MATH. SCAND. 18 (1966), 93–96

BOUNDED RADIAL VARIATION AND DIVERGENCE OF POWER SERIES

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It has long been known that continuity of the function

$$f(z) = \sum a_n z^n \tag{1}$$

on the closure of the unit disk D does not imply convergence of the series $\sum a_n$. (The earliest counterexample was constructed by L. Fejér [1]; see also Landau [6, Section 3].) On the other hand, if $\sum n\,|a_n|^2 < \infty$, then the power series in (1) converges at each point of the unit circle C at which f has a radial limit (see [2], [3], [6, Section 13]).

In a private conversation, P. and V. Turán raised the question whether convergence on C of the power series is still assured if we replace the hypothesis of a finite Dirichlet integral with the assumption that f maps the radii of D onto curves of bounded length. The theorem in the present note shows that the answer is negative; but it leaves open the question whether the series in (1) converges everywhere on C if $a_n \to 0$ and f is univalent and maps each radius of D onto a curve of finite length.

H. S. Shapiro [7] recently showed that the assumption of bounded variation of f on $[0,1]$ permits no substantial relaxation of the restriction on $\{a_n\}$ in the classical Tauberian theorem. Corresponding to each positive ε he exhibited a divergent series $\sum a_n$ such that $a_n = O(n^{\varepsilon-1})$ and such that the function (1) has finite variation on the segment $[0,1]$. P. B. Kennedy and P. Szüsz [5] have extended Shapiro's result by showing that if $\varphi(n) \to \infty$, then there exists a divergent series $\sum a_n$ such that $n|a_n| < \varphi(n)$ for all n and such that the function (1) is bounded and monotonic on the segment $[0,1]$. Our theorem strengthens Shapiro's result in almost exactly the same way; but unlike the example of Kennedy and Szüsz, our function is not monotonic on $[0,1]$; on the other hand, it is continuous in $D \cup C$ and has uniformly bounded radial variation in D.

THEOREM. *If $\varphi(n) > 0$ and $\varphi(n) \to \infty$, then there exists a divergent series $\sum a_n$ such that*

Received October 16, 1965.

This note was written with support from the National Science Foundation (GP-3466).

$$n|a_n| < \varphi(n), \qquad n = 0, 1, \ldots, \tag{2}$$

and such that the function (1) *is continuous in* $D \cup C$ *and has uniformly bounded variation on the radii of* D.

The building block in our construction is the modified Fejér polynomial $F(z,n,p)$, obtained by deletion of $2p$ terms from the middle of the polynomial

$$F(z,n) = \frac{1}{n} + \frac{z}{n-1} + \ldots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \ldots - \frac{z^{2n-1}}{n}.$$

Since $F(z,n,p) = F(z,n) - z^{n-p}F(z,p)$ and

$$|F(z,n)| < 2\int_0^\pi \frac{\sin t}{t}\,dt = M$$

on C (see [4, Section 3]), we see at once that $|F(z,n,p)| < 2M$ on C.

Without loss of generality, we may assume that φ is increasing. We choose a sequence $\{n_j\}$ such that $n_{j+1} > 3n_j$ and

$$\sum (\log \varphi(n_j))^{-\frac{1}{2}} < \infty \tag{3}$$

(in the final stage of the proof, we shall replace $\{n_j\}$ with a sufficiently thin subsequence). We define the integers p_j by the formula

$$p_j = [n_j/\varphi(n_j)], \tag{4}$$

and we construct the function

$$f(z) = \sum_{j=1}^{\infty} \frac{z^{n_j} F(z, n_j, p_j)}{\log \varphi(n_j)}. \tag{5}$$

Let $f(z) = \sum a_n z^n$ and $s_n = \sum_0^n a_k$. Since

$$s_{3n_j} = 0, \quad j = 1, 2, \ldots, \qquad \text{and} \qquad \lim_{j\to\infty} s_{2n_j} = 1,$$

the series $\sum a_k$ diverges. Because the polynomials $F(z,n,p)$ are uniformly bounded on C, condition (3) implies that f is continuous on $D \cup C$.

Concerning condition (2), we observe that for coefficients a_n arising from the jth term in (5), the quantity $n|a_n|$ has its maximum at the beginning of the block of negative coefficients, and that this maximum therefore has the value

$$\frac{2n_j + p_j + 1}{(p_j + 1)\log \varphi(n_j)}.$$

By (4), the numerator is less than $3n_j$, and if we impose the additional condition $\varphi(n_1) > e^3$, it follows from (4) that (2) is satisfied for all n.

It remains to show that we can choose the sequence $\{n_j\}$ so that f has uniformly bounded radial variation. To this end, we observe first that the radial variation in D of any polynomial $\sum b_n z^n$ does not exceed $\sum |b_n|$. This implies that the maximum radial variation of the jth polynomial in (5) is less than 3.

Next we remark that by the rule of Descartes, the derivative of $z^n F(z,n,p)$ has only one zero on the segment $[0,1]$, so that the total variation of the jth term on $[0,1]$ is at most $4M/\log\varphi(n_j)$. This implies not only—by virtue of (3)—that f has finite variation on the segment $[0,1]$, but also that for each n_j we can find some sector A_j of D, bisected by the segment $[0,1]$, in which the radial variation of the jth term of (5) is less than $5M/\log\varphi(n_j)$ (hence less than $5\pi(\log\varphi(n_j))^{-\frac{1}{2}}$; the use of the larger bound will be more convenient, later).

To complete the discussion of the radial variation, we need a careful estimate of the integral (along the radius of $e^{i\theta}$, for $0<|\theta|\leqq\pi$) of the absolute value of $[z^nF(z,n,p)]'$. The derivative has the value

$$\frac{nz^{n-1}}{n}+\frac{(n+1)z^n}{n-1}+\dots+\frac{(2n-p-1)z^{2n-p-2}}{p+1}-$$
$$-\left[\frac{(2n+p)z^{2n+p-1}}{p+1}+\dots+\frac{(3n-1)z^{3n-2}}{n}\right].$$

Since the negative coefficients form a numerically decreasing sequence, the absolute value at $z=re^{i\theta}$ of the terms in brackets is by Abel's summation formula less than

$$\pi(2n+p)r^{2n+p-1}/|\theta|\,. \tag{6}$$

The absolute values of the terms in the first half of the polynomial $(z^nF)'$ do not necessarily form a monotonic sequence. However, for each value r, the difference

$$\frac{(n+k)r^{n+k-1}}{n-k}-\frac{(n+k+1)r^{n+k}}{n-k-1}$$
$$=\frac{r^{n+k-1}}{(n-k)(n-k-1)}[(n+k)(n-k-1)-r(n+k+1)(n-k)]$$

changes sign at most once as k runs through the values $0,1,\dots,n-p-2$. Therefore the corresponding polynomial consists of one or two sections in each of which the absolute values of the terms form a monotonic sequence, and consequently its absolute value is less than $2\pi N/|\theta|$, where N denotes the modulus of the greatest term in the polynomial. Clearly, $N<2nr^{n-1}$, and taking account of the quantity (6), we see that

$$\left| \frac{d}{dz} [z^n F(z,n,p)] \right| < \frac{\pi}{|\theta|} [(2n+p)r^{2n+p-1} + 4nr^{n-1}] ,$$

at $z = re^{i\theta}$, $0 < |\theta| \leqq \pi$. Integrating the right member over the range $0 \leqq r \leqq 1$, we deduce that the radial variation of the jth term in (5) is less than

$$\frac{5\pi}{|\theta| \log \varphi(n_j)} .$$

In particular, when $(\log \varphi(n_j))^{-\frac{1}{2}} < |\theta| \leqq \pi$, then the variation of the jth term on the corresponding radius is less than $5\pi(\log \varphi(n_j))^{-\frac{1}{2}}$.

We have shown that the radial variation of the jth term in (5) is bounded by 3, and that it is bounded by $5\pi(\log \varphi(n_j))^{-\frac{1}{2}}$ inside of the sector A_j and outside of the sector B_j defined by the inequality $|\arg z| < (\log \varphi(n_j))^{-\frac{1}{2}}$. We now suppose that the sequence $\{n_j\}$ increases so rapidly that the sectors $B_1, A_1, B_2, A_2, \ldots$ form a nested sequence, and we consider any radius R of D on which, for some index k, the kth term of (5) has variation greater than $5\pi(\log \varphi(n_k))^{-\frac{1}{2}}$. Since R must lie inside of B_k and outside of A_k, it lies inside of $A_1, A_2, \ldots, A_{k-1}$ and outside of B_{k+1}, $B_{k+2}, \ldots$. This implies that the variation of f on R is less than

$$3 + 5\pi \sum_{j \neq k} (\log \varphi(n_j))^{-\frac{1}{2}} ,$$

and the proof of the theorem is complete.

REFERENCES

1. L. Fejér, *Über gewisse Potenzreihen an der Konvergenzgrenze*, Sitzungsber. math.-phys. Kl. Königl. Bayer. Akad. Wiss. 1910, no. 3, 1–17.
2. L. Fejér, *La convergence sur son cercle de convergence d'une série de puissance effectuant une représentation conforme du cercle sur le plan simple*, C. R. Acad. Sci. Paris 156 (1913), 46–49.
3. L. Fejér, *Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene*, Mathematische Abhandlungen, Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum am 6. August 1914 gewidmet von Freunden and Schülern, Berlin, 1914.
4. F. Herzog and G. Piranian, *Some properties of the Fejér polynomials*, Proc. Amer. Math. Soc. 7 (1956), 379–386.
5. P. B. Kennedy and P. Szüsz, *On a bounded increasing power series*, Proc. Amer. Math. Soc. (to appear).
6. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Zweite Auflage, Berlin, 1929.
7. H. S. Shapiro, *A remark concerning Littlewood's Tauberian theorem*, Proc. Amer. Math. Soc. 16 (1965), 258–259.

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