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ON HOMOLOGICAL DIMENSIONS OF RINGS WITH COUNTABLY GENERATED IDEALS

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1. Introduction.

Let R be an arbitrary, not necessarily commutative ring with an identity element. A left R-module $A \neq 0$ is said to have homological dimension n, denoted $\mathrm{l.dh}_R(A)$, if A has a projective resolution of the form

$$\ldots 0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to A \to 0$$

but no such of this type with fewer non-zero terms. The left global dimension of R, denoted $\mathrm{l.gl.dim}\,R$, is defined as $\sup \mathrm{l.dh}_R(A)$, where A ranges over all left R-modules, and is characterized by the property that $\mathrm{l.gl.dim}\,R \leqq n$, if and only if $\mathrm{Ext}_R^{n+1}(A,B)=0$ for all left R-modules A and B.

The right global dimension of R, denoted $\mathrm{r.gl.dim}\,R$, is defined analogously by means of right R-modules. In general, $\mathrm{l.gl.dim}\,R$ and $\mathrm{r.gl.dim}\,R$ do not coincide. Probably, the simplest counterexample (cf. Small [5]) is the ring R of 2×2 matrices

$$\begin{pmatrix} h & q_1 \\ 0 & q_2 \end{pmatrix}, \qquad h \in Z, \quad q_1, q_2 \in Q\,,$$

for which one finds $\mathrm{r.gl.dim}\,R=1$ and $\mathrm{l.gl.dim}\,R=2$.

If, however, R is assumed to be left and right Noetherian, the two dimensions coincide and are both equal to the weak global dimension of R, denoted $\mathrm{w.gl.dim}\,R$, which is the smallest integer n for which $\mathrm{Tor}_{n+1}^R(A,B)=0$ for all right R-modules A and all left R-modules B (cf. Northcott [4, theorem 20, p. 154]).

From the definition of $\mathrm{w.gl.dim}\,R$ it readily follows that for any ring R one has

$$\mathrm{w.gl.dim}\,R \leqq \mathrm{l.gl.dim}\,R \quad \text{and} \quad \mathrm{w.gl.dim}\,R \leqq \mathrm{r.gl.dim}\,R\,.$$

It is the purpose of this note to show that

$$\mathrm{l.gl.dim}\,R \leqq \mathrm{w.gl.dim}\,R + 1\,,$$

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if all the left ideals of R are countably generated. By the corresponding result for right R-modules and the left-right symmetry of the weak global dimension it follows that for any ring R whose left ideals and right ideals are countably generated, one has

$$|\text{l.gl.dim} R - \text{r.gl.dim} R| \leqq 1 .$$

In view of the just mentioned example this result is, in a certain sense, best possible.

2. The connection between l.gl.dim *R* and w.gl.dim *R*.

Before stating the main result we shall prove the following two lemmas.

LEMMA 1. *Let R be a ring whose left ideals are countably generated. Then any submodule of a countably generated left R-module is itself countably generated.*

PROOF. It apparently suffices to show that any submodule A of a finitely generated left R-module M is countably generated. We shall prove this last statement by induction on the number n of generators of M. If $n=1$, then $M \cong R/L$ for some left ideal L of R. Therefore $A \cong J/L$ for a suitable left ideal J containing L. Since J is countably generated, so is J/L. Suppose now that the statement has been proved for any module generated by fewer than n elements. If M is generated by the n elements $m_1,\ldots,m_n$, that is

$$M = Rm_1 + \ldots + Rm_n ,$$

let $B = Rm_1$; then $A \cap B \subseteq Rm_1$ is countably generated and

$$A/A \cap B \cong (A+B)/B \subseteq M/B$$

is countably generated by the inductive assumption, since M/B is generated by $(n-1)$ elements. Consequently A is countably generated.

REMARK. The lemma immediately implies the following analogue of Hilbert base theorem. If the commutative ring R has countably generated ideals, and S is a ring unitary over R and countably generated over R (as a ring), then the ideals of S are countably generated. In fact, if the elements y_i form a countable set of generators for S over R, then the power products $y_{i_1}^{\nu_1} \ldots y_{i_n}^{\nu_n}$ will be a countable set of generators of S, viewed as an R-module.

LEMMA 2. *If A is a countably generated flat left R-module, where R is a ring whose left ideals are countably generated, then* $\text{l.dh}_R(A) \leqq 1$.

PROOF. Since A is countably generated there is a short exact sequence

$$0 \to B \to F \xrightarrow{\varphi} A \to 0$$

where F is a free left R-module with a countable base $e_1, e_2, \ldots$, and B, here regarded as a submodule of F, is countably generated because of lemma 1.

Let C be an arbitrary submodule of B generated by finitely many elements $c_1, \ldots, c_\nu \in B$. From Bourbaki [1, exercise 23, p. 65) it follows that there exists a homomorphism u from F to B such that $u(c_i) = c_i$, $1 \leqq i \leqq \nu$.

If $c_i = \sum_j r_{ij} e_j$, $r_{ij} \in R$, $1 \leqq j \leqq \mu$ say, then the elements $\bar{c}_j = u(e_j) \in B$ satisfy the relations

$$c_i = \sum_j r_{ij} \bar{c}_j, \qquad 1 \leqq i \leqq \nu,\ 1 \leqq j \leqq \mu .$$

Let α^C be the endomorphism of F defined by $\alpha^C(e_j) = e_j - \bar{c}_j$ for $1 \leqq j \leqq \mu$ and $\alpha^C(e_j) = e_j$ for $j > \mu$. Then we have

$$\alpha^C(c) = 0 \quad \text{for all } c \in C . \tag{1}$$

The module generated by the elements $\bar{c}_j, 1 \leqq j \leqq \mu$ will be denoted by $\bar{C}$.

Since B is countably generated it may be written as the ascending union of finitely generated submodules B_n

$$B = \bigcup_{n=1}^{\infty} B_n, \qquad B_1 \subseteqq B_2 \subseteqq \ldots \subseteqq B_n \subseteqq \ldots .$$

We shall now inductively define finitely generated submodules C_n of B in the following way:

$$C_1 = B_1,\ C_2 = \bar{C}_1 + B_2,\ \ldots,\ C_n = \bar{C}_{n-1} + B_n .$$

Here we have

$$C_1 \subseteqq \bar{C}_1 \subseteqq C_2 \subseteqq \bar{C}_2 \subseteqq \ldots \subseteqq C_n \subseteqq \bar{C}_n \subseteqq C_{n+1} \subseteqq \ldots, \tag{2}$$

$$\bigcup_{n=1}^{\infty} C_n = B .$$

Let $\alpha^{(n)}$ be the endomorphism α^{C_n}. For the mappings $\alpha^{(n)}$ we have

(i) $\varphi \alpha^{(n)} = \varphi$,
(ii) $\alpha^{(n)}(c) = 0$ for all $c \in C_n$,
(iii) $\alpha^{(n)} \alpha^{(m)} = \alpha^{(n)}$ for $n > m$.

(i) follows from the definition of $\alpha^{(n)}$, and (ii) follows from (1). To prove (iii) we remark that, according to the definition of $\alpha^{(n)}$ we have in any case $\alpha^{(m)}(e_j)=e_j+c$, where $c\in\bar{C}_m\subseteq C_n$. By (ii) $\alpha^{(n)}(c)=0$, hence $\alpha^{(n)}(\alpha^{(m)}(e_j))=\alpha^{(n)}(e_j)$. Now (iii) is obvious since $\alpha^{(n)}\alpha^{(m)}$ and $\alpha^{(n)}$ have the same values on the base elements e_j of F.

Let P be the direct sum of a countable set of copies of F, here regarded as the set of all sequences $\{x_1,x_2,\ldots,x_n,\ldots\}$, where $x_n\in F$ and $x_n=0$ for almost all n. We claim that the following sequence

$$0\to P\xrightarrow{\chi}P\xrightarrow{\Phi}A\to 0 \tag{3}$$

is exact. Here Φ is defined by

$$\Phi\{x_1,\ldots,x_i,\ldots\} = \varphi(\sum x_i)$$

and χ by

$$\chi\{x_1,x_2,\ldots,x_n,\ldots\} = \{x_1,x_2-\alpha^{(1)}x_1,\ldots,x_n-\alpha^{(n-1)}x_{n-1},\ldots\}\,.$$

It is evident, that Φ is surjective and χ injective. From (i) it follows that $\operatorname{Im}\chi\subseteq\operatorname{Ker}\Phi$. To prove the converse inclusion let $\{x_1,\ldots,x_i,\ldots\}$ be an element of $\operatorname{Ker}\Phi$, so that $\sum x_i$ belongs to $\operatorname{Ker}\varphi=B$. Because of (2), $\sum x_i\in C_n$ for a suitable n. Thus (ii) implies $\alpha^{(n)}(\sum x_i)=0$. By (iii) we may assume that $x_i=0$ for $i>n$. By repeated use of (iii) it is readily checked that

$$\begin{aligned}\chi\{x_1,\alpha^{(1)}x_1+x_2,\alpha^{(2)}(x_1+x_2)+x_3,\ldots,\alpha^{(n-1)}(x_1+\ldots+x_{n-1})+x_n,0,0,\ldots)\\ = \{x_1,x_2,\ldots,x_n,0,0,\ldots\}\,.\end{aligned}$$

This proves the inclusion $\operatorname{Ker}\Phi\subseteq\operatorname{Im}\chi$.

As a direct sum of free modules, P is free, in particular projective. The existence of the exact sequence (3) implies $\mathrm{l.dh}_R(A)\leqq 1$.

We are now able to prove

THEOREM 1. *For an arbitrary ring R whose left ideals are countably generated one has*

$$\mathrm{l.gl.dim}\,R \leqq \mathrm{w.gl.dim}\,R+1\,.$$

PROOF. We may obviously assume that $\mathrm{w.gl.dim}\,R=n<\infty$, since otherwise there is nothing to prove. To show $\mathrm{l.gl.dim}\,R\leqq n+1$ it suffices to prove $\mathrm{l.dh}_R(C)\leqq n+1$ for any cyclic left R-module $C=R/L$, L being a left ideal of R (cf. Northcott [4, theorem 15, p. 141]).

By assumption L is countably generated and is therefore the homomorphic image of a countably generated free R-module F_1 with a kernel K_1 which is countably generated because of lemma 1. Continuing this way we get the following short exact sequences

$$(*)\qquad \begin{array}{llll} 0 \to L & \to R & \to C & \to 0\,, \\ 0 \to K_1 & \to F_1 & \to L & \to 0\,, \\ 0 \to K_2 & \to F_2 & \to K_1 & \to 0\,, \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ 0 \to K_{n-1} & \to F_{n-1} & \to K_{n-2} & \to 0\,, \end{array}$$

where the F's are free modules with countable bases and the K's are countably generated by lemma 1. For any right R-module A we obtain by repeated use of the connecting homomorphisms for Tor an isomorphism

$$\mathrm{Tor}_1^R(A,K_{n-1}) \cong \mathrm{Tor}_{n+1}^R(A,C) = 0\,,$$

since $\mathrm{w.gl.dim}\,R = n$. Consequently K_{n-1} is a flat countably generated left R-module. By lemma 2, K_{n-1} is the quotient of two projective R-modules

$$0 \to P_{n+1} \to P_n \to K_{n-1} \to 0\,.$$

Combining this short exact sequence with (*) we get a long exact sequence

$$0 \to P_{n+1} \to P_n \to F_{n-1} \to \dots \to F_1 \to R \to C\,.$$

This makes it clear that $\mathrm{l.dh}_R(C) \leqq n+1$.

By passage to R's opposite ring we get a similar result for the right global dimension of R. Taking into account the right-left symmetry of the weak global dimension we obtain

COROLLARY 1. *For an arbitrary ring R whose left ideals and right ideals are countably generated we have*

$$|\mathrm{l.gl.dim}\,R - \mathrm{r.gl.dim}\,R| \leqq 1\,.$$

In particular this inequality holds if R has only countably many elements.

REMARK. It is easy to give examples of rings, for which all right ideals, but not all left ideals are countably generated. Actually, let S be a (commutative) principal ideal domain, whose quotient field K, regarded as a S-module is not countably generated (for instance $S = F[X]$, where F is a field with a non-countable set of elements). The matrix ring

$$R = \begin{pmatrix} s & k_1 \\ 0 & k_2 \end{pmatrix}, \qquad s \in S, \quad k_1, k_2 \in K\,,$$

is a right principal ideal ring, while the left ideal consisting of the matrices

$$\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \qquad k \in K\,,$$

is not countably generated.

By examples of this sort, however, one does not obtain rings whose left and right global dimensions differ by more than 1, and it appears to be an open question how great values the difference

$$|\text{l.gl.dim}\, R - \text{r.gl.dim}\, R|$$

actually can assume.

Since any left (or right) semi-hereditary ring R has $\text{w.gl.dim}\, R \leqq 1$ (cf. Cartan–Eilenberg [2] VI 2.9), we get

Corollary 2. *Let R be a (left or right) semi-hereditary ring. If R's left (right) ideals are countably generated, then* $\text{l.gl.dim}\, R \leqq 2$ ($\text{r.gl.dim}\, R \leqq 2$).

In particular, if R is a Prüfer ring with countably generated ideals, we have (cf. Cartan–Eilenberg [2, VII.5]) that $\text{gl.dim}\, R \leqq 1$ or $\text{gl.dim}\, R = 2$, according as R is Noetherian (hence Dedekind) or not. On the other hand it has been proved by Kaplansky (unpublished) that a valuation ring R containing non-countably generated ideals has $\text{gl.dim}\, R > 2$. This means that, even in the commutative case, the assumption of the countable generation of the ideals is essential for the validity of theorem 1.

We shall finish this note by showing that for any $n > 0$ there exists a commutative integral domain R (with countably generated ideals) for which $\text{w.gl.dim}\, R = n$ and $\text{gl.dim}\, R = n+1$. For $n = 1$ we can take R as the ring of all algebraic integers, since these form a non-Noetherian Prüfer ring. For any $n > 1$ the polynomial ring $R[x_1, \ldots x_{n-1}]$ will then be an example of the desired kind. In fact, we have (cf. MacLane [3, theorem 4.2, p. 210])

$$\text{gl.dim}\, R[x_1, \ldots, x_{n-1}] = 2 + (n-1) = n+1 .$$

Moreover, as proved in the appendix below, one has

$$\text{w.gl.dim}\, R[x_1, \ldots, x_{n-1}] = 1 + (n-1) = n .$$

3. Appendix.

We shall here sketch a proof of the following theorem which may be known, but for which the author has not been able to find any reference.

Theorem 2. *For any not necessarily commutative ring R one has*

$$\text{w.gl.dim}\, R[X] = \text{w.gl.dim}\, R + 1 .$$

Proof. If $\text{w.gl.dim}\, R = n$ we shall prove $\text{w.gl.dim}\, R[X] \geqq n+1$ and $\text{w.gl.dim}\, R[X] \leqq n+1$. These inequalities follow from the lemmas 3 and 4 below.

LEMMA 3. *Let $A \neq 0$ be a left R-module with* $\text{w.l.dh}_R(A) = m$. *If A is made into an $R[X]$-module by setting $f(x)a = f(0)a$ for any $f(x)$ in $R[X]$, then*

$$\text{w.l.dh}_{R[X]}(A) \geqq m+1 .$$

PROOF. For $m=0$ we have to prove that A is not $R[X]$ flat. Let φ be the injective homomorphism from $R[X]$ to $R[X]$ defined by $\varphi f(x) = xf(x)$. The kernel of the mapping

$$R[X] \otimes_{R[X]} A \xrightarrow[\varphi \otimes 1_A]{} R[X] \otimes_{R[X]} A$$

is $R[X] \otimes_{R[X]} A \cong A \neq 0$; hence $\text{w.l.dh}_{R[X]} A > 0$.

For $m=1$ let

$$0 \to K \to F \to A \to 0$$

be an exact sequence of $R[X]$-modules where F is $R[X]$-free. If $\text{w.l.dh}_{R[X]}(A) \leqq 1$, K would be $R[X]$-flat, and hence, what is readily checked, K/XK would be R-flat. Since $XF \subseteqq K$ and $XF/XK \cong A$, we have short exact sequences

$$0 \to K/XF \to F/XF \to A \to 0 \tag{4}$$

$$0 \to A \to K/XK \to K/XF \to 0 . \tag{5}$$

Since $m = \text{w.l.dh}_R(A) = 1$ and F/XF is R-free, we infer from (4) that K/XF is R-flat and hence from (5) that A is R-flat, contradicting $\text{w.l.dh}_R(A) = 1$. This shows that $\text{w.l.dh}_{R[X]}(A) \geqq 2$.

For $m > 1$ let

$$0 \to K' \to F' \to A \to 0$$

be a short exact sequence of R-modules with F' being R-free. We proceed by induction on m. Obviously

$$\text{w.l.dh}_R(K') = m-1 ,$$

so the inductive assumption implies

$$\text{w.l.dh}_{R[X]}(K') \geqq m ,$$

and hence, since $\text{w.l.dh}_{R[X]}(F') = 1$, we get $\text{w.l.dh}_{R[X]}(A) \geqq m+1$.

LEMMA 4. *Let $A \neq 0$ be any left $R[X]$-module, which, viewed as an R-module, has* $\text{w.l.dh}_R(A) = m$; *then*

$$\text{w.l.dh}_{R[X]}(A) \leqq m+1 .$$

PROOF. For $m=0$ we have to show that $\text{w.l.dh}_{R[X]}(A) \leqq 1$, if A is R-flat. Now, for any flat left R-module A the tensor product $R[X] \otimes_R A$, made into a left $R[X]$-module in the obvious way, is $R[X]$-flat. Indeed, for an arbitrary right $R[X]$-module M there are isomorphisms (cf. Northcott [4, 8.3, theorem 1])

$$T(M) = M \otimes_{R[X]} (R[X] \otimes_R A) \cong (M \otimes_{R[X]} R[X]) \otimes_R A \cong M \otimes_R A .$$

Since these isomorphisms are natural in M, $T(M)$ is an exact functor of M, that is, $R[X] \otimes_R A$ is $R[X]$-flat.

Lemma 4 is proved in the case $m=0$ by showing that there are $R[X]$-homomorphisms ψ and φ such that

$$(6) \qquad 0 \to R[X] \otimes_R A \xrightarrow{\psi} R[X] \otimes_R A \xrightarrow{\varphi} A \to 0$$

is exact. In fact, if φ and ψ are defined by

$$\varphi(f(x) \otimes a) = f(x)a, \qquad f(x) \in R[X] ,$$
$$\psi(x^\nu \otimes a) = x^\nu \otimes (xa) - x^{\nu+1} \otimes a ,$$

it is not hard to see that (6) is exact.

For $m>0$ let

$$0 \to K_m \to F_{m-1} \to \ldots \to F_0 \to A \to 0$$

be an exact sequence of $R[X]$-modules, where $F_0,\ldots,F_{m-1}$ are $R[X]$-free, in particular R-free. By repeated use of the connecting homorphisms for Tor we get

$$\operatorname{Tor}_1^R(B,K_m) \cong \operatorname{Tor}_{m+1}^R(B,A) = 0$$

for any right R-module B, that is, $\mathrm{w.l.dh}_R(K_m)=0$. By the just settled case $m=0$ we have

$$\mathrm{w.l.dh}_{R[X]}(K_m) \leqq 1 ,$$

and hence for any right $R[X]$-module M,

$$0 = \operatorname{Tor}_2^{R[X]}(M,K_m) \cong \operatorname{Tor}_{m+2}^{R[X]}(M,A);$$

that is, $\mathrm{w.l.dh}_{R[X]}(A) \leqq m+1$.

Remark. By combination of lemma 3 and lemma 4 it is readily seen that the inequalities in these lemmas actually are equalities. A slight modification of lemma 3 shows that theorem 2 also holds if $\mathrm{w.gl.dim}\, R = \infty$.

Added in proof. An example of a ring whose left and right global dimensions differ by 2 has just been given in Lance W. Small, *Hereditary rings*, Proc. Nat. Acad. Sci. U.S.A. 55 (1966), 25–27. This means that the assumption of the countable generation of the ideals is essential for the validity of not only theorem 1 but also corollary 1.

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