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THE VITALI-HAHN-SAKS THEOREM FOR VON NEUMANN ALGEBRAS

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1. Introduction.

Our aim is to give an operator-theoretic generalization of the Vitali–Hahn–Saks theorem (see [2, pp. 158–159]). Indeed, our theorem will give somewhat more information than the ordinary measure-theoretic version, as it gives the limit functional as a pointwise limit on all of $\mathscr{A}$, where $\mathscr{A}$ is the von Neumann algebra relative to which we formulate the theorem.

Consider first the following more general situation:

Let E be a Banach-space, and E^* its dual. Let K be a w^*-closed convex subset of the unit ball B_1^* of E^*. Then K is w^*-compact, and it is the w^*-closed span of its set of extreme points $\partial_e K$ (Krein–Milman theorem). Suppose that E^* is the norm-closed linear span of K. Now, let $\{x_n\}_{n\in N}$ be a sequence in E which converges pointwise on $\partial_e K$, that is for every $\varphi \in \partial_e K$ the limit $\lim_{n\to\infty}\varphi(x_n)$ exists as a finite number and thus defines a function $\hat{x}$ on $\partial_e K$. We may now ask: Does $\{x_n\}$ converge on all of K or on all of E^*? And will $\hat{x}$ be extendable to a representing functional for an element x in E such that

$$\varphi(x) = \lim_{n\to\infty}\varphi(x_n) \quad \text{for all } \varphi \in E^*\,?$$

A partial answer to this question is provided by the theorem of Rainwater ([6, p. 999]) which states that if $K = B_1^*$, and under the additional requirements that $\{x_n\}$ is bounded and converges pointwise on $\partial_e K$ to an element x which is assumed to be in E, then $\lim_{n\to\infty}\varphi(x_n) = \varphi(x)$ for all $\varphi \in E^*$.

Easily available counterexamples show that this is the best that can be hoped for in this general setting. For instance, take $E = C[0,1]$, and let $\{x_n\}$ be any sequence of continuous functions in E converging pointwise on $[0,1]$ to a discontinuous function. Since $[0,1]$ can be identified with the extreme points of the unit ball in E^* this shows that the assumption that the limit shall be an element of E can not be dropped. Likewise,

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the assumption that $\{x_n\}$ shall be bounded is necessary: Let $\{x_n\}$ converge pointwise to 0 on $[0,1]$ in such a way that

$$\int_0^1 x_n(s)\,ds = 1 \quad \text{for all } n=1,2,\ldots .$$

This integral is an element of E^* (in fact, with norm 1), so $\{x_n\}$ will not converge weakly.

Nevertheless, in the proper setting for von Neumann algebras the problem will have a positive solution, without the assumptions occuring in the Rainwater theorem.

In what follows, $\mathscr{A}$, $\mathscr{B}$ will denote von Neumann algebras. By $\mathscr{A}_*$, $\mathscr{B}_*$ wee denote their pre-duals, $\mathscr{A}^*$, $\mathscr{B}^*$ their norm-duals respectively. $\mathscr{P}$ will denote the set of projections in a von Neumann algebra $\mathscr{A}$. By $\mathscr{A}^+$, $\mathscr{A}^H$ and $\mathscr{A}_1$ wee denote the positive elements, the hermitian elements and the elements of norm less than or equal to one in $\mathscr{A}$, respectively, and $\mathscr{A}_1{}^H$ is defined as $\mathscr{A}_1 \cap \mathscr{A}^H$, and $\mathscr{A}_1{}^+$ as $\mathscr{A}_1 \cap \mathscr{A}^+$. We say that a linear functional on $\mathscr{A}$ is *normal* if it is continuous on $\mathscr{A}_1$ when the latter is equipped with the weak operator topology. A linear functional on $\mathscr{A}$ is normal if and only if it can be represented as an element of $\mathscr{A}_*$. ([1, ch. I, § 3, theorem 1, p. 40]).

In the general context outlined above, we now take $E = \mathscr{A}_*$, so $E^* = \mathscr{A}$. For K we choose $\mathscr{A}_1{}^+$, and note that $\partial_e K$ is equal to $\mathscr{P}$, a result which is due to Kadison [7]. In this setting our version of the Vitali–Hahn-Saks-theorem is the precise solution of the problem. The reader will also observe that the measure-theoretic version of this theorem can be interpreted in exactly the same way. Indeed, it is just a special case of our theorem.

We wish to thank prof. R. Kadison for calling our attention to the fact that each commutative von Neumann algebra is identifiable with a measure-theoretic picture ([8, part II, theorem 5, p. 32, and part I, theorem 1, p. 5]). This made considerable simplifications of the proofs possible.

2. A principle of uniform boundedness.

If $\mathscr{B}$ is a commutative von Neumann algebra, then there exists a locally compact space S and a positive measure μ on S with support S such that the spaces $\mathscr{B}$ and $L_C^\infty(S,\mu)$ are linearly isometric. Here $L_C^\infty(S,\mu)$ denotes the space of all complex valued, essentially bounded functions on S, where two functions are identified when they are equal almost everywhere. Moreover there is an isometric isomorphism of the pre-dual

$\mathscr{B}_*$ of $\mathscr{B}$ onto $L^1_{\mathrm{C}}(S,\mu)$, the integrable functions on S (identified as for L^∞). If φ is a normal functional on $\mathscr{B}$ (i.e. an element of $\mathscr{B}^*$) and $\hat{\varphi}$ is the corresponding function in $L^1_{\mathrm{C}}(S,\mu)$, then

$$(2.1) \qquad \varphi(A) = \int_S \hat{\varphi}(s)\hat{A}(s)\, d\mu(s), \qquad s \in S ,$$

for every $A \in \mathscr{B}$, when $\hat{A}$ is the function in $L^\infty_{\mathrm{C}}(S,\mu)$ corresponding to A. ([1, ch. I, § 7, pp. 112–120], [8, part II, theorem 5, p. 32, part I, theorem 1, p. 5]).

Let A be a self-adjoint operator in a von Neumann algebra $\mathscr{A}$, and let $\mathscr{B}$ be the commutative von Neumann sub-algebra of $\mathscr{A}$ it generates. Suppose now that $\mathscr{F}$ is a family of normal linear functionals on $\mathscr{A}$ which is pointwise bounded on the projections in $\mathscr{A}$. A fortiori $\mathscr{F}$ is then pointwise bounded on the projections in $\mathscr{B}$.

By the representation of $\mathscr{B}$ as $L^\infty_{\mathrm{C}}(S,\mu)$ for some S and μ, this transfers to the statement that for each measurable set $E \subseteqq S$ there is a constant $K(E) < \infty$ such that

$$(2.2) \qquad \left| \int_E \hat{\varphi}(s)\, d\mu(s) \right| < K(E); \qquad s \in S ,$$

for all $\hat{\varphi} \in L^1_{\mathrm{C}}(S,\mu)$ corresponding to members of $\mathscr{F}$. Then it follows, by a theorem of Nikodym ([2, ch. IV, 9.8, p. 309]) tht we can find a constant $K < \infty$ such that

$$(2.3) \qquad \left| \int_E \hat{\varphi}(s)\, d\mu(s) \right| < K, \qquad s \in S ,$$

for all measurable sets E in S and the same class of functions $\{\hat{\varphi}\}$. By standard measure theory it immediately follows that the L^1-norms of the elements of $\{\hat{\varphi}\}$ must be uniformly bounded. Hence, by the isometric character of the map $\varphi \to \hat{\varphi}$ we obtain in particular that the set $\{\varphi(A) : \varphi \in \mathscr{F}\}$ is bounded. But then, by the Banach–Steinhaus theorem and the fact that every operator in $\mathscr{A}$ can be written as the linear sum of two self-adjoint operators, it follows that $\mathscr{F}$ is uniformly bounded on bounded sets in $\mathscr{A}$. Therefore we have proved:

THEOREM 1. *If $\mathscr{F}$ is a family of normal functionals on a von Neumann algebra $\mathscr{A}$, which is pointwise bounded on the projections in $\mathscr{A}$, then $\mathscr{F}$ is uniformly bounded on bounded sets of $\mathscr{A}$.*

3. The Vitali-Hahn-Saks theorem.

Let $\mathscr{A}$ be a von Neumann algebra and let φ be a linear functional on $\mathscr{A}$. We say that φ is *completely additive* if for any family $\{P_\gamma\}_{\gamma\in\Gamma}$ of mutually orthogonal projections in $\mathscr{A}$, we have

$$\varphi(\textstyle\sum_{\gamma\in\Gamma}P_\gamma) = \sum_{\gamma\in\Gamma}\varphi(P_\gamma) . \tag{3.1}$$

Dixmier has proved that if φ is positive, then complete additivity is equivalent to normality ([1, p. 65, exc. 9]). More generally, Sakai ([4, footnote p. 440]) observed that this equivalence still holds when φ is bounded. In particular, for φ bounded, the condition (3.1) is equivalent to the requirement that if $\{P_\gamma\}_{\gamma\in\Gamma}$ is any downward directed, monotone net of commuting projections in $\mathscr{A}$ such that $\mathrm{glb}\{P_\gamma\}_{\gamma\in\Gamma}=0$, then it shall follow that $\varphi(P_\gamma) \to 0$; $\gamma \in \Gamma$.

Therefore, and in analogy with the corresponding concept for measures, we say that a family $\mathscr{F}$ of bounded linear functionals on $\mathscr{A}$ is *uniformly completely additive* on $\mathscr{A}$ if for any $\varepsilon>0$ we can find an index $\gamma_0 \in \Gamma$ such that if $\gamma \geqq \gamma_0$, then $|\varphi(P_\gamma)|<\varepsilon$ for all $\varphi\in\mathscr{F}$. Here $\{P_\gamma\}_{\gamma\in\Gamma}$ is commutative and descending to zero as above.

We now state our version of the Vitali–Hahn–Saks theorem.

THEOREM 2. *Let $\{\varphi_n\}_{n\in N}$ be a sequence of normal linear functionals on $\mathscr{A}$, and suppose that for every projection $P\in\mathscr{A}$, $\lim_{n\to\infty}\varphi_n(P)$ exists as a finite complex number, which we denote by $\varphi(P)$. Then*:

(i) *φ has a unique extension to all of $\mathscr{A}$ as an element of $\mathscr{A}^*$, and $\lim\varphi_n(A)$ exists and is equal to $\varphi(A)$ for every $A\in\mathscr{A}$.*

(ii) *φ is completely additive, and consequently normal.*

(iii) *The restrictions $\{\varphi_n \mid \mathscr{P}\cap\mathscr{B}\}_{n\in N}$ is equicontinuous in 0 with respect to the relativized weak operator topology on any commutative von Neumann sub-algebra $\mathscr{B}\subseteq\mathscr{A}$.*

(iv) *The family $\{\varphi_n\}_{n\in N}$ is uniformly completely additive.*

PROOF. The family $\{\varphi_n\}_{n\in N}$ is obviously pointwise bounded on the projections in $\mathscr{A}$, so that by Theorem 1 we can conclude that it is uniformly bounded on bounded sets in $\mathscr{A}$. By spectral-theory $\{\varphi_n\}$ converges on a norm-dense set in $\mathscr{A}^H$, and thus by uniform boundedness on all of $\mathscr{A}^H$, and hence on all of $\mathscr{A}$. We then put

$$\varphi(A) = \lim_{n\to\infty}\varphi_n(A), \qquad A\in\mathscr{A} ,$$

and φ becomes linear, bounded and is the only possible extension of the original φ defined on the projections with these properties. This completes the proof of (i).

Next, let $\mathscr{B}$ be any commutative von Neumann sub-algebra of $\mathscr{A}$, and let $L^\infty_{\mathrm{C}}(S,\mu)$ be a function-algebra corresponding to it as in § 2. For every $n=1,2,\ldots$, let ν_n be the measure defined by

$$\nu_n(E) = \int_E \hat{\varphi}_n(s)\, d\mu(s), \qquad s \in S\,,$$

when $\hat{\varphi}_n$ is the function in $L^1_{\mathrm{C}}(S,\mu)$ which corresponds to φ_n, and E is any μ-measurable set in S. Then define the measure ν by

$$\nu = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\nu_n|}{1+|\nu_n|(S)}\,.$$

Here $|\nu_n|$ denotes the total variation of the measure ν_n. Then ν is absolutely continuous with respect to μ and therefore determines a function $\eta \in L^1_{\mathrm{C}}(S,\mu)$, $\eta = d\nu/d\mu$. Now, let E be any μ-measurable set, and let P_E be the projection in $\mathscr{B}$ which corresponds to χ_E, the characteristic function of the set E. Then

$$\lim_{n\to\infty} \nu_n(E) = \lim_{n\to\infty} \int_S \hat{\varphi}_n(s)\chi_E(s)\, d\mu(s) = \lim_{n\to\infty} \varphi_n(P_E)$$

exists as a finite complex number. Moreover, each ν_n is absolutely continuous with respect to ν, so by the measure-theoretic Vitali–Hahn–Saks theorem we know that for any given $\varepsilon > 0$ there is a $\delta > 0$ such that for all μ-measurable sets E satisfying $\nu(E) < \delta$ we shall have $\nu_n(E) < \varepsilon$, $n = 1,2,\ldots$ ([2, ch. III, 7.2, p. 158]). But since ν corresponds to the L^1-function η, this is by the relation (2.1) exactly the same as saying that $\{\varphi_n\}$ is equicontinuous on $\mathscr{P}\cap\mathscr{B}$ in 0 with respect to the $\sigma(\mathscr{B},\mathscr{B}_*)$-topology. Now this topology will coalesce with the weak operator-topology, relativized from $\mathscr{A}$ to $\mathscr{P}\cap\mathscr{B}$ ([1, ch. I, § 3.3, p. 36]). Hence (iii) is proved.

(iv) follows immediately from (iii), since we need only consider the commutative von Neumann algebra generated by the family $\{P_\gamma\}_{\gamma\in\Gamma}$ in question, and note that $P_\gamma \to 0$ with respect to the weak operator-topology.

(ii) follows at once from (iv) and the remarks preceding the theorem.

We do not know whether the family $\{\varphi_n\}_{n\in N}$ actually is weakly equicontinuous on $\mathscr{P}$ in 0 (cf. (iii) in the theorem above). However, the family $\{\varphi_n\}$ will be equicontinuous with respect to the Mackey-topology $\tau(\mathscr{A},\mathscr{A}_*)$, on all of $\mathscr{A}$. This can be seen as follows: $\mathscr{A}_*$ is a Banach-space with dual $\mathscr{A}$, and therefore the $\sigma(\mathscr{A}_*,\mathscr{A})$-closed, convex, circled extension of the sequence $\{\varphi_n\}$ (which is relatively $\sigma(\mathscr{A}_*,\mathscr{A}$-compact) must be $\sigma(\mathscr{A}_*,\mathscr{A})$-compact ([3, 17.12, p. 159]).

The Mackey-topology $\tau(\mathscr{A},\mathscr{A}_*)$ for $\mathscr{A}$ is given as the topology of uniform convergence on the class of convex, circled, $\sigma(\mathscr{A}_*,\mathscr{A})$-compact subsets of $\mathscr{A}_*$, so in particular $\{\varphi_n\}$ must be equicontinuous on $\mathscr{A}$ with respect to this topology.

An affirmative answer to the question above will therefore be obtained if we can prove that the restrictions to P of the Mackey-topology $\tau(\mathscr{A},\mathscr{A}_*)$ and the weak operator topology respectively, determine equivalent neighbourhood systems around 0. This is true when $\mathscr{A}$ is commutative, and due to a recent result of Sakai, we are also able to state it for von Neumann algebras of finite type.

THEOREM 3. *Let $\mathscr{A}$ be a von Neumann algebra of finite type, and let the sequence $\{\varphi_n\}_{n\in N}$ be as in the premises of Theorem 2. Then $\{\varphi_n \mid \mathscr{A}_1{}^+\}_{n\in N}$ is equicontinuous in 0 with respect to the weak operator-topology. In particular $\{\varphi_n \mid \mathscr{P}\}_{n\in N}$ is equicontinuous in 0.*

PROOF. In any von Neumann algebra, finite or not, we have for $A \in \mathscr{A}$, A positive: $\varphi(A^2) \leqq \varphi(A)\cdot\|A\|$; $\varphi \geqq 0$, $\varphi \in \mathscr{A}_*$. The s-topology for a von Neumann algebra $\mathscr{A}$ is determined by the family of semi-norms:

$$\{p_\varphi(A) = [\varphi(A^*A)]^{\frac{1}{2}},\ \varphi \in \mathscr{A}_*,\ \varphi \geqq 0\}, \qquad A \in \mathscr{A}\ .$$

Sakai [5], has proved that for von Neumann algebras of finite type, the $\tau(\mathscr{A},\mathscr{A}_*)$-topology will be equivalent to the s-topology on bounded sets of $\mathscr{A}$. Then, since the weak operator topology and w^*-topology for $\mathscr{A}$ (as the dual of $\mathscr{A}_*$) also coalesce, it follows by the considerations preceding the theorem and the inequality starting the proof, that the theorem is true.

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