxes. The curve ∂M can be represented by a function

$$y = y(x), \qquad 0 < x < a \leqq +\infty ,$$

which is positive, convex, decreasing ($\inf y = 0$) and with left and right hand derivatives at every point of its domain of definition. The derivative is negative and increasing.

With a suitable chosen set function F one can show (see [2]) that the space

$$A_F = \{a \mid F[M(a)] < \infty\}$$

with the norm

$$\|a\|_{A_F} = F[M(a)]$$

becomes an interpolation space. Here we shall only deal with the special case (see [2])

$$F[M] = F_{\alpha,\beta,\gamma,\delta}[M] = \left(\int_{\partial M} x^\alpha y^\beta |dx|^\gamma \, |dy|^\delta\right)^{1/(\alpha+\beta+1)} \tag{1.6}$$

$$= \left(\int_{\partial M} x^\alpha y^\beta |y'|^\delta dx\right)^{1/(\alpha+\beta+1)}$$

with $\alpha,\beta,\gamma,\delta \geqq 0$, $\gamma+\delta=1$. We write

$$A_{\alpha,\beta,\gamma,\delta} = A_{F\alpha,\beta,\gamma,\delta} .$$

Remark. Arduini [1] has shown that $A_{\alpha,\beta,\gamma,\delta}$ only depends on two parameters, viz. $\alpha+\gamma$ and $\beta+\delta$. This result will follow in a new way from our theorem 3.1.

b. Peetre's definition of interpolation-spaces between A_0 and A_1. We introduce

$$K(t,a) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \qquad a \in A_0 + A_1, \ 0 < t < \infty . \tag{1.7}$$

$K(t,a)$ is a positive, concave and increasing function with left and right hand derivatives at every point. If we further assume (1.4) and (1.5) we get

$$\lim_{t\to+\infty} t^{-1}K(t,a) = 0 \quad \text{and} \quad \lim_{t\to+0} K(t,a) = 0 .$$

With a suitable chosen functional Φ one can show (see [6]) that the space

$$A_\Phi = \{a \mid a \in A_0 = A_1, \Phi[K(t,a)] < \infty\}$$

with the norm

$$\|a\|_{A_\Phi} = \Phi[K(t,a)]$$

becomes an interpolation space. Here we shall only deal with the special case

$$\Phi[\varphi] = \Phi_{\theta,p}[\varphi] = \left(\int_0^\infty (t^{-\theta}\varphi)^p\, t^{-1}\, dt\right)^{1/p} \tag{1.8}$$

with $0 < \theta < 1$, $1 \leqq p \leqq \infty$. We write

$$A_{\theta,p} = A_{\Phi_{\theta,p}}\,.$$

Remark. The conditions (1.4) and (1.5) are necessary for $F_{\alpha,\beta,\gamma,\delta}[M] < \infty$ and $\Phi_{\theta,p}[K] < \infty$.

2. The connection between $\partial M(a)$ and $K(t,a)$.

From the definition (1.7) of $K(t,a)$ it follows that the connection between the points (x,y) on $\partial M(a)$ and $K(t,a)$ is given by a kind of Legendre transform

$$K(t,a) = \inf_{(x,y)\in\partial M(a)}(x+ty) \tag{2.1}$$

or with the inverse transform

$$x = \sup_{0<t<\infty}\big(K(t,a)-ty\big)\,. \tag{2.2}$$

If we regard y as a function of x, we also get, at those points where y is differentiable,

$$K(t,a) = x - y/y', \qquad t = -1/y'\,. \tag{2.3}$$

At the points where y is not differentiable, we can give a meaning to (2.3) by letting y' take any value between the derivative on the left and on the right at the point. At those points where $K(t,a)$ is differentiable we get

$$x = K(t,a) - t\,k(t,a), \qquad y = k(t,a)\,, \tag{2.4}$$

where

$$k(t,a) = \frac{dK(t,a)}{dt}.$$

As above we can give a meaning to (2.4), even when $K(t,a)$ is not differentiable.

3. The equivalence between $A_{\alpha,\beta,\gamma,\delta}$ and $A_{\theta,p}$.

Our aim is to show the following theorem.

THEOREM 3.1. *If* $\alpha+\gamma=p(1-\theta)$ *and* $\beta+\delta=p\theta$, *then* $A_{\alpha,\beta,\gamma,\delta}=A_{\theta,p}$ *and the norms* $\|\cdot\|_{A_{\alpha,\beta,\gamma,\delta}}$ *and* $\|\cdot\|_{A_{\theta,p}}$ *are equivalent.*

We first prove the theorem for a special M. We assume that ∂M is a curve, which does not touch the co-ordinate axes or is asymptotic to any of them. Let $(0,y_0)$ and $(x_0,0)$ be those points where ∂M meets the co-ordinate axes. Then

$$|dy/dx|_{x=0+} < \infty \quad \text{and} \quad |dx/dy|_{y=0+} < \infty$$

and we put

$$|dy/dx|_{x=0-} = |dx/dy|_{y=0-} = +\infty .$$

To a ∂M of this type there corresponds a $K(t,a)$ which is defined for $t \geqq 0$ and for which, according to (2.1), (2.2) and (2.4),

$$(3.1)\quad \begin{cases} K(0) = 0, \qquad k(0) = y_0 < \infty , \\ \text{and} \\ k(t) = 0 \quad \text{for} \quad t > |y'(x_0-0)|^{-1} = |dx/dy|_{y=0+} . \end{cases}$$

NOTATIONS. In the sequel we denote equivalence of norms by $\sim$. We omit θ and p in $\Phi_{\theta,p}$ and write $F_{\alpha,\beta}$ instead of $F_{\alpha,\beta,\gamma,\delta}$. We also often omit one or both variables in $K(t,a)$ and $k(t,a)$. Note that in the sequel always holds $\alpha+\gamma=p(1-\theta)$, $\beta+\delta=p\theta$ and $\gamma+\delta=1$ but sometimes not $\alpha,\beta,\gamma,\delta \geqq 0$.

LEMMA 3.1. *We have* $\Phi[K] \sim \Phi[tk] \sim \Phi[K-tk]$.

PROOF. We get

$$(\Phi[K])^p = \int_0^\infty t^{-p\theta-1} K^p \, dt$$

$$= [-(p\theta)^{-1} t^{-p\theta} K^p]_0^\infty + \theta^{-1} \int_0^\infty t^{-p} K^{p-1} k \, dt ,$$

but from (1.4), (1.5) and (2.1) it follows that

$$K \leqq t \sup\nolimits_{(x,y)\in\partial M} y = t y_0 ,$$

so that

$$t^{-\theta} K \leqq t^{1-\theta} y_0 \to 0 \quad \text{as } t \to 0$$

and

$$K \leqq \sup\nolimits_{(x,y)\in\partial M} x = x_0 ,$$

so that

$$t^{-\theta}K \leqq t^{-\theta}x_0 \to 0 \quad \text{as } t \to \infty .$$

Hence we have

$$(\Phi[K])^p = \theta^{-1} \int_0^\infty t^{-p\theta} K^{p-1} k \, dt .$$

From Hölder's inequality we get

$$\theta(\Phi[K])^p = \int_0^\infty t^{-p\theta} K^{p-1} k \, dt$$

$$\leqq \left(\int_0^\infty t^{-p\theta-1} (tk)^p \, dt \right)^{1/p} \left(\int_0^\infty t^{-p\theta-1} K^p \, dt \right)^{1-1/p}$$

or

$$\Phi[K] \leqq \theta^{-1} \Phi[tk] .$$

The concavity of $K(t)$ gives us that $K(t) \geqq tk(t)$ so that

$$\Phi[K] \geqq \Phi[tk] ,$$

and the equivalence between $\Phi[K]$ and $\Phi[tk]$ is proved.

The rest of the assertion is proved in an analogous manner.

Lemma 3.2. *We have*

$$\Phi[tk] = F_{0,p-1}[M] \, (1-\theta)^{-1/p} \quad \textit{and} \quad \Phi[K-tk] = F_{p-1,0}[M] \, \theta^{-1/p} .$$

Proof. We get

$$(\Phi[tk])^p = \int_0^\infty t^{p(1-\theta)-1} k^p(t) \, dt$$

$$= p^{-1}(1-\theta)^{-1} \int_0^\infty k^p(t) \, d(t^{p(1-\theta)})$$

$$= -p^{-1}(1-\theta)^{-1} \left(\int_0^\infty t^{p(1-\theta)} \, d\big(k^p(t)\big) + \left[t^{p(1-\theta)} k^p(t)\right]_0^\infty \right).$$

The last equality follows from a partial integration in a Stieltjes integral. The last term will disappear; for according to (3.1) $k(t)$ is bounded and $k(t) = 0$ for sufficiently large t. If we now change variables in the Stieltjes integral and let k be a continuous variable, we get

$$(\Phi[tk])^p = -p^{-1}(1-\theta)^{-1} \int_{k(0)}^0 t^{p(1-\theta)} \, d(k^p)$$

$$= (1-\theta)^{-1} \int_0^{k(0)} t^{p(1-\theta)} k^{p-1} \, dk .$$

After another change of variables, where we let y be the independent variable, we get because of $k=y$ and $t=|y'|^{-1}=|dx/dy|$

$$(\Phi[tk])^p = (1-\theta)^{-1}\int_0^{y_0} y^{p-1}\,|dx/dy|^{p(1-\theta)}\,dy = (1-\theta)^{-1}(F_{0,p-1}[M])^p\,.$$

The second part of the assertion is proved in exactly analogous manner.

An immediate consequence of lemmas 3.1 and 3.2 is the following

COROLLARY 3.1. *We have* $F_{0,p-1}[M]\sim F_{p-1,0}[M]\sim\Phi[K]$.

LEMMA 3.3. *There is a constant* C *which only depends on* p, θ *and* α *such that for all* α *with* $0\leqq\alpha\leqq p-1$

$$F_{\alpha,\beta}[M] \leqq CF_{0,p-1}[M],\quad F_{\alpha,\beta}[M] \leqq CF_{p-1,0}[M],\quad F_{\alpha,\beta}[M] \leqq C\Phi[K]\,.$$

PROOF. According to corollary 3.1 it is enough to show one of the inequalities. From Hölder's inequality we get

$$\begin{aligned}(F_{\alpha,\beta}[M])^p &= \int_0^{x_0} x^\alpha y^\beta |y'|^\delta\,dx\\ &\leqq \left(\int_0^{x_0} y^{p-1}\,|y'|^{1-p(1-\theta)}\,dx\right)^{\beta/(p-1)}\left(\int_0^{x_0} x^{p-1}\,|y'|^{p\theta}\,dx\right)^{\alpha/(p-1)}\\ &= (F_{0,p-1}[M])^{p\beta/(p-1)}\,(F_{p-1,0}[M])^{p\alpha/(p-1)}\\ &\sim (F_{0,p-1}[M])^p\,.\end{aligned}$$

Lemma 3.3 gives us one of the inequalities of the equivalence between the norms $F_{\alpha,\beta}[M]$ and $\Phi[K]$. To prove the remaining inequality we need the following lemma.

LEMMA 3.4. *If* $0\leqq\alpha\leqq p-1$ *and* $\delta\geqq 0$, *then*

$$F_{\alpha,\beta}[M] \geqq \beta^{1/p}\,(\alpha+1)^{-1/p}\,F_{\alpha+1,\beta-1}[M]\,.$$

PROOF. We get

$$\begin{aligned}(\alpha+1)\int_0^{x_0} x^\alpha y^\beta\,|y'|^\delta\,dx-\beta\int_0^{x_0} x^{\alpha+1}y^{\beta-1}\,|y'|^{\delta+1}\,dx\\ &= \int_0^{x_0} [d/dx(x^{\alpha+1}y^\beta)]\,|y'|^\delta\,dx\\ &= [x^{\alpha+1}y^\beta\,|y'|^\delta]_0^{x_0} - \int_0^{x_0} x^{\alpha+1}y^\beta\,d/dx(|y'|^\delta)\,dx\,.\end{aligned}$$

Here is $[x^{\alpha+1}y^{\beta}|y'|^{\delta}]_0^{x_0}=0$ and the last integral $\leqq 0$ for $x,y\geqq 0$, and $|y'|^{\delta}$ is decreasing. Thus we get the lemma.

In order to prove the remaining inequality of the norm equivalence, let k be an integer such that

$$\alpha+k\leqq p-1\leqq\alpha+k+1.$$

By replacing α in lemma 3.4 by $\alpha,\alpha+1,\ldots,\alpha+k$ we get a sequence of inequalities

$$F_{\alpha+l,\beta-l}[M] \geqq (\beta-l)^{1/p}(\alpha+l+1)^{-1/p}F_{\alpha+l+1,\beta-l-1}[M], \qquad l=0,1\ldots,k,$$

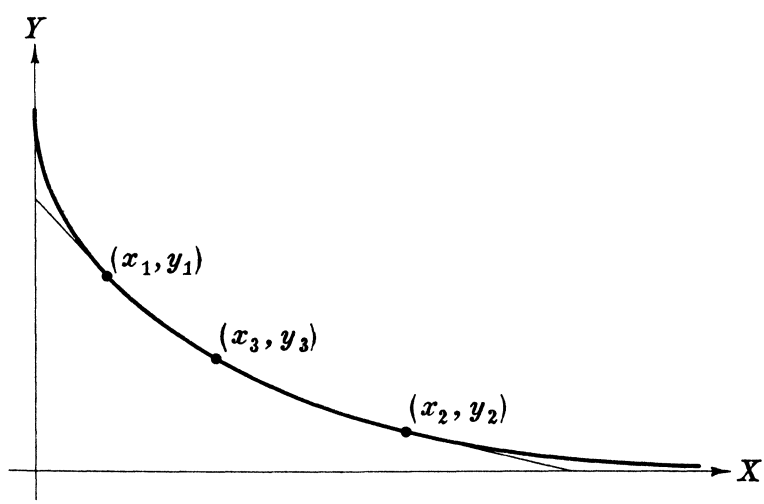
which together give

$$F_{\alpha,\beta}[M] \geqq CF_{\alpha+k+1,\beta-k-1}[M], \tag{3.2}$$

where C is a constant >0 that only depends on α, p and θ. Hölder's inequality and (3.2) give us now the last inequality of the norm equivalence

$$\begin{aligned}(F_{p-1,0}[M])^p &= \int_0^{x_0} x^{p-1}|y'|^p\,dx \\ &\leqq \left(\int_0^{x_0} x^{\alpha}y^{\beta}|y'|^{\delta}\,dx\right)^{1-\beta/(k+1)}\left(\int_0^{x_0} x^{\alpha+k+1}y^{\beta-k-1}|y'|^{\delta+k+1}dx\right)^{\beta/(k+1)} \\ &= (F_{\alpha,\beta}[M])^{p-p\beta/(k+1)}\,(F_{\alpha+k+1,\beta-k-1}[M])^{p\beta/(k+1)} \\ &\leqq C^{-p\beta/(k+1)}(F_{\alpha,\beta}[M])^p\,.\end{aligned}$$

Note that $0\leqq\beta/(k+1)\leqq 1$.



Finally, we investigate the remaining case, when the coordinate axes are tangent to or asymptotic to ∂M. Let (x_1, y_1) and (x_2, y_2) be two points of ∂M (neither of them being an endpoint of ∂M) with $x_1 < x_2$. Draw the tangent to ∂M (or a line with a slope between $y'(x_1 - 0)$ and $y'(x_1 + 0)$) at the point (x_1, y_1) and let that part of ∂M for which $x \leqq x_1$ be replaced by the tangent. At the point (x_2, y_2) we make the corresponding construction, i.e. we replace the part of ∂M for which $y \leqq y_2$ by the tangent at the point (x_2, y_2). See the figure on p. 51.

We call the new boundary ∂M^*. To ∂M^* there belongs a $K(t,a)$, which we call $K^*(t,a)$. On ∂M^* and $K^*(t,a)$ we can apply what we have proved above. Thus we know that there are constants C_1 and C_2 with $0 < C_1 \leqq C_2 < \infty$, depending only on α, p and θ, such that

$$C_1 \Phi[K^*] \leqq F[M^*] \leqq C_2[K^*] . \tag{3.3}$$

If we now let $x_1 \to 0$ and $y_2 \to 0$, ∂M^* will tend pointwise to ∂M, and from the definition (2.1) of K we see that $K^*(t,a)$ tends increasingly to $K(t,a)$. Then according to the Beppo Levi theorem $\Phi[K^*]$ tends to $\Phi[K]$. We also have $F[M^*] \to F[M]$; for if (x_3, y_3) is a point on ∂M between (x_1, y_1) and (x_2, y_2), then

$$(F[M])^p = \int_0^{x_3} x^\alpha y^\beta |dy/dx|^\delta \, dx + \int_0^{y_3} x^\alpha y^\beta |dx/dy|^\gamma \, dy ,$$

and when x_1 decreases towards 0, y^β and $|dy/dx|^\delta$ increase in the first integral, while the second is unchanged. When y_2 decreases towards 0, x and $|dx/dy|^\gamma$ increase in the second integral, while the first is unchanged. Passing to the limit in (3.3) we thus get

$$C_1 \Phi[K] \leqq F[M] \leqq C_2 \Phi[K] ,$$

and theorem 3.1 is proved.

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