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TOPOLOGICAL SEMILOOPS

G. S. McCARTY, Jr.

1. Introduction.

Let X be a locally compact, locally connected Hausdorff space, and let G be a group of homeomorphisms of X. Arens [1] has shown that the compact-open topology on G is the smallest topology with which G is a topological transformation group on X; that is, the evaluation $G \times X \to X: (g,x) \to g(x)$ is continuous and G is a Hausdorff topological group. For a fixed member e of X, the *projection* $\pi: G \to X: \pi(g) = g(e)$ is continuous. If X is such a space and $e \in X$, and if there is a group G of homeomorphisms of X such that the projection π possesses a *cross-section* $\sigma: X \to G$ (that is, $\pi \circ \sigma$ is the identity on X and $\sigma(e) = 1$) then we define X to be a *topological semiloop* with *identity* e (abbreviated tsl). In case X is a manifold it has also been called *suitable* [2], [3]. If X and X' are tsl's and $\mu: X \to X'$ is an open map which "preserves products", so

$$\mu(\sigma(x_1)(x_2)) = \sigma'(\mu(x_1))(\mu(x_2)) ,$$

then μ is called a *morphism* of tsl's.

Clearly every topological loop [4] (satisfying the local conditions above) is a tsl; and every tsl is an H-space [6], with the product $x_1 x_2 = \sigma(x_1)(x_2)$ (which is continuous by the exponential law of mapping spaces [6, Theorem III 9.9]). An example of a tsl whose product is not that of a loop is given for the real interval $X = (-1,1)$ by

$$\sigma(x)(y) = x + y - x|y| ;$$

there is no $x \in X$ such that $\sigma(x)(\frac{1}{2}) = 0$. Few topologically nontrivial examples are known of loops which are not groups; the 7-sphere S^7 is one, with the Cayley multiplication.

In § 2 a sufficient condition is found that a homeomorphism may be lifted through a covering map. This is used to establish the following results.

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THEOREM 1. *If X is a locally path-connected* tsl *with identity e, Y is connected and $\varkappa\colon Y \to X$ is a covering map, $\varkappa(f)=e$, then there exists a unique* tsl *structure on Y with identity f such that $\varkappa$ is a morphism.*

COROLLARY 1. *If, in Theorem 1, X is a topological loop under the* tsl *product, so is Y.*

These facts generalize the theorem of Hofmann [4, Satz 6.6] that the universal covering space of a topological loop is again a topological loop. If X is a topological group, so is every covering space Y; this improves a classical result in the sense that no universal covering space need exist for X.

In § 3 quotients of tsl structures are constructed. A sub-tsl A of a tsl X is termed *normal* if it is the kernel of some morphism. Let $\mathscr{G}$ denote the group of homeomorphisms generated by the left multiplications of X, and if $\pi\colon \mathscr{G} \to X$ is the evaluation at e, let $\mathscr{P}=\pi^{-1}(e)$. We shall write $\bar{x}$ for $\sigma(x)$ and $\bar{X}$ for $\sigma(X)$.

THEOREM 2. *A subset A of a* tsl *X is a normal* sub-tsl *of X iff there exists a closed normal subgroup $\mathscr{K}$ of $\mathscr{G}$ such that*

$$\bar{X}^{-1}\bar{X} \cap \mathscr{K}\mathscr{P} \subset \mathscr{K} \quad \textit{and} \quad \pi(\mathscr{K}) = A .$$

COROLLARY 2. *Let A be a normal* sub-tsl *of a* tsl *X. Then X/A is a topological loop iff for all w, $x \in X$ both*

$$\bar{x}^{-1}\bar{X}^{-1}\bar{X}\bar{x} \cap \mathscr{K}\mathscr{P} \subset \mathscr{K} \quad \textit{and} \quad \bar{w}^{-1}\bar{X}\,\bar{x}^{-1} \cap \mathscr{K}\mathscr{P} \neq \emptyset .$$

These latter two conditions are satisfied if A is a normal sub-*loop of the loop X.*

EXAMPLES. The center of the 7-sphere S^7 under Cayley multiplication is $A=\{\pm 1\}$, and $S^7/A=P^7$, the projective 7-plane, is a topological loop. Paige has shown [9] that P^7 is simple. Paige defines in [9] the 8-dimensional Cayley algebra which is not a division algebra over the real field. The multiplicative loop of elements of norm 1 has center $\{\pm 1\}$, and the quotient loop is a simple Moufang topological loop which can be shown to be a manifold homeomorphic to the direct product of the projective 3-space with a 4-plane, $P^3 \times R^4$; it is not a group. The same construction over the complex field yields a simple topological loop-manifold of dimension 14 which is a 7-plane bundle over P^7 (we have not shown the bundle to be trivial).

We remark that the corollaries above justify the definition of a tsl; the fundamental structure seems to be that of one-sided inversion. The author hopes to discuss the purely algebraic notion of semiloop in a later note.

2. Covering spaces of a topological semiloop.

We first list some definitions. Let X be a tsl with identity e. The subgroup $\mathscr{G}=\langle\bar{X}\rangle$ of G generated by $\bar{X}$ is called the *group associated with* the tsl X; clearly, no generality is lost if we assume $G=\mathscr{G}$ in the definition of a tsl X. The *inner mapping group* is the subgroup $\mathscr{P}=\pi^{-1}(e)$ of $\mathscr{G}$; $\mathscr{P}$ is closed, and π is open (since a cross-section exists). Thus X is homeomorphic to the quotient space $\mathscr{G}/\mathscr{P}$ of left cosets of $\mathscr{P}$ in $\mathscr{G}$, and $\mathscr{G}$ is homeomorphic to the topological product $\bar{X}\times\mathscr{P}$. Each element $g\in\mathscr{G}$ has a unique expression of the form $g=\bar{x}p$ for some $x\in X$, $p\in\mathscr{P}$. The product x_1x_2 of two elements of X is just that unique element of X such that $\bar{x}_1\bar{x}_2=\overline{(x_1x_2)}p$ for some $p\in\mathscr{P}$. If x^{-1} is the unique right inverse of x in X, $xx^{-1}=e$, then $x^{-1}=\pi(\bar{x}^{-1})$ and inversion is continuous. The cross-section σ is just a continuous choice of left coset representatives of $\mathscr{P}$ in $\mathscr{G}$ (compare Hudson [5]).

We remark that π is an H-map iff $\mathscr{P}$ is homotopy-normal in $\mathscr{G}$ (see [8] for definition); if so, then the above right inverse map on X is a left homotopy inverse as well, and the product in X is homotopy-associative.

We preceed the proof of Theorem 1 by a lemma which offers a sufficient condition that a homeomorphism of a base space may be lifted to a homeomorphism of its covering space. The reader is referred to Hu [6] for basic facts about covering maps.

LEMMA 1. *Let Y be connected space, X be a locally compact, locally path-connected, Hausdorff space and $\varkappa\colon Y\to X$ a regular covering map. Let $\varkappa(f)=e$, $\varkappa(y)=x$, and let g be a homeomorphism of X with $g(e)=x$. If g lies in the path-component of* 1 *in the group of homeomorphisms of X then there exists a unique homeomorphism h of Y such that $\varkappa\circ h=g\circ\varkappa$ and $h(f)=y$.*

PROOF. Since both $\varkappa$ and $\varkappa'=g^{-1}\circ\varkappa$ are covering maps, by [6, Theorem III 16.4] there exists a unique covering *map* $h\colon Y\to Y$ such that

$$\varkappa\circ h=g\circ\varkappa \quad \text{and} \quad h(f)=y$$

iff

$$\varkappa_*\pi_1(Y,f)\subset g_*{}^{-1}\varkappa_*\pi_1(Y,y)\,.$$

But $\varkappa_*\pi_1(Y,y)$ is the image of $\varkappa_*\pi_1(Y,f)$ under the translation along a path in X covered by some path from f to y in Y. Further, the effect of $g_*{}^{-1}$ is that of translation, along the image in X of the homeotopy of g with the identity map on X [7, Remark 5.21]. (Trivially, Y is locally compact, locally connected and Hausdorff whenever X is.) The composition of these two translations sends $\varkappa_*\pi_1(Y,f)$ to a conjugate of itself in $\pi_1(X,e)$, and $\varkappa$ is regular, so

$$g_*^{-1}\varkappa_*\pi_1(Y,y) = \varkappa_*\pi_1(Y,f)\,.$$

Similarly, the interchange of the roles of the covers $\varkappa$ and $\varkappa'$ yields a covering map $h'\colon Y \to Y$ such that

$$\varkappa \circ h' = g^{-1} \circ \varkappa \quad \text{and} \quad h'(y) = f\,.$$

Therefore $h'h$ is that unique map, the identity on Y, such that

$$\varkappa \circ h'h = \varkappa \quad \text{and} \quad h'h(f) = f\,,$$

and h is a homeomorphism.

Proof of Theorem 1. Let X be an H-space with multiplication $m\colon X \times X \to X$ and identity e, and let $\varkappa\colon Y \to X$ be a covering map with $\varkappa(f)=e$. Then

$$\varkappa \times \varkappa\colon \; Y \times Y \to X \times X$$

is a covering map; we wish to show that there exists a multiplication $n\colon Y \times Y \to Y$ with identity f, such that $\varkappa$ is an H-map; i.e., the following diagram commutes:

$$\begin{array}{ccc} Y \times Y & \xrightarrow{\;\;n\;\;} & Y \\ {\scriptstyle \varkappa\times\varkappa}\downarrow & & \downarrow{\scriptstyle \varkappa} \\ X \times X & \xrightarrow{\;\;m\;\;} & X \end{array}$$

But a map n will exist covering m iff

$$m_*(\varkappa \times \varkappa)_*\pi_1(Y \times Y) \subset \varkappa_*\pi_1(Y)\,.$$

And

$$(\varkappa \times \varkappa)_*\pi_1(Y \times Y) = \varkappa_*\pi_1(Y) \times \varkappa_*\pi_1(Y) \quad \text{in} \quad \pi_1(X) \times \pi_1(X);$$

since products in $\pi_1(X)$ are defined by m, m_* carries $\varkappa_*\pi_1(Y) \times \varkappa_*\pi_1(Y)$ onto $\varkappa_*\pi_1(Y)$.

For each $y \in Y$, define the map $\bar{y}\colon Y \to Y$ by $\bar{y}(z) = n(y,z)$; if $\varkappa(y) = x$ then $\bar{y}$ clearly lifts the homeomorphism $\bar{x}$ of X. Lemma 1 says that $\bar{y}$ is a homeomorphism; it applies since $\pi_1(X)$ is abelian and thus $\varkappa$ is regular.

By the exponential law, the function τ on Y into the group of all homeomorphisms of Y, $\tau(y) = \bar{y}$, is continuous; hence Y is a tsl. The uniqueness of τ is clear; Theorem 1 is proved.

PROOF OF COROLLARY 1. Let σ and $\tilde{\sigma}$ be the tsl structures for the topological loop X corresponding to the left and right multiplications of X with $\bar{x}=\sigma(x)$, $\tilde{x}=\tilde{\sigma}(x)$. Use Theorem 1 to find the tsl structures τ and $\tilde{\tau}$ on Y lifting σ and $\tilde{\sigma}$, respectively, and let $\bar{y}=\tau(y)$, $\tilde{y}=\tilde{\tau}(y)$. Now $\tilde{y}$ is the unique map on Y such that $\varkappa\circ\tilde{y}=\tilde{x}\circ\varkappa$ and $\tilde{y}(f)=y$. But define $\hat{y}\colon Y\to Y\colon \hat{y}(z)=\bar{z}(y)$; $\hat{y}$ is clearly continuous; and, if $\varkappa(y)=x$ and $\varkappa(z)=w$,

$$\varkappa\circ\hat{y}(z) = \varkappa\circ\bar{z}(y) = \overline{w}(x) = \tilde{x}(w) = \tilde{x}(\varkappa(z)) .$$

Consequently, $\varkappa\circ\hat{y}=\tilde{x}\circ\varkappa$, and also $\hat{y}(e)=\bar{e}(y)=y$; so $\hat{y}=\tilde{y}$, and Y has the multiplication of a loop. Since inversion and multiplication in Y are continuous (see remarks beginning this section), Y is a topological loop and $\varkappa$ is an open morphism of Y onto X; i.e., a quotient morphism: Corollary 1 is proved.

3. The Proof of Theorem 2.

If X is a tsl and $A\subset X$ then A is a *normal sub-tsl* of X if A is the kernel of some morphism defined on X; that is, if there exists a morphism μ: $X\to Y$ of X into some tsl Y with identity f and $A=\mu^{-1}(f)$. It is easy to show that the image of μ is a tsl; therefore we assume μ is onto, and thus Y has the quotient topology. If $\mathscr{G}=\langle\overline{X}\rangle$ and $\mathscr{H}=\langle\overline{Y}\rangle$ are the groups associated with X and Y, then μ induces a morphism of groups $\theta\colon\mathscr{G}\to\mathscr{H}$ which may be defined by

$$\theta(g)\mu(x) = \mu(g(x)) .$$

To show that θ is a well-defined function we must prove that if $\mu(x_1)=\mu(x_2)$ then $\mu(g(x_1))=\mu(g(x_2))$. But μ is a morphism; thus

$$\mu(\bar{x}(x_1)) = \overline{\mu(x)}\mu(x_1) ,$$

or $\mu\circ\bar{x}=\overline{\mu(x)}\circ\mu$. Also

$$\mu = \mu\circ\bar{x}\circ\bar{x}^{-1} = \overline{\mu(x)}\circ\mu\circ\bar{x}^{-1} ,$$

or $\mu\circ\bar{x}^{-1}=\overline{\mu(x)}^{-1}\circ\mu$; hence, if $\varepsilon=\pm 1$ then $\mu\circ\bar{x}^{\varepsilon}=\overline{\mu(x)}^{\varepsilon}\circ\mu$. Now g has an expression of the form

$$g = \bar{x}_1^{\varepsilon_1}\circ\bar{x}_2^{\varepsilon_2}\circ\ldots\circ\bar{x}_n^{\varepsilon_n},$$

and

$$\mu\circ g(x) = \mu\circ\bar{x}_1^{\varepsilon_1}\circ\ldots\circ\bar{x}_n^{\varepsilon_n} = \overline{\mu(x_1)}^{\varepsilon_1}\circ\ldots\circ\overline{\mu(x_n)}^{\varepsilon_n}\circ\mu(x);$$

this shows θ to be well defined. That θ preserves products is trivial:

$$\begin{aligned}\theta(g)\circ\theta(h)\circ\mu(x) &= \theta(g)\circ\mu\circ h(x)\\ &= \mu\circ g\circ h(x) = \theta(g\circ h)\circ\mu(x) .\end{aligned}$$

And θ is continuous since the action of $\mathscr{G}$ on Y, just as that of $\mathscr{G}$ on X, is admissible. The kernel $\mathscr{K}$ of the morphism θ is thus a closed normal subgroup of $\mathscr{G}$. If $k \in \mathscr{K}$ then $\mu \circ k(x) = \mu(x)$; in particular,

$$\mu \circ k(e) = \mu \circ \pi(k) = f \quad \text{so} \quad \pi(k) \in A\,.$$

Conversely, $a \in A$ implies $\theta(\bar{a}) \circ \mu(x) = \overline{\mu(a)}\,\mu(x)$ and thus $a \in \pi(\mathscr{K})$; hence $\pi(\mathscr{K}) = A$. Now let $g \in \bar{X}^{-1}\bar{X} \cap \mathscr{K}\mathscr{P}$, so g has the forms $g = \bar{x}^{-1}\bar{x}' = kp$. Then

$$\mu \circ g = \mu \circ \bar{x}^{-1}\bar{x}' = \bar{y}^{-1}\bar{y}' \circ \mu \quad \text{for} \quad y, y' \in Y\,,$$

and

$$\mu \circ g(e) = \mu \circ kp(e) = \mu \circ k(e) = f = \bar{y}^{-1}\bar{y}'\mu(e)\,.$$

Hence

$$y' = \bar{y}\bar{y}^{-1}\bar{y}'(f) = \bar{y}(f) = y$$

and so

$$\mu \circ g = \mu \quad \text{and} \quad g \in \mathscr{K}\,.$$

Furthermore, θ is continuous onto the Hausdorff space $\mathscr{H}$, so $\mathscr{K} = \theta^{-1}(1)$ is closed.

Conversely, let X be a tsl with associated group $\mathscr{G}$, and let $\mathscr{K}$ be a closed normal subgroup of $\mathscr{G}$. Let θ, η and μ be the natural maps defined by the following commutative diagram

$$\begin{array}{ccccc}
\mathscr{G} & \xrightarrow{\ \theta\ } & \mathscr{G}/\mathscr{K} & \xrightarrow{\ \varphi\ } & \mathscr{H} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \varrho} \\
X = \mathscr{G}/\mathscr{P} & \xrightarrow[\ \mu\ ]{} & \mathscr{G}/\mathscr{K}\mathscr{P} & = & Y
\end{array}$$

We identify X with $\mathscr{G}/\mathscr{P}$, define $Y = \mathscr{G}/\mathscr{K}\mathscr{P}$, and let $\mathscr{H}$ be the group $\mathscr{G}/\mathscr{K}$ of homeomorphisms of Y furnished with the compact-open topology, so that φ is the identity function on the set $\mathscr{G}/\mathscr{K}$. Now notice that $\bar{X}$ is closed in $\mathscr{G} = \bar{X} \times \mathscr{P}$, and hence

$$\bar{A} = \mathscr{K} \cap \bar{X}, \qquad A = \sigma^{-1}(\bar{A}), \qquad \mathscr{K}\mathscr{P} = \pi^{-1}(A)$$

are all closed. Thus $Y = \mathscr{G}/\mathscr{K}\mathscr{P}$ is Hausdorff (the relation on $\mathscr{G}$ of belonging to the same coset of $\mathscr{K}\mathscr{P}$ is closed in $\mathscr{G} \times \mathscr{G}$). The map $\mu = \eta \circ \theta \circ \pi^{-1}$ is open; therefore Y is locally compact and locally connected, being the continuous open image of a space X having these properties. Since $\mathscr{G}$ is a topological transformation group on Y, and $\mathscr{K}$ is in the kernel of this action, $\mathscr{G}/\mathscr{K}$ also acts admissibly on Y. But the compact-open topology is the smallest on $\mathscr{G}/\mathscr{K}$ for which this is true; thus φ is continuous, and so is ϱ.

Now the cross-section σ of X in $\mathscr{G}$ induces the cross-section $\tau\colon Y \to \mathscr{H}$:

$$\tau = \varphi \circ \theta \circ \sigma \circ \mu^{-1} \quad \text{and} \quad \tau(\bar{x}\mathscr{K}\mathscr{P}) = \bar{x}\mathscr{K} \in \mathscr{H} .$$

Since $\mathscr{G}=\bar{X}\mathscr{P}$, each $y \in Y$ has the form $y=\bar{x}\mathscr{K}\mathscr{P}$ for some $\bar{x} \in \bar{X}$; if also $y=\bar{x}'\mathscr{K}\mathscr{P}$ then $\bar{x}^{-1}\bar{x}' \in \mathscr{K}\mathscr{P}$ and the condition on $\mathscr{K}$ that $\bar{X}^{-1}\bar{X} \cap \mathscr{K}\mathscr{P} \subset \mathscr{K}$ implies $\bar{x}' \in \bar{x}\mathscr{K}$: consequently τ is well defined. The continuity of τ is obvious, as are the facts that $\varrho \circ \tau = 1 \in \mathscr{H}$ and $\tau(1\mathscr{K}\mathscr{P})=\mathscr{K}$, the identity in $\mathscr{H}$. Thus Theorem 2 is proved.

Proof of Corollary 2. Since X/A is a tsl, and thus T_2, locally compact and locally connected, the right inversion function will be continuous if it is defined at all. But right multiplication by $\bar{x}\mathscr{K}\mathscr{P}$ is onto iff for all w there is a v with

$$\bar{v}\mathscr{K}\bar{x}\mathscr{K} \subset \bar{w}\mathscr{K}\mathscr{P}$$

iff

$$\bar{v}\bar{x} \in \bar{w}\mathscr{K}\mathscr{P}$$

iff

$$\bar{v} \in \bar{w}\mathscr{K}\mathscr{P}\bar{x}^{-1}$$

iff

$$\bar{X} \cap \bar{w}\mathscr{K}\mathscr{P}\bar{x}^{-1} \neq \emptyset .$$

Similarly, right multiplication by $\bar{x}\mathscr{K}\mathscr{P}$ is one to one iff, for all w in X,

$$\bar{v}\mathscr{K}\bar{x}\mathscr{K} \subset \bar{w}\mathscr{K}\mathscr{P} \quad \text{and} \quad \bar{u}\mathscr{K}\bar{x}\mathscr{K} \subset \bar{w}\mathscr{K}\mathscr{P}$$

implies $\bar{u} \in \bar{v}\mathscr{K}$. That is,

$$\text{iff } \bar{x}^{-1}\bar{u}^{-1}\bar{v}\bar{x} \in \mathscr{K}\mathscr{P} \quad \text{implies} \quad \bar{u}^{-1}\bar{v} \in \mathscr{K};$$

equivalently,

$$\text{iff } \bar{x}^{-1}\bar{X}^{-1}\bar{X}\bar{x} \cap \mathscr{K}\mathscr{P} \subset \bar{x}^{-1}\mathscr{K}\bar{x} = \mathscr{K} .$$

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UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA, U.S.A.