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SYMMETRY IN REAL BANACH ALGEBRAS

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1. Introduction.

This paper deals with three, seemingly unrelated, problems of the theory of normed algebras over the real field: the characterization of algebras of complex type, the theory of real group algebras and the concept of a Šilov boundary. There is a unifying feature, however. It is well known that the classical representation theory of Gelfand for semi-simple, commutative complex Banach algebras has straight-forward generalization to real algebras [9], [6]. In that case, however, the possibility of complex conjugation induces a certain symmetry in the representation. It turns out that in the problems dealt with here, the solutions depend to a large extent on systematic consideration of this symmetry.

Section 2 deals with the characterization of algebras of complex type. Conditions, under which a complex multiplication will have to be continuous, are given. It is shown that even if an algebra has only complex homomorphisms it need not be of complex type; a necessary condition is that its Gelfand space consists of two disjoint symmetric parts (Theorem 2.6). Section 4, which deals with some problems of group algebras directly suggested by this result, contains an example (4.1) that shows that not even this stronger condition is sufficient. But we also give an affirmative result on algebras of complex type: an abstract characterization of the full complex-function algebra $C_0(\Omega)$ as a real Banach algebra (Theorem 2.7).

Section 3 deals mainly with real L^1-algebras over locally compact abelian groups. If they are regarded as real Banach algebras, we can identify the Gelfand space with the character group; the conjugation map then becomes the forming of inverse. Then the idea of reality conditions [6, Section 6] is applied to this situation. Using a result by Beurling and Helson [2] we can characterize a class of groups whose real group algebras are R_3. Finally we get a complete description of those groups that have strictly real (R_4) group algebras.

It is a somewhat distressing fact that the Šilov boundary fails to exist even for very well-behaved real algebras of functions, including most of

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those arising from the Gelfand representation of a commutative real Banach algebra. In sections 5 and 6 a more suitable concept of boundary is introduced, based on a certain "symmetrization" of the given set of functions. We can prove that this boundary exists for all real algebras of functions that separate points (Theorem 6.1), that it exists as soon as the Šilov boundary exists and can be only slightly bigger (Theorem 6.2), and that it agrees with the Šilov boundary for complex algebras.

For the reader's convenience we recall the reality conditions, introduced in [6, Ch. II]. A Banach algebra A is said to be of *complex type* if scalar multiplication can be extended from the real to the complex field (or rather from $R \times A$ to $C \times A$), making A a complex Banach algebra,

R_1, of *real type*, if A is not of complex type;
R_2 if A does not contain any subalgebra of complex type with identity;
R_3 if $\operatorname{ixp} \alpha x$ $(=1-\exp \alpha x)$ is an unbounded function on the real line for every $x \in A$, $x \neq 0$;
R_4 if $-x^2$ has quasi-inverse in A for every $x \in A$.

2. Algebras of complex type and the Gelfand space.

In this section, A will denote a real, commutative, semi-simple Banach algebra. The set of non-zero continuous homomorphisms of A into C (the complex numbers) is called the *Gelfand space* and is denoted Φ_A. On Φ_A is defined an involutoric homeomorphism τ, the *conjugate mapping*, through $\tau\varphi(x) = \overline{\varphi(x)}$. Further $\Phi_A{}^R$ denotes the "real" part of Φ_A, i.e. those $\varphi \in \Phi_A$ so that $\varphi(A) = R$ (the real numbers), and $\Phi_A{}^C$ the complement of $\Phi_A{}^R$ [6, Section 3]. There exists a continuous isomorphism, $x \to \hat{x}$, of A onto a subalgebra of $C_0(\Phi_A)$, the algebra of all complex continuous functions on the locally compact space Φ_A that tend to 0 at infinity [6, Section 4].

A real Banach algebra can be made into a complex Banach algebra (is of *complex type*) if and only if there exists a continuous linear operator J on A, satisfying $-J^2 = \text{identity}$ and $J(xy) = Jx \cdot y = x \cdot Jy$ for all $x, y \in A$ [6, Section 6].

It is easy to see that, for a normed algebra with identity, a complex multiplication must necessarily be continuous [5, p. 249]. We make two observations, generalizing this in slightly different directions. It should be noted that without any assumption on the "non-degeneration" of multiplication the result is no longer true. In 2.1 and 2.2 A is not required to be commutative.

PROPOSITION 2.1. *Let A be a real Banach algebra with an approximate identity. Then any complex multiplication on A is continuous.*

PROOF. Such a multiplication is given by a linear operator J, as above. We show that J is closed. Assume $x_n \to x$, $Jx_n \to y$ and let $\{e_\lambda\}_{\lambda\in\Lambda}$ be the approximate identity. Then $Jx_n \cdot e_\lambda = x_n \cdot Je_\lambda$ as $n \to \infty$ gives $ye_\lambda = x \cdot Je_\lambda = Jx \cdot e_\lambda$.

Since $\{e_\lambda\}$ is approximate identity we have $Jx = y$, J is closed and the closed-graph theorem shows that J is continuous.

PROPOSITION 2.2. *Let A be a normed algebra with the property that the set of (left or right) topological divisors of 0 is not all of A. Then any complex multiplication on A is continuous.*

PROOF. Let a be an element which is not a left topological divisor of 0. For a sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = 0$ we have

$$\lim_{n\to\infty} Jx_n \cdot a = \lim_{n\to\infty} x_n \cdot Ja = 0$$

and $\lim Jx_n = 0$; otherwise a would be a left topological zero devisor.

For a Banach algebra of complex type it is clear that $\Phi_A = \Phi_A{}^C$ or, equivalently, τ has no fixed points. One can naturally ask whether the converse is true and the following example shows it is not; an algebra may fail to be of complex type even if all of its canonical homomorphisms are complex.

EXAMPLE 2.3. Let Ω be a connected compact Hausdorff space and σ a fixed-point free involution on Ω (a particular example is then the n-sphere S^n, with σ = reflexion in the origin). Let A be the algebra of all complex-valued functions that are continuous on Ω and satisfy $f(\sigma w) = \overline{f(w)}$ for all $w \in \Omega$. Then clearly $\Phi_A = \Phi_A{}^C$. But A does not contain a square root of minus the identity; such a function would have to be constant, $+i$ or $-i$, since Ω is connected, but this conflicts with the conjugation requirement. (The space of maximal ideals in A is Ω, identified by σ; in particular for $\Omega = S^n$ and σ = reflection in 0 it is projective space P^n [6, p. 246].)

We now proceed to show that, for an algebra of complex type, the Gelfand space consists of two disjoint homeomorphic subsets. An auxiliary result, although completely elementary, is stated separately.

PROPOSITION 2.4. *Let V be a topological vector space over the real numbers and V' a closed subspace of codimension $\geqq 2$. Then the complement of V' is connected.*

Since a vector space over the complex numbers can be regarded as a real space of twice the dimension we have

COROLLARY 2.5. *Let V be a topological vector space over the complex numbers and V' a proper closed subspace. Then the complement of V' is connected.*

THEOREM 2.6. *Let A be a real Banach algebra of complex type. Then $\Phi_A = \Phi_A' \cup \Phi_A''$ where Φ_A' and Φ_A'' are disjoint and homeomorphic by τ.*

PROOF. We can take Φ_A' to be the canonical homomorphisms of A as a complex algebra

$$\Phi_A' = \{\varphi\,;\ \varphi \in \Phi_A,\ \varphi(ix) = i\varphi(x) \text{ for all } x \in A\}$$

and

$$\Phi_A'' = \tau\Phi_A' = \{\varphi\,;\ \varphi \in \Phi_A,\ \varphi(ix) = -i\varphi(x) \text{ for all } x \in A\}\,.$$

Hence Φ_A' and Φ_A'' are disjoint. Now take an arbitrary $\varphi \in \Phi_A$. For every $x \in A$

$$\varphi(ix)^2 = \varphi(-x^2) = -\varphi(x)^2\,,$$

hence $\varphi(ix) = \pm i\varphi(x)$. But $\varphi \neq 0$ on the complement of a closed subspace (ideal) of codimension 2, hence on a connected open set (Proposition 2.4). The function

$$x \to \varphi(ix)/\varphi(x)$$

is continuous on this set and then must be a constant. Thus $\varphi \in \Phi_A' \cup \Phi_A''$ and the proof is complete.

Granted the result of Theorem 2.6, one could further ask whether the converse is true. More precisely, if $\Phi_A = \Phi_A' \cup \Phi_A''$, two disjoint parts with $\tau\Phi_A' = \Phi_A''$, can then a complex multiplication be introduced via $\hat{J}$, so that

$$\hat{J}\hat{x}(\varphi) = i\hat{x}(\varphi), \quad \varphi \in \Phi_A', \qquad \text{and} \qquad \hat{J}\hat{x}(\varphi) = -i\hat{x}(\varphi), \quad \varphi \in \Phi_A''\,?$$

The examples given in section 4 show that this is not possible in general and the question whether $\hat{J}\hat{x}$ belongs to $\hat{A}$ for all x can be a very delicate question in analysis.

We conclude this section with an affirmative result where the intrinsic symmetry of Φ_A, together with an involution, yields an abstract characterization of algebras consisting of all continuous *complex* functions on some locally compact space that tend to 0 at infinity, regarded as *real* Banach algebras. Hence it falls in line with [3], [9, Theorem 4.2.]: algebras of all complex functions, regarded as complex Banach algebras, and [1], [6, Section 15]: algebras of all real functions, regarded as real Banach algebras.

THEOREM 2.7. *Let A be a real commutative Banach algebra with an involution so that*

(1) $\|x\|^2 \leqq \alpha\|x^*x+y^*y\|$, *$\alpha$ constant.*

(2) *A contains an antihermitian element k which does not belong to any maximal modular ideal.*

*Then A is homeomorphically *-isomorphic to $C_0(\Omega)$, Ω a locally compact space.*

PROOF. It is a known result by Arens and Kaplansky [1] that an algebra satisfying (1) is homeomorphically *-isomorphic (under the Gelfand map $x \to \hat{x}$) to $C_0(\Phi_A, \tau)$ the algebra of *all* complex-valued continuous functions on Φ_A that tend to 0 at infinity and satisfy $f(\tau\varphi)=\overline{f(\varphi)}$; involution is here pointwise complex conjugation. Since $k= -k^*$ the function $\hat{k}$ takes only imaginary values but it is always $\neq 0$. Since $k(\tau\varphi)= -k(\varphi)$ the subsets

$$\Phi_A' = \{\varphi \, ; \, \varphi \in \Phi_A, \, i\hat{k}(\varphi) > 0\}, \qquad \Phi_A'' = \{\varphi \, ; \, \varphi \in \Phi_A, \, i\hat{k}(\varphi) < 0\}$$

form a disjoint partition of Φ_A in two parts, homeomorphic by τ. For arbitrary $f \in C_0(\Phi_A, \tau)$ we define Jf by

$$Jf(\varphi) = \begin{cases} if(\varphi), & \varphi \in \Phi_A' \, , \\ -if(\varphi), & \varphi \in \Phi_A'' \, . \end{cases}$$

Then $Jf \in C_0(\Phi_A, \tau)$ and J is an isometry in the sup norm. Hence J defines a complex multiplication and since $\overline{Jf(\varphi)}= -J\overline{f(\varphi)}$ involution is conjugate linear. The desired function algebra, F, consists of the functions from $C_0(\Phi_A, \tau)$ restricted to Φ_A'. The map $C_0(\Phi_A, \tau) \to F$ is an isometric isomorphism and since F contains all real-valued functions and admits multiplication by i, the conclusion follows.

REMARK. If A has identity, the argument can be simplified somewhat. Then (2) implies that k has an inverse h, and since $|\hat{h}| \in C_0(\Phi_A, \tau)$ the element $\hat{k}\,|\hat{h}|$ is an imaginary unit for $C_0(\Phi_A, \tau)$.

3. Real group algebras.

Harmonic analysis on groups is concerned with some locally compact (mostly abelian) group G, its Haar measure and the complex algebras $L^1(G)$ and $M(G)$, consisting of all complex-valued integrable functions and all bounded complex measures, respectively, under convolution multiplication. In this section we will particularly study $L_R{}^1(G)$ and $M_R(G)$, the set of real integrable functions and real bounded measures, respectively. They will be regarded as real Banach algebras and will particularly be considered from the point of view of the Gelfand representation and the reality conditions [6, Ch. II].

The group G is always assumed to be locally compact and abelian, and its dual group is denoted $\hat{G}$. The process of complexification is as defined in [9, p. 6], [6, p. 243] and the Gelfand space and conjugation map τ as in [6, p. 245]. We first collect some elementary facts that can be obtained form known results for $L^1(G)$.

LEMMA 3.1.
(a) *The complexification of* $L_R^1(G)$ *is homeomorphically isomorphic to* $L^1(G)$.
(b) $L_R^1(G)$ *is semi-simple.*
(c) *The Gelfand space of* $L_R^1(G)$ *can be identified with the dual group* $\hat{G}$.
(d) *The Gelfand representation* $x \to \hat{x}$ *is given by* $\hat{x}(\chi) = \int \chi(g)x(g)dg$, $\chi \in \hat{G}$ (*dg is normalized Haar measure on* G).
(e) *The conjugation map* τ *on* $\hat{G}$ *is given by* $\tau\chi = -\chi$.

PROOF. (a) is verified by direct computation. Since $L_R^1(G)$ is a (real) subalgebra of $L^1(G)$, (b) follows from the fact that $L^1(G)$ is commutative and semi-simple. Even (c) follows from (a), noting the facts that the Gelfand space of a real Banach algebra is in one-to-one correspondence with the maximal ideal space of the complexification [9, Theorem 3.1.4] and that $L^1(G)$ has $\hat{G}$ as its maximal ideal space. In the same way, (d) follows from the theory of $L^1(G)$. For (e) finally, we note that

$$\hat{x}(\tau\chi) = \overline{\hat{x}(\chi)} = \int \overline{\chi(g)}\,x(g)\,dg = \int (-\chi)(g)\,x(g)\,dg = \hat{x}(-\chi)$$

from which follows $\tau\chi = -\chi$.

REMARK 3.2. From (a) in Lemma 3.1 it follows that if, for two groups G_1, G_2, $L_R^1(G_1)$ and $L_R^1(G_2)$ are isomorphic (as algebras), this also holds for $L^1(G_1)$ and $L^1(G_2)$. In the other direction this is not true. If G is a finite group with n elements, we have $L^1(G) \cong C^n$ but $L_R^1(G) \cong R^k \oplus C^l$ where $k-1 =$ number of elements of order 2 and $2l = n-k$. Since there is already for $n=4$ one group with $k=4$ (the Klein group $K_4 = Z_2 \oplus Z_2$) and one with $k=2$ (the cyclic group Z_4) we see that $L_R^1(G)$ *is more efficient than* $L^1(G)$ *in distinguishing groups.*

We next turn to the reality conditions R_1 through R_4[6, Section 6] and their relation to properties of G. We immediately notice that $0 \in \hat{G}$ maps onto the reals, hence there is always one "real" element in the Gelfand space and we have

PROPOSITION 3.3. $L_R^1(G)$ *is* R_1 *for every* G.

Assume that A is a semi-simple commutative real Banach algebra such that

(a) Φ_A is connected,

(b) A is R_1 *or* A does not have identity.

It is then easy to see that A is R_2. For real group algebras, with the same assumptions, we arrive at the stronger conclusion R_3, which follows (Corollaries 3.5, 3.6) from the next theorem.

For the result on R_3, we make use of the following theorem by Beurling and Helson [2]: *Let G be such that $\hat{G}$ is connected. If $\mu \in M(G)$ has the property that $\|\mu^n\| \leqq k, n$ an arbitrary integer, then μ is a point mass and $\|\mu\| = 1$.*

We let $M_r(G)$ denote the real subalgebra of $M(G)$ that consists of all μ, so that $\int d\mu$ is real.

THEOREM 3.4. *Let G be such that $\hat{G}$ is connected. Then $M_r(G)$ is R_3.*

PROOF. Let $\mu \in M_r(G)$ and assume that $\exp \alpha\mu$ is a bounded function of α. From the Beurling–Helson theorem it follows that $\exp \alpha\mu$ is a point mass for each α, its mass equal to ± 1. Since the distance between any two such elements is equal to 2 and $\exp \alpha x$ is a continuous function of α it follows that it is a constant. Hence $\exp \alpha x = e$ for all α and $x = 0$.

Since the R_3 property is inherited by subalgebras, we immediately get

COROLLARY 3.5. *If G is such that $\hat{G}$ is connected,* then *$M_R(G)$ and $L_R^1(G)$ are R_3.*

In view of the fact [5, Corollary 24,35] that $\hat{G}$ is connected if and only if $G = R^n \oplus F$ with F discrete and torsion-free, n an integer $\geqq 0$, the results 3.4 and 3.5 are really statements about R and discrete, torsion-free groups.

It is clear from 3.4 that $M(G)$, for $\hat{G}$ connected, is "almost R_3" in the sense that $\exp \alpha\mu$ bounded implies that μ is a scalar multiple of i times the identity. From this follows directly

COROLLARY 3.6. *If G is non-discrete and $\hat{G}$ connected, then $L^1(G)$ is R_3.*

We next turn to the condition R_4 (strict reality, the spectrum of every element is real) and obtain a complete characterization of R_4 group algebras.

THEOREM 3.7. *For the algebra $L_R^1(G)$ to be strictly real it is necessary and sufficient that every element $\neq 0$ in G is of order* 2.

PROOF. The algebra $L_R^1(G)$ is strictly real if and only if the conjugation map is the identity map on the Gelfand space. According to Lemma

3.1 (e) this is equivalent to $\chi+\chi=0$ for all χ in $\hat{G}$. But since $\chi(2x)=(2\chi)(x)$ this is again equivalent to every element $\neq 0$ of G being of order 2.

The group with two elements is denoted Z_2 and, for any cardinal $\varkappa$, $Z_2^\varkappa$ stands for the direct product of $\varkappa$ copies of Z_2 with the product topology.

THEOREM 3.8. *If G is a compact group, the algebra $L_R^1(G)$ is strictly real if and only if G is homeomorphically isomorphic to $Z_2^\varkappa$ for some cardinal $\varkappa$.*

PROOF. This follows from Theorem 3.7 and [5, Theorem 25.9].

REMARK 3.9. It is not necessary to assume G abelian to have the results 3.7 and 3.8. For arbitrary G we have that $L_R^1(G)$ is semi-simple, hence if it is R_4 it is also commutative [7, p. 405]. Then $\hat{G}$ is abelian and G abelian.

Finally, it can be pointed out that the characterization of $L_R^1(G)$ as R_3 or R_4 opens up the prospect of studying the group of quasi-invertible elements of these algebras, in particular its relation to the set of idempotents [6, Sections 9, 10].

4. Complex type and group algebras.

In this section we apply the ideas of section 2 to the particular structure of a group algebra. It is clear from 3.1 (e) that if a group G is such that $\hat{G}$ has no elements of order 2, and removal of the point 0 from the Gelfand space $\hat{G}$ makes $\hat{G}$ consist of disjoint symmetric parts, the remaining algebra is at least a candidate for complex type, by Theorem 2.6. The results obtained indicate that in most cases rather subtle considerations still remain before the question of complex type can be resolved.

The following example shows that the converse of Theorem 2.6 is not true.

EXAMPLE 4.1. We let A be the subalgebra of $L_R^1(R)$ (R the group of real numbers in the natural topology) that consists of all functions with zero mean value,

$$\int_{-\infty}^{\infty} x(t)\,dt = 0 .$$

From Lemma 3.1 it follows that Φ_A can be identified with R except 0 and that the Gelfand map $x \to \hat{x}$ is given by

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} e^{2\pi i\omega t}\, x(t)\, dt .$$

Since 0 is excluded it follows from Lemma 3.1 (e) that the positive and negative half-axis form a partition of Φ_A as in Theorem 2.6 and also the only one possible. To a complex multiplication on A would then correspond a mapping $\hat{J}$ on $\hat{A}$, satisfying

$$\hat{J}\hat{x}(\omega) = i \operatorname{sign}(\omega)\hat{x}(\omega) .$$

We show by a concrete example that for $x \in A$, $\hat{J}\hat{x}$ need not belong to $\hat{A}$, and hence A is not of complex type. Let $\alpha = \alpha_1 - \alpha_2$ where

$$\alpha_1(\omega) = [\log(e+|\omega|)]^{-1} \qquad \text{and} \qquad \alpha_2(\omega) = \exp(-\pi\omega^2) .$$

Since α_1 is monotonous and convex for $\omega \geqq 0$, α_1 is the Fourier transform of a summable function [10, Theorems 124 and 10], hence $\alpha \in \hat{A}$. Then $\hat{J}\alpha$ is purely imaginary and, if it belongs to $\hat{A}$, the transform of an (a.e.) odd, summable function. It is easy to see, however, that for any $g \in L^1(R)$

$$\left| \int_1^N \omega^{-1} \left(\int_{-\infty}^{\infty} g(t) \sin 2\pi\omega t \, dt \right) dw \right| < 2\pi^2 \|g\|$$

for all $N \geqq 1$. Since

$$\int_1^N [\omega \log(e+\omega)]^{-1} \, d\omega$$

increases without bound with N, while

$$\int_1^N \omega^{-1} \exp(-\pi\omega^2) \, d\omega$$

stays bounded, it follows that α is not the transform of a summable function.

We now turn to the circle group T and let $A_0{}^1(T)$ denote the subalgebra of $L_R{}^1(T)$ which consists of all functions with mean value zero. The Gelfand space of $A_0{}^1(T)$ can be identified with the integers, Z, except 0, and, in contrast to Example 4.1, we have many different possibilities for the partition of Theorem 2.6. In spite of this we can prove the following result.

THEOREM 4.2. *$A_0{}^1(T)$ is not of complex type.*

PROOF. Assume that $A_0{}^1(T)$ is of complex type. Its Gelfand space $Z - \{0\}$ is denoted Z'. From Theorem 2.3 it follows that there exists a set $\Phi' \subset Z'$ such that $\Phi' \cup (-\Phi') = Z'$ and Φ', $-\Phi'$ are disjoint. The complex multiplication gives rise to a mapping $\hat{J}$ on $\hat{A}$. If we define a function λ on Z' by

$$\lambda(n) = i, \quad n \in \Phi', \qquad \lambda(n) = -i, \quad n \in -\Phi',$$

$\hat{J}$ can be described by

$$\hat{J}\hat{x} = \lambda\hat{x} .$$

The question whether $\lambda\hat{x} \in \hat{A}_0{}^1$ for all $x \in A_0{}^1$ is of course equivalent to the same question for L^1 and it is known [12, p. 176] that this is the case if and only if λ is the Fourier–Stieltjes transform of a measure on T. Let $\beta = \frac{1}{2}(1 + i\lambda)$. Then β is a function on Z with the values 0 and 1, moreover $\beta(n) = 1 - \beta(-n)$ for $n \neq 0$, we can take $\beta(0) = 0$. But if such a sequence were the transform of a measure, it would be equal to a periodic sequence except at finitely many places [4]. For n a sufficiently large multiple of the period we would then have

$$\beta(n) = \beta(-n) ,$$

which implies $2\beta(n) = 1$, a contradiction. Hence $A_0{}^1(T)$ is not of complex type.

In contrast to the negative results 4.1 and 4.2, we can turn to the L^2 theory of compact groups and get affirmative results.

For a compact abelian group G we know that $L^2(G) \subset L^1(G)$ but also that the maximal modular ideals are obtained by restriction [8, p. 161]. Hence we have all the facts of Lemma 3.1 for the real algebra $L_R{}^2(G)$. Further, let $A_0{}^2(G)$ denote the subalgebra of $L_R{}^2(G)$ that consists of all functions with mean value zero, $\int x(g)\,dg = 0$.

Theorem 4.3. *Let G be an infinite compact group such that $\hat{G}$ has no element of order 2. Then $A_0{}^2(G)$ is of complex type and there are 2^γ different ways of introducing complex scalars, $\gamma = \operatorname{card}\hat{G}$.*

Proof. It is a standard result for L^2-algebras that $L^2(G)$ is homeomorphically isomorphic to $l^2(\hat{G})$, the set of all complex functions on the (discrete) space $\hat{G}$ such that

$$\|\alpha\|^2 = \sum_{\chi \in \hat{G}} |\alpha(\chi)|^2 < \infty .$$

The Gelfand space of $A_0{}^2(G)$ is $\hat{G}_0 = \hat{G} - \{0\}$, and since it is also a (real) subalgebra of $L^2(G)$, it follows that $\hat{A}_0{}^2 \subset l^2(\hat{G}_0)$. For $\hat{x} \in \hat{A}_0$, moreover

$$\overline{\hat{x}(\chi)} = \hat{x}(-\chi) \qquad \text{for all} \quad \chi \in \hat{G}_0$$

holds. The assumption that no $\chi \in \hat{G}$ is of order 2 guarantees that for each χ there is an $x \in A_0{}^2$ so that $\chi(x)$ is non-real. Hence, since $\hat{A}_0{}^2$ is a closed real subalgebra of $l^2(\hat{G})$ it must consist of all those complex functions in $l^2(\hat{G}_0)$ that satisfy the conjugation condition.

To each unordered pair $(\chi, -\chi)$, $\chi \in \hat{G}_0$, assign arbitrarily a first member, and let Φ' be the set of all first members. The mapping $\alpha \to \alpha_c$ where

$$\alpha_c(\chi) = \begin{cases} i\alpha(\chi), & \chi \in \Phi' , \\ -i\alpha(\chi), & \chi \in -\Phi' , \end{cases}$$

is an isometry and preserves conjugation, hence $\alpha \to \alpha_c$ defines a complex multiplication on $A_0{}^2(G)$. It is also clear that the choice of first members in the pairs can be done in 2^γ different ways, all giving rise to different maps $\alpha \to \alpha_c$. (For G finite it is easy to check that there are $2^{\frac{1}{2}(\gamma-1)}$ different possibilities.)

By applying the result just given to $G=T$ and comparing with Theorem 4.2, we see that the question of introducing complex scalars can be a delicate problem in analysis. There is also reason to regard this question in general as an abstract counterpart of problems on conjugate Fourier series and related topics.

We finally turn to the integer group, Z, where the situation is somewhat different. The Gelfand representation of $L_R{}^1(Z)$ leads to a subalgebra $W_R(T)$ of the Wiener algebra $W(T)$. $W_R(T)$ consists of all functions on T that have real Fourier coefficients forming absolutely convergent series. Here two points have to be removed from the Gelfand space to make it eligible under 2.6. Let $W_S(T)$ be the subalgebra of $W_R(T)$ consisting of those functions on $[-\pi,\pi]$ that satisfy $f(0)=f(\pm\pi)=0$. The elements of $W_S(T)$ are of the form f_1+if_2 where f_1 is even and f_2 odd. If this algebra is of complex type Jf_1 is a function equal to if_1 on $[0,\pi]$ and equal to $-if_1$ on $[-\pi,0]$. But it is a consequence of general results by Wik [11, p. 96], that Jf_1 does not in general belong to $W_S(T)$.

5. A modified Šilov boundary, symmetric case.

Throughout this section Ω will denote a locally compact Hausdorff space and $C_0(\Omega)$ the class of functions on Ω that are complex-valued, continuous and tend to 0 at infinity.

Let D be a subset of $C_0(\Omega)$. A set $\Sigma \subset \Omega$ is called *maximizing* (for D) if it is closed and

$$\max_{\omega\in\Sigma}|f(\omega)| = \max_{\omega\in\Omega}|f(\omega)|$$

for all $f \in D$. The intersection of all maximizing sets we call the Šilov *sub-boundary* and denote it $\underline{\partial^S}\Omega$. If $\underline{\partial^S}\Omega$ is maximizing, the Šilov boundary is said to exist and it is denoted $\partial^S\Omega$. It is a known result that if D is the image under the Gelfand representation of a complex Banach algebra or, more generally, is an arbitrary complex algebra that separates points, $\partial^S\Omega$ exists [9, p. 133].

The corresponding is not true, however, for the more general case of a *real* algebra, as a simple example shows [9, p. 311]. For every algebra of the class defined in our Example 2.3 the sub-boundary $\underline{\partial^S\Omega}$ is in fact empty.

In generalizing the notion of Šilov boundary we will use a special class of function sets D, whose chief representative is the image under the Gelfand representation of a real Banach algebra [6, Section 4]. For given Ω let τ be an involutoric ($\tau\circ\tau=$identity) homeomorphism on Ω. By a τ-conjugate function f we mean one that satisfies $f(\tau\omega)=\overline{f(\omega)}$ for all $\omega\in\Omega$. For a set $D\subset C_0(\Omega)$ consisting only of τ-conjugate functions we define $\partial\Omega$ as the intersection of all *τ-invariant* maximizing sets. If Ω is maximizing we say that the boundary exists and denote it $\partial\Omega$.

For any subset $D\subset C_0(\Omega)$ the weakest topology on Ω making all $f\in D$ continuous is called the *D-topology*. D is said to *separate points* if for given ω, $\sigma\in\Omega$, $\omega\neq\sigma$, there exist $f, f_0\in D$, so that $f(\omega)\neq f(\sigma)$, and $f_0(\omega)\neq 0$. If D separates points, the D-topology is Hausdorff and it is easy to prove the following

PROPOSITION 5.1. *If $D\subset C_0(\Omega)$ separates points, the D-topology coincides with the given topology on Ω.*

The result showing the significance of $\partial\Omega$ will now be given.

THEOREM 5.2. *Let D be a τ-conjugate real subalgebra of $C_0(\Omega)$ that separates points. Then $\partial\Omega$ exists.*

PROOF. By Proposition 5.1 we can use the D-topology for Ω in the sequel.

A standard argument, using the local compactness together with Zorn's lemma, proves the existence of minimal τ-invariant maximizing sets. Now let Γ be such a minimal set and Σ an arbitrary τ-invariant closed set that does not contain Γ. We will show the existence of a $g\in D$, so that g does not assume its maximum on Σ, which proves that Γ is unique minimal, hence $\Gamma=\partial\Omega$.

We remark, for use in the proof, that for $h>k>0$ and z a complex number

(3) $|z^2+h^2|<hk$ implies $|z-ih|<k$ or $|z+ih|<k$,

(4) $|z(\pm)ih|<k$ implies $|z(\mp)ih|>2h-k$.

There exists a point $\omega_0\in\Gamma-\Sigma$ and, since Σ is closed, a neighborhood of ω_0 that does not intersect Σ and is of the form $V = \bigcap_{i=1}^n V_i$ where

$$V_i = \{\omega \; ; \; |f_i(\omega) - f_i(\omega_0)| < \varepsilon\}$$

with $f_i \in D$. Since V does not intersect Σ, this is also true of $V \cup \tau V$.

We first construct a τ-invariant neighborhood of a special form, contained in $V \cup \tau V$. Assume

$$\mathrm{Im} f_i(\omega_0) = b_i \neq 0 \quad \text{for} \quad 1 \leqq i \leqq k$$

and

$$f_i(\omega_0) \text{ real} \quad \text{for} \quad k+1 \leqq i \leqq n .$$

Put $b = \min_{1 \leqq i \leqq k} |b_i|$. We can assume $\varepsilon < b/2$, which in particular means that V_i and τV_i are disjoint for $1 \leqq i \leqq k$. The set

$$V_{ij} = \{\omega \; ; \; |[f_i(\omega) - f_i(\omega_0) + f_j(\omega) - f_j(\omega_0)][f_i(\omega) - \overline{f_i(\omega_0)} + f_j(\omega) - \overline{f_j(\omega_0)}]| < \varepsilon b\}$$

is τ-invariant and a neighborhood of both ω_0 and $\tau\omega_0$. We now define the τ-invariant neighborhood

$$V_0 = \Big(\bigcap_{i,j=1}^{k} V_{ij} \Big) \cap \Big(\bigcap_{i=k+1}^{n} V_i \Big)$$

and proceed to show that

$$V_0 \subset V \cup \tau V .$$

If $\omega \in V_0$, then in particular $\omega \in V_{ii}$, $1 \leqq i \leqq k$. From (3) it then follows that

$$|f_i(\omega) - f_i(\omega_0)| < \frac{\varepsilon b}{4|b_i|} < \frac{\varepsilon}{4} \qquad \text{or} \qquad |f_i(\omega) - \overline{f_i(\omega_0)}| < \frac{\varepsilon}{4} ,$$

that is, $\omega \in V_i \cup \tau V_i$. Assume that $\omega \in V_i$ for a certain i and let $j \neq i$, $1 \leqq j \leqq k$. We know that $\omega \in V_j \cup \tau V_j$. If $\omega \in \tau V_j$ we would have, by repeated use of (4),

$$\begin{aligned} |[f_i(\omega) - f_i(\omega_0) + f_j(\omega) - f_j(\omega_0)][f_i(\omega) - \overline{f_i(\omega_0)} + f_j(\omega) - f_j\overline{(\omega_0)}]| & \\ \geqq (-\varepsilon + 2|b_j| - \varepsilon)(2|b_i| - \varepsilon - \varepsilon) & \\ \geqq 4(b-\varepsilon)^2 > b^2 > 2\,\varepsilon\, b , & \end{aligned}$$

which conflicts with the requirement that $\omega \in V_{ij}$. Hence $\omega \in V_i$ implies $\omega \in V$. In the same way $\omega \in \tau V_i$ implies $\omega \in \tau V$, and we know that

$$V_0 = \tau V_0 \subset V \cup \tau V ,$$

in particular that V_0 does not intersect Σ. The set V_0 is of the form

$$V_0 = \{\omega \; ; \; |t_\nu(\omega) - T_\nu| < \varepsilon_\nu, \; \nu = 1, \ldots, N\} ,$$

where $t_\nu \in D$ and T_ν are *real* constants.

Since Γ is minimal and $\Lambda = \Gamma - V_0$ a proper closed subset of Γ, there exists an element $s \in D$, so that

$$\max_\Lambda |s(\omega)| < \max_{\Gamma, \Omega} |s(\omega)| = M .$$

The function g is now defined by $g = (M^{-1}S)^n$ where n shall be chosen so that

$$\max_\Lambda |g(\omega)| < \delta = (\min_\nu \varepsilon_\nu)(2 \max_{\omega, \nu} |f_\nu(\omega)| + 1)^{-1} .$$

Now consider the function $s_\nu = (t_\nu - T_\nu)g \in D$. For an $\omega \in \Lambda$ we have

$$|s_\nu(\omega)| = |(t_\nu(\omega) - T_\nu) g(\omega)| \leqq 2 \max |t_\nu(\omega)| \delta < \varepsilon_\nu .$$

For $\omega \in V_0$ the same inequality holds, thanks to the definition of V_0. Hence we have $|s_\nu(\omega)| < \varepsilon_\nu$ on Γ, and the same holds throughout Ω. Let ω' be a point with $|g(\omega')| = 1$. Then

$$|t_\nu(\omega') - T_\nu| = |S_\nu(\omega')| < \varepsilon_\nu$$

and $\omega' \in V_0$. Hence every maximum point of g lies in V_0, which is outside of Σ. This is a contradiction and the proof is finished.

6. The boundary in the general case.

We now proceed to define a boundary $\partial\Omega$ which will exist more often than $\partial^S\Omega$, in particular for all real subalgebras of $C_0(\Omega)$ that separate points.

Let D be an arbitrary set of functions from $C_0(\Omega)$ that separates points. We will construct an isomorphic set of τ-conjugate functions. ("Isomorphism" here means a one-to-one mapping which preserves the algebra operations defined in D). Let Ω_{12} be the disjoint union of Ω with itself, and denote the canonical embeddings $\omega \to \omega_1$ and $\omega \to \omega_2$. We map D onto a subset of $C_0(\Omega_{12})$ by $f \to f_{12}$, where

$$f_{12}(\omega_1) = f(\omega), \qquad f_{12}(\omega_2) = \overline{f(\omega)} .$$

This subset, D_{12}, consists of conjugate functions, relative to the involution $\omega_1 \to \omega_2$, $\omega_2 \to \omega_1$ on Ω_{12}, but does not in general separate points. Hence let $\tilde{\Omega}$ be the sets of constancy for D_{12}, $\omega_{1(2)} \to \tilde{\omega}_{1(2)}$ the natural identification map and $f_{12} \to \tilde{f}$ the restriction of f_{12} to $\tilde{\Omega}$. The elements of $\tilde{\Omega}$ are single points ω_1 or ω_2 and pairs (ω_1, σ_2) where $f(\omega) = \overline{f(\sigma)}$ for all $f \in D$. Clearly $\tilde{\Omega}$ is locally compact and has an involution, induced by the involution on Ω_{12}. Now $\tilde{D}$ is a set of conjugate functions that separates points and if $\tilde{D}$ has a boundary $\partial\tilde{\Omega}$ we define

$$\partial\Omega = \{\omega \; ; \; \tilde{\omega}_1 \in \partial\tilde{\Omega}\} = \{\omega \; ; \; \tilde{\omega}_2 \in \partial\Omega\} .$$

If D is already a τ-conjugate set every element of $\tilde{\Omega}$ is a pair of the form $(\omega, \tau\omega)$ and it is clear that the boundary just defined agrees with the one given originally for τ-conjugates sets.

The main theorem now follows from this definition and Theorem 5.2:

THEOREM 6.1. *For any real subalgebra of $C_0(\Omega)$ that separates points $\partial\Omega$ exists.*

It remains to establish the relation between $\partial\Omega$ and $\partial^S\Omega$ and to substantiate the claim that the former exists more often.

THEOREM 6.2. *Assume that, for a set $D \subset C_0(\Omega)$ that separates points, the Šilov boundary $\partial^S\Omega$ exists. Then $\partial\Omega$ exists and*

$$\partial\Omega = \partial^S\Omega \cup \Sigma_0 ,$$

where Σ_0 is homeomorphic to a subset of $\partial^S\Omega$ that has no interior, regarded as a subset of $\partial^S\Omega$.

PROOF. We first define the set

$$\Sigma = \{\omega \; ; \; \omega \in \partial^S\Omega, \exists\, \omega_0 \neq \omega \text{ so that } f(\omega_0) = \overline{f(\omega)} \text{ for all } f \in D\}$$

and show that Σ has empty interior in $\partial^S\Omega$. Assume that V is a non-void set contained in Σ and relatively open in $\partial^S\Omega$. For $\omega \in \Sigma$ the point ω_0 is uniquely determined since D separates points. Hence we have a map $\omega \to \omega_0$ which, in the D-topology restricted to $\Sigma \cup \Sigma_0$, is a homeomorphism. Hence V can be chosen to that $V \cap V_0$ is empty, and also $V \cap \overline{V}_0$ empty. The set $(\partial^S\Omega - V) \cup \overline{V}_0$ is closed and, since the values taken on V appear conjugated on V_0, also maximizing. But then $\underline{\partial^S\Omega} \subset \partial^S\Omega - V$ which is a contradiction; Σ contains no inner points.

The smallest involution invariant subset of $\tilde{\Omega}$ containing the image of $\partial^S\Omega$ under the map $\omega \to \tilde{\omega}_1$ is

$$\Delta_1 = \{\alpha \; ; \; \alpha = \{\omega_1\}, \omega \in \partial^S\Omega\} \cup \{\alpha \; ; \; \alpha = \{\omega_2\}, \omega \in \partial^S\Omega\} \cup$$
$$\cup \{\alpha \; ; \; \alpha = (\omega_1, \sigma_2), \omega \in \partial^S\Omega \textit{ or } \sigma \in \partial^S\Omega\} .$$

Let Δ be some involution-invariant maximizing set in $\tilde{\Omega}$. Then

$$\underline{\Delta} = \{\omega \; ; \; \tilde{\omega}_1 \in \Delta\}$$

is a maximizing set for D, hence $\underline{\Delta} \supset \partial^S\Omega$. This implies $\Delta \supset \Delta_1$ and since Δ_1 is maximizing we know that $\partial\tilde{\Omega}$ exists and is equal to Δ_1. Since we now have

$$\partial\Omega = \{\omega \; ; \; \tilde{\omega}_1 \in \Delta_1\}$$

it is clear that $\partial^S\Omega \subset \partial\Omega$ and

$$\partial\Omega - \partial^S\Omega = \{\omega\ ;\ \omega \in \partial^S\Omega, \exists\, \sigma \in \partial^S\Omega, \sigma \neq \omega, f(\sigma) = \overline{f(\omega)} \text{ for all } f \in D\},$$

which is exactly Σ_0. Hence

$$\partial\Omega = \partial^S\Omega \cup \Sigma_0 ,$$

and the proof is finished.

COROLLARY 6.3. *For a complex subalgebra of $C_0(\Omega)$ that separates points $\partial\Omega = \partial^S\Omega$.*

PROOF. In this case it is immediate that Σ is empty.

It should finally be pointed out that in the case of an algebra consisting only of real-valued function neither concept of boundary is of any use, since we have

THEOREM 6.4. *Let D be a real subalgebra of $C_0(\Omega)$, containing only real-valued functions and separating points. Then $\partial^S\Omega$ exists and $\partial^S\Omega = \partial\Omega = \Omega$.*

PROOF. For the existence of $\partial^S\Omega$ the proof of Theorem 3.3.1 in [9] can be restated, step by step, for real scalars. Then, by the Stone–Weierstrass theorem, the algebra is dense in $C_0{}^R(\Omega)$, the algebra of *all* real functions in $C_0(\Omega)$, and the argument of Theorem 3.3.2 in [9] gives the conclusion.

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