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ON GAUSSIAN MEASURES EQUIVALENT TO WIENER MEASURE II

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1. Introduction.

Let P_r be a Gaussian probability measure determined by a covariance function $r(s,t)$ (see [2, pp. 71–74] but note that we are assuming the mean function to be zero). Similarly, let P_{w_σ} denote the familiar Wiener measure determined by the covariance function $w_\sigma(s,t) \equiv \sigma^2 \min(s,t)$. The problem is to determine *simple* and *easily checked* conditions on r which guarantee that P_r and P_{w_σ} are equivalent (mutually absolutely continuous with respect to each other). An earlier paper of the author [12] offered one solution to this problem though it may be questioned whether the conditions given there are simple and easily checked. Be that as it may, the paper did suggest a way of attacking the problem—by studying the linear transformations of the Wiener process and especially how Wiener measure transforms under them. In the present paper, we pursue this route further and arrive at a solution that certainly meets the criterion that we have set up. We show (Theorem 8) that sufficient conditions for equivalence are that

(i) $r(0,t)=0$,

(ii) $\sigma^{-2}\, \partial^2[r(s,t)-w_\sigma(s,t)]/\partial t \partial s$ exists, is square integrable and has smallest eigenvalue greater than -1.

In § 2–§ 4, we introduce the material that is required to establish the main results on equivalence which follow in § 5. In particular, in § 2 we solve a special nonlinear integral equation, in § 3 we study a representation problem for Gaussian processes and in § 4 we generalize Woodward's theorem on the transformation of Wiener integrals. Our theorems are stated in pairs. The first theorem in a pair assumes that a certain kernel is of bounded variation and allows a strong conclusion while the second assumes only that this kernel is square integrable and conse-

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quently offers a somewhat weaker conclusion. The final section of the paper is devoted to examples.

2. An integral equation.

Let $\{C_0, B, P_{w_\sigma}\}$ be the Wiener process, i.e., let $C_0 \equiv C_0(0,b)$ be the class of all continuous real valued functions x defined on $I \equiv [0,b]$ with $x(0)=0$, let B be the smallest Borel field of subsets of C_0 with respect to which $x(t)$ is measurable for each (fixed) $t \in I$ and let P_{w_σ} be the Gaussian probability measure on B determined by the covariance function

$$w_\sigma(s,t) \equiv \sigma^2 w(s,t) \equiv \sigma^2 \min(s,t) .$$

We assume that the mean function is identically zero.

There are good reasons for believing that all Gaussian processes whose measures are equivalent to Wiener measure may be obtained as linear transformations of the Wiener process of the form

$$y(t) = x(t) + \int_0^b \int_0^t K(u,s)\, du\, dx(s) . \tag{2.0}$$

[In fact, in private correspondance with the author, Hiroshi Sato has stated that this is a theorem which he has proved.] A major problem and one which we shall solve in § 3 is therefore: to determine conditions on $r(s,t)$ which guarantee that the corresponding process $\{y(t), 0 \leqq t \leqq b\}$ has the representation (2.0). Now if (2.0) holds, then

$$\begin{aligned} r(s,t) &= E\{y(s)y(t)\} \\ &= E\left\{x(s)x(t) + \int_0^b\int_0^t x(s)K(u,v)\, du\, dx(v) + \right. \\ &\quad \left. + \int_0^b\int_0^s x(t)K(w,z)\, dw\, dx(z) + \int_0^b\int_0^b\int_0^s\int_0^t K(u,v)K(w,z)\, du\, dw\, dx(v)\, dx(z)\right\} \\ &= \sigma^2\left[w(s,t) + \int_0^s\int_0^t K(u,v)\, du\, dv + \int_0^t\int_0^s K(w,z)\, dw\, dz + \right. \\ &\quad \left. + \int_0^b\int_0^s\int_0^t K(u,v)K(w,v)\, du\, dw\, dv\right] . \end{aligned}$$

Therefore

$$\sigma^{-2}\, \partial^2[(s,t) - w_\sigma(s,t)]/\partial t \partial s = K(t,s) + K(s,t) + \int_0^b K(s,v)K(t,v)\, dv . \tag{2.1}$$

If we assume that $K(s,t)$ is symmetric and let $F(s,t)$ denote the left side of (2.1), then this equation takes the form

$$(2.2)\qquad 2K(s,t) = F(s,t) - \int_0^b K(s,u)K(t,u)\,du$$

which we shall study as an integral equation in K with F regarded as a known function.

Our first attempt at a solution of (2.2) will be by means of Picard's method of successive approximations. Accordingly, let $K_0(s,t)=0$ and let

$$(2.3)\qquad 2K_n(s,t) = F(s,t) - \int_0^b K_{n-1}(s,u)K_{n-1}(t,u)\,du$$

for $n=1,2,\ldots$.

Following [6, p. 148], let $\boldsymbol{L}^2$ denote the space of real valued functions K on $I\times I$ for which

$$|K| \equiv \left[\int_0^b\int_0^b K^2(s,t)\,ds\,dt\right]^{\frac12} < \infty ,$$

and let L^2 be the space of real valued functions f on I for which

$$\|f\| \equiv \left[\int_0^b f^2(s)\,ds\right]^{\frac12} < \infty .$$

Lemma 1. *Let $F(\cdot,\cdot)\in \boldsymbol{L}^2$ with $m\equiv|F|\leqq 1$. Let*

$$B(s) = \|F(s,\cdot)\| = \left[\int_0^b F^2(s,t)\,dt\right]^{\frac12}$$

which by Fubini's theorem is finite for almost all s. If $K_n(s,t)$ is as above, then

$$(2.4)\qquad |K_n| \leqq m ,$$

$$(2.5)\qquad \|K_n(s,\cdot)\| \leqq B(s) ,$$

$$(2.6)\qquad |K_{n+1}(s,t)-K_n(s,t)| \leqq B(s)B(t)\,m^{n-1} ,$$

these inequalities holding for $n=1,2,\ldots$.

These inequalities may be proved by induction in the order given using the triangle inequality for norms and Schwarz's inequality at appropriate places.

Before stating our first theorem, we need to recall a definition. Let BVH denote the class of function K on $I \times I$ which are of bounded variation in the sense of Hardy–Krause. Thus $K \in \text{BVH}$ if

(i) there exists $(s_0, t_0) \in I \times I$ such that $K(s_0, t)$ and $K(s, t_0)$ are of bounded variation on I,

(ii) $\text{Var}_{I\times I} K(s,t) \equiv \sup \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta_{ij}(K) < \infty$

where

$$\Delta_{ij}(K) = |K(s_i,t_j) - K(s_i,t_{j-1}) + K(s_{i-1},t_{j-1}) - K(s_{i-1},t_j)| .$$

We remark that $K \in \text{BVH}$ implies that both $\text{Var}_{s\in I} K(s,t)$ and $\text{Var}_{s\in I} K(t,s)$ exist for each t and are bounded as functions of t on I.

THEOREM 1. *Let $F \in \text{BVH}$ and be symmetric with $|F| < 1$. Then there exists $K \in \text{BVH}$ satisfying* (2.2) *everywhere.*

PROOF. Since $F \in \text{BVH}$ it is bounded and hence $B(s) \equiv \|F(s,\cdot)\|$ is finite for all s. Thus by (2.6) of Lemma 1

$$|K_2(s,t) - K_1(s,t)| + |K_3(s,t) - K_2(s,t)| + \dots \\ \leqq B(s)B(t)[1 + m + m^2 + \dots] < \infty .$$

Hence

$$K(s,t) \equiv \lim_{n\to\infty} K_n(s,t) = K_1(s,t) + [K_2(s,t) - K_1(s,t)] + \dots$$

exists everywhere on $I \times I$. That K satisfies (2.2) everywhere follows by taking limits in (2.3). We must show that $K \in \text{BVH}$.

Choose a constant C such that $\text{Var}_{I\times I} F(s,t) \leqq C$ and $\text{Var}_{s\in I} F(s,t) \leqq C$. It follows as we shall show presently that

(i) $\|\text{Var}_{s\in I} K_n(s,\cdot)\| \leqq Cb^{\frac{1}{2}}$,

(ii) there exists $t_0 \in I$ such that $\text{Var}_{s\in I} K_n(s,t_0) \leqq C$,

(iii) $\text{Var}_{I\times I} K_n(s,t) \leqq \frac{1}{2}[C + C^2 b]$,

each of these inequalities holding for $n = 1, 2, \dots$. Assuming these facts for the moment, we note that

$$\sum_{i=1}^{m} \sum_{j=1}^{m'} \Delta_{ij}(K) = \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{m'} \Delta_{ij}(K_n) \leqq \liminf_{n\to\infty} \operatorname*{Var}_{I\times I} K_n(s,t)$$

so that $\text{Var}_{I\times I} K(s,t) \leqq \frac{1}{2}[C + C^2 b]$. Similarly

$$\operatorname*{Var}_{s\in I} K(t_0,s) = \operatorname*{Var}_{s\in I} K(s,t_0) \leqq \liminf_{n\to\infty} \operatorname*{Var}_{s\in I} K_n(s,t_0) \leqq C$$

from which we conclude that $K \in \text{BVH}$.

We proceed to the proofs of (i)–(iii). (i) follows easily by induction. To see (ii), we note first that there exists $t_0 \in I$ such that $\|F(\cdot,t_0)\| \leqq b^{-\frac{1}{2}}$

for otherwise $|F|>1$. Using this and the symmetry of $K_n(s,t)$, it follows by induction that $\|K_n(\cdot,t_0)\| \leqq b^{-\frac{1}{2}}$, $n=1,2,\ldots$. Thus

$$\begin{aligned}
2 \operatorname*{Var}_{s\in I} K_n(s,t_0) &= \operatorname*{Var}_{s\in I}\Big[F(s,t_0) - \int_0^b K_{n-1}(s,u)K_{n-1}(t_0,u)\,du\Big] \\
&\leqq \operatorname*{Var}_{s\in I} F(s,t_0) + \int_0^b \operatorname*{Var}_{s\in I} K_{n-1}(s,u)\,|K_{n-1}(t_0,u)|\,du \\
&\leqq C + \|\operatorname*{Var}_{s\in I} K_{n-1}(s,\cdot)\|\,\|K_{n-1}(t_0,\cdot)\| \\
&\leqq C + Cb^{\frac{1}{2}}b^{-\frac{1}{2}} \\
&= 2C\,.
\end{aligned}$$

Finally

$$\begin{aligned}
2 \operatorname*{Var}_{I\times I} K_{n+1}(s,t) &= \operatorname*{Var}_{I\times I}\Big[F(s,t) - \int_0^b K_n(s,u)K_n(t,u)du\Big] \\
&\leqq C + \sup \sum_{i=1}^{m}\sum_{j=1}^{m'} \int_0^b |K_n(s_i,u)-K_n(s_{i-1},u)|\,|K_n(t_j,u)- \\
&\qquad\qquad - K_n(t_{j-1},u)|\,du \\
&\leqq C + \int_0^b [\operatorname*{Var}_{s\in I} K_n(s,u)]^2\,du \\
&\leqq C + C^2 b
\end{aligned}$$

which establishes (iii) and completes the proof of the theorem.

The norm condition of Theorem 1 is a severe one but we have been unable to remove it. However, we can do somewhat better if we look for solutions of our integral equation in L^2 rather than BVH. To that end, let $\{\varphi_k,\lambda_k\}$ be an eigen system for the symmetric kernel $F(s,t)$, (see [10, pp. 112–115] for the definition of what is there called a characteristic system but note that our eigen values are the reciprocals of the characteristic values used there, i.e.,

$$\lambda_k\varphi_k(t) = \int_0^b F(t,s)\varphi_k(s)\,ds\,.$$

Finally, let

$$\lambda_1^- = \inf_{\|x\|=1} \int_0^b\int_0^b F(t,s)\,x(t)\,x(s)\,ds\,dt\,. \tag{2.7}$$

This is equivalent to saying that λ_1^- is the smallest negative eigen value ($\lambda_1^-=0$ if there are no negative eigen values).

THEOREM 2. *Let $F \in L^2$ and be symmetric with $\lambda_1^- \geqq -1$. Then there is a function $K \in L^2$ satisfying* (2.2) *almost everywhere.*

PROOF. We know [10, p. 115] that we may expand F in the mean convergent series

$$F(s,t) = \sum_j \lambda_j \varphi_j(s) \varphi_j(t) .$$

Let

$$K(s,t) = \sum_j \mu_j \varphi_j(s) \varphi_j(t) ,$$

where $\mu_j = -1 + (1+\lambda_j)^{\frac{1}{2}}$ and note that μ_j is real since $\lambda_1^- \geqq -1$. The convergence of $\sum \lambda_j^2$ guarantees that of $\sum \mu_j^2$ and hence the mean convergence of the series for K. Moreover

$$\int_0^b K(s,u)K(t,u)\, du = \sum_j \mu_j^2 \varphi_j(s\varphi)_j(t) .$$

The result now follows from the fact that $2\mu_j = \lambda_j - \mu_j^2$.

3. The representation problem.

THEOREM 3. *Let $r(s,t)$ be a covariance function satisfying*

(3.0) $$r(0,t) = 0 \textit{ for } t \in I \equiv [0,b] ,$$

(3.1) $$F(s,t) = \sigma^{-2}\, \partial^2[r(s,t) - w_\sigma(s,t)]/\partial t \partial s \textit{ exists on } I \times I ,$$

(3.2) $$F \in \mathrm{BVH} \textit{ and } |F| < 1 .$$

Then $r(s,t)$ determines a Gaussian process $\{y(t), 0 \leqq t \leqq b\}$ which has the representation

(3.3) $$y(t) = x(t) + \int_0^b \int_0^t K(u,s)\, du dx(s)$$

where K is the solution of (2.2) *guaranteed by Theorem* 1 *and $\{x(t), 0 \leqq t \leqq b\}$ is the Wiener process with covariance function $w_\sigma(s,t)$.*

PROOF. If we define a Gaussian process $y(t)$ by (3.3), it follows from the calculation at the beginning of section 2 that its covariance function, call it $\varrho(s,t)$, is given by

$$\varrho(s,t) = w_\sigma(s,t) + \sigma^2 \int_0^s \int_0^t 2K(v,w) + \int_0^b K(v,u)K(w,u)\, du\, dw dv$$

$$= w_\sigma(s,t) + \sigma^2 \int_0^b \int_0^t F(v,w)\, dw dv .$$

But using (3.0) and (3.1) it is easy to show that

$$\sigma^2 \int_0^s\int_0^t F(v,w)\, dw dv = r(s,t) - w_\sigma(s,t)$$

and thence that $\varrho(s,t)) = r(s,t)$ from which the theorem follows.

We remark that in the result above there is no difficulty with the interpretation of the integral

$$J \equiv \int_0^b\int_0^t K(u,s)\, du dx(s) .$$

The fact that $K \in \mathrm{BVH}$ implies that $\int_0^t K(u,s)du$ is of bounded variation in s and so J exists as a standard Riemann–Stieltjes integral. If we require only that $K \in \boldsymbol{L}^2$, this is no longer true. However, J still exists as a stochastic integral in the sense of Doob [2, pp. 430–431]. If K is symmetric, this stochastic integral may be represented in a convenient way as an infinite series. To be specific, let $\{\varphi_j, \mu_j\}$ be an eigen system for K and let

$$N_j(x) = \int_0^b \varphi_j(s)\, dx(s) , \tag{3.4}$$

this integral also being interpreted as a stochastic integral [2, pp. 427–428]. Now $\{N_j(x)\}$ forms a sequence of independent Gaussian variables each with zero mean and variance σ^2 and so $\{N_j^2(x) - \sigma^2\}$ is a sequence of independent random variables each with zero mean and variance $3\sigma^2$. By a well known theorem of Kolmogorov [4, p. 236], it follows that $\sum \mu_j^2 [N_j^2(x) - \sigma^2]$ converges except on a null set N of C_0 and hence so does $\sum \mu_j^2 N_j^2(x)$. Excluding N, the Riesz–Fisher theorem guarantees that $\sum \mu_j N_j(x) \varphi_j(u)$ converges in mean to a function in L^2 (and hence in L^1). It is not difficult to show that for almost all $x \in C_0$

$$\begin{aligned} \int_0^b\int_0^t K(u,s)\, du dx(s) &= \int_0^t \sum_j \mu_j N_j(x) \varphi_j(u)\, du \\ &= \sum_j \mu_j N_j(x) \int_0^t \varphi_j(u)\, du . \end{aligned} \tag{3.5}$$

With the interpretation of the Stieltjes integral, we may state the following generalization of Theorem 3.

THEOREM 4. *Let $r(s,t)$ be a covariance function satisfying* (3.0), (3.1), $F \in \boldsymbol{L}^2$ *and* $\lambda_1^- \geqq -1$. *Then $r(s,t)$ determines a Gaussian process $\{y(t), 0 \leqq t \leqq b\}$ which has the representation*

$$y(t) = x(t) + \int_0^b \int_0^t K(u,s)\, du dx(s),$$

where K is the solution to (2.2) *guaranteed by Theorem* 2 *and* $\{x(t), 0 \leqq t \leqq b\}$ *is the Wiener process.*

PROOF. Let φ_j, λ_j and μ_j be as in the proof of Theorem 2 and $N_j(x)$ as in (3.4). Then

$$E\{x(t)N_j(x)\} = \sigma^2 \int_0^t \varphi_j(u)\,du$$

and

$$E\{y(s)y(t)\} =$$

$$= \lim_{n\to\infty} E\left\{x(s)x(t) + \sum_{j=1}^n \mu_j x(s) N_j(x) \int_0^t \varphi_j(u)\,du + \right.$$

$$\left. + \sum_{j=1}^n \mu_j x(t)\, N_j(x) \int_0^s \varphi_j(u)\,du + \sum_{j,k=1}^n \mu_j \mu_k N_j(x) N_k(x) \int_0^s \int_0^t \varphi_j(u)\varphi_k(v)\,dv du \right\}$$

$$= w_\sigma(s,t) + \lim_{n\to\infty} \sum_{j=1}^n \sigma^2(2\mu_j + \mu_j^2) \int_0^s \int_0^t \varphi_j(u)\varphi_j(v)\,dv du$$

$$= w_\sigma(s,t) + \sigma^2 \int_0^s \int_0^t F(u,v)\,dv du = r(s,t)\,.$$

We remark that Theorem 4 yields as a byproduct sufficient conditions on r which insure that it determines a process with continuous sample functions. Moreover, the result thus obtained does not follow from (or even overlap) the general results of Loeve [4, p. 520] concerning this matter.

4. Woodward's theorem and a generalization.

We begin by recalling the definition of the Fredholm determinant for a kernel K.

$$(4.0)\quad d(\lambda; K) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_0^b \cdots \int_0^b \begin{vmatrix} K(s_1, s_1) \ldots K(s_1, s_n) \\ \cdots \qquad \cdots \\ K(s_n, s_1) \ldots K(s_n, s_n) \end{vmatrix} ds_1 \ldots ds_n$$

$d(\lambda; K)$ always exists for a kernel $K \in \mathrm{BVH}$ but unfortunately may not exist for $K \in L^2$ since $K(s,s)$ may not even be measurable in this case. However if we let $K^*(s,t) = K(s,t)$ for $s \neq t$ and $K^*(s,s) = 0$, then $d(\lambda; K^*)$

does exist and so we define the modified Fredholm determinant $\delta(\lambda; K)$ by $\delta(\lambda; K) = d(\lambda; K^*)$. We mention that if $\{\varphi_j, \mu_j\}$ is an eigen system for K, then [1, p. 217]

$$\delta(\lambda; K) = \prod_j (1+\lambda\mu_j)\exp(-\lambda\mu_j) . \tag{4.1}$$

We state now a special case of Woodward's theorem (see [13] and also Theorems 2 and 3 of [12]).

THEOREM 5 (Woodward). *Let* $K \in \mathrm{BVH}$ and $d(-1; K) \neq 0$. *Then the transformation of the Wiener process defined by*

$$(Tx)(t) = x(t) + \int_0^b \int_0^t K(u,s)\, du\, dx(s)$$

is 1–1 *from* C_0 *onto* C_0. *Moreover if* G *is integrable,*

$$E\{G(x)\} = |d(-1; K)|\, E\{G(Tx)\Phi(x; K)\} \tag{4.2}$$

where

$$\Phi(x; K) = \tag{4.3}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\int_0^b\int_0^b \left[K(s,t)+K(t,s)+\int_0^b K(u,s)K(u,t)\,du\right] dx(s)\,dx(t)\right\}.$$

Alternatively if H *is integrable,*

$$E\{H(Tx)\} = |d(-1; K)|^{-1} E\{H(x)\Phi(x; K^{-1})\}, \tag{4.4}$$

where K^{-1} *is the Volterra reciprocal kernel corresponding to* K, *that is, a kernel satisfying*

$$K^{-1}(s,t)+K(s,t) = -\int_0^b K^{-1}(s,u)K(u,t)\,du = -\int_0^b K(s,u)K^{-1}(u,t)\,du . \tag{4.5}$$

Our plan is to generalize this theorem to an arbitrary symmetric kernel $K \in \boldsymbol{L}^2$. Accordingly, let $\{\varphi_j, \mu_j\}$ be an eigen system for K and $N_j(x)$ as in (3.4). Then formally

$$d(-1; K) = \delta(-1; K)\exp\left(\sum_j \mu_j\right)$$

and

$$\Phi(x; K) = \exp\left[-\frac{1}{2\sigma^2}\sum_j (2\mu_j+\mu_j^2)N_j^2(x)\right].$$

Neither of these expressions need exist. However if we remove the second factor from $d(-1; K)$ and insert it in $\Phi(x; K)$, both become meaningful. More explicitly, let us define

(4.6) $$\Psi(x; K) = \exp\left\{-\frac{1}{2\sigma^2}\sum_j [2\mu_j(N_j^2(x)-\sigma^2)+\mu_j^2 N_j^2(x)]\right\}.$$

Applying the theorem of Kolmogorov referred to earlier [4, p. 236] and using the fact that $\sum \mu_j^2$ converges, we see that $\Psi(x; K)$ is well defined for almost all $x \in C_0$. Thus, we are led to

THEOREM 6. *Let $K \in L^2$ and be symmetric and suppose that $\delta(-1; K) \neq 0$. Then the transformation of the Wiener process defined by*

$$(Tx)(t) = x(t) + \int_0^b\int_0^t K(u,s)\,du\,dx(s)$$

(see (3.5)) is essentially 1-1 from C_0 onto C_0. Moreover if G is integrable,

(4.7) $$E\{G(x)\} = \delta(-1; K)\, E\{G(Tx)\,\Psi(x; K)\}\,.$$

Alternatively if H is integrable,

(4.8) $$E\{H(Tx)\} = \delta(-1; K^{-1})\, E\{H(x)\,\Psi(x; K^{-1})\}$$

where

$$K^{-1}(s,t) = -\sum_j \frac{\mu_j}{1+\mu_j}\varphi_j(s)\varphi_j(t)\ ,$$

$\{\varphi_j, \mu_j\}$ being an eigen system for K.

This theorem is closely related to a more abstract result in [9] which however is not formulated correctly (see [8, p. 465]). Furthermore, it presumably can be made to follow from [7, Theorem 3] where the general setting is quite different though when properly looked at equivalent to ours (see [8]). Instead of trying to base the proof of Theorem 6 on these papers, we have chosen to rely on Woodward's theorem (which we wanted anyway for Theorem 7) and standard measure theory arguments.

PROOF OF THEOREM 6. By saying that T is essentially 1–1 from C_0 onto C_0, we mean that there exists a transformation T^{-1} satisfying $T^{-1}Tx = TT^{-1}x = x$ for almost all $x \in C_0$. That this is the case follows from the formula

$$(T^{-1}x)(t) = x(t) - \int_0^t \sum_j \frac{\mu_j}{1+\mu_j} N_j(x)\,\varphi_j(u)\,du\ .$$

We shall prove our theorem first for K of finite rank. More precisely, let

$$K(s,t) = \sum_{j=1}^N \mu_j\varphi_j(s)\varphi_j(t)$$

where the φ_j's are orthonormal and belong to L^2. For each j, let $\{\varphi_{jn}\}$ be a sequence of functions with continuous derivatives such that

$$\varphi_j(s) = \underset{n\to\infty}{\text{l.i.m.}}\, \varphi_{jn}(s)$$

and let

$$K_n(s,t) = \sum_{j=1}^{N} \mu_j \varphi_{jn}(s)\varphi_{jn}(t) .$$

Then $K_n \in \mathrm{BVH}$ and

$$d(-1; K_n) = \delta(-1; K_n) \exp\left[\int_0^b K_n(s,s)\,ds\right] = B_n \delta(-1; K_n) \exp\left(\sum_{j=1}^{N} \mu_j\right)$$

where

$$B_n = \exp\left[\int_0^b K_n(s,s)\,ds - \sum_{j=1}^{N} \mu_j\right].$$

When $n \to \infty$,

$$B_n \to 1, \quad \delta(-1; K_n) \to \delta(-1; K)$$

and

$$\exp\left(\sum_{j=1}^{N} \mu_j\right) \Phi(x; K_n) \to \Psi(x; K) ,$$

these facts being trivial to verify except possibly for the one about δ for which the reader may see [1].

Next let χ be the set characteristic function of the quasi-interval

$$A = \{x \in C_0 \colon \; x(t_m) < a_m, \; m = 1, 2, \ldots, M\}$$

and let

$$\chi_\varepsilon(x) = \prod_{m=1}^{M} \chi_{\varepsilon m}(x) ,$$

where

$$\chi_{\varepsilon m}(x) = \begin{cases} 1 & x(t_m) \leqq a_m - \varepsilon , \\ [a_m - x(t_m)]/\varepsilon & a_m - \varepsilon \leqq x(t_m) \leqq a_m , \\ 0 & a_m \leqq x(t_m) , \end{cases}$$

so that $\chi_\varepsilon(x) \uparrow \chi(x)$ as $\varepsilon \to 0^+$. Finally, let

$$\Lambda_\varepsilon(x) = \begin{cases} 1 & \sup_n [\Phi(x; K_n)] \leqq 1/\varepsilon , \\ 0 & \text{otherwise} , \end{cases}$$

and note that $\Lambda_\varepsilon(x) \uparrow 1$ as $\varepsilon \to 0^+$.

Now $\delta(-1; K) \neq 0$ implies that $d(-1; K_n) \neq 0$ for large n and so we may apply Woodward's theorem to the kernel K_n with T_n defined in the obvious manner. We obtain

$$E\{\chi_\varepsilon(x)\} = |d(-1;K_n)|\, E\{\chi_\varepsilon(T_n x)\Phi(x;K_n)\}$$
$$\geqq B_n|\delta(-1;K_n)|\, E\left\{\Lambda_\varepsilon(x)\chi_\varepsilon(T_n x)\exp\left(\sum_{j=1}^{N}\mu_j\right)\Phi(x;K_n)\right\}.$$

In this inequality let $n \to \infty$ and $\varepsilon \to 0^+$ in that order noting that we may pass the limits inside the expected values by bounded convergence and monotone convergence respectively. The results is

$$E\{\chi(x)\} \geqq |\delta(-1;K)|\, E\{\chi(Tx)\Psi(x;K)\}. \tag{4.9}$$

We observe next that

$$0 < E\{\Psi(x;K)\} = [\delta(-1;K)]^{-1}$$

as may be checked by direct calculation using the fact that $N_1, N_2, \ldots, N_N$ are independent Gaussian variables with means zero and variances σ^2. Hence applying (4.9) to $1-\chi$, we have

$$\begin{aligned} 1 - E\{\chi(x)\} &= E\{1-\chi(x)\} \\ &\geqq \delta(-1;K)\, E\{[1-\chi(Tx)]\Psi(x;K)\} \\ &= 1 - \delta(-1;K)\, E\{\chi(Tx)\Psi(x;K)\}. \end{aligned}$$

Thus

$$E\{\chi(x)\} \leqq \delta(-1;K)\, E\{\chi(Tx)\Psi(x;K)\}$$

from which we conclude that (4.9) is acutally an equality. This equality readily extends to arbitrary measurable set characteristic functions and thence to integrable functions. Hence we have proved (4.7), however, so far only for certain kernels of finite rank.

Moving on to the general symmetric L^2 kernel K, we know that it has the mean convergent expansion

$$K(s,t) = \sum_{j=1}^{\infty}\mu_j\varphi_j(s)\varphi_j(t),$$

where $\{\varphi_j, \mu_j\}$ is its eigen system. In other words, our general K may be approximated in mean by the kernels of finite rank that we have just studied.

We find it convenient to shift our notation and let $K_N(s,t)$ now be defined by

$$K_N(s,t) = \sum_{j=1}^{N}\mu_j\varphi_j(s)\varphi_j(t).$$

The corresponding transformation T_N is then

$$(T_N x)(t) = x(t) + \int_0^b \int_0^t K_N(u,s)\,du\,dx(s)$$

$$= x(t) + \int_0^t \sum_{j=1}^N \mu_j N_j(x)\varphi_j(u)\,du$$

and

$$(T_N^{-1}x)(t) = x(t) - \int_0^t \sum_{j=1}^N \frac{\mu_j}{1+\mu_j} N_j(x)\varphi_j(u)\,du\,.$$

Let χ and χ_ε be as originally introduced and let

$$\Gamma_\varepsilon(x;K) = \begin{cases} 1 & \Psi(x;K) \leqq (1-\varepsilon)/\varepsilon\,, \\ 1-a & \Psi(x;K) = a+(1-\varepsilon)/\varepsilon, \quad 0 \leqq a \leqq 1\,, \\ 0 & \Psi(x;K) \geqq 1/\varepsilon\,, \end{cases}$$

so that $\Gamma_\varepsilon(x;K)\uparrow 1$ as $\varepsilon \to 0^+$. One may show that

$$\Psi(T_N^{-1}x;K_N) \to \Psi(T^{-1}x;K)$$

and consequently that

$$\Gamma_\varepsilon(T_N^{-1}x;K_N) \to \Gamma_\varepsilon(T^{-1}x;K) \text{ as } N \to \infty\,.$$

Also

$$\Gamma_\varepsilon(x;K_N) \to \Gamma_\varepsilon(x;K) \qquad \text{and} \qquad \delta(-1;K_N) \to \delta(-1;K)\,.$$

Thus, letting $N \to \infty$ and $\varepsilon \to 0^+$ in the already proved result

$$E\{\chi_\varepsilon(x)\Gamma_\varepsilon(T_N^{-1}x;K_N)\} = \delta(-1;K_N)\,E\{\chi_\varepsilon(T_N x)\Gamma_\varepsilon(x;K_N)\Psi(x;K_N)\}$$

(obtained after letting $G(x)=\chi_\varepsilon(x)\Gamma_\varepsilon(T_N^{-1}x;K_N)$ in (4.7)), we have by bounded convergence and monotone convergence

$$E\{\chi(x)\} = \delta(-1;K)\,E\{\chi(Tx)\Psi(x;K)\}\,.$$

This equality extends easily to any measurable set characteristic function χ and thence to an arbitrary integrable function G, thereby yielding (4.7) in the general case. To get (4.8) we may write (4.7) with T replaced by T^{-1} and then let $G(x)=H(Tx)$. This completes the proof of the theorem.

5. Gaussian measures equivalent to Wiener measure.

THEOREM 7. *Let $r(s,t)$ be a covariance function satisfying*

(5.0) $$r(0,t) = 0 \quad \textit{for} \quad t \in I \equiv [0,b]\,,$$

(5.1) $$F(s,t) \equiv \sigma^{-2}\, \partial^2[r(s,t) - w_\sigma(s,t)]/\partial t \partial s \quad \textit{exists on } I \times I\,,$$

(5.2) $$F \in \mathrm{BVH} \textit{ and } |F| < 1\,.$$

Then the Gaussian measures determined by $r(s,t)$ *and* $w_\sigma(s,t) \equiv \sigma^2 \min(s,t)$ *are equivalent and*

(5.3) $$dP_r/dP_{w_\sigma} = [d(-1;F)]^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}\int_0^1\!\!\int_0^1 F^{-1}(s,t)\, dx(s)dx(t)\right].$$

where F^{-1} *is the Volterra reciprocal kernel corresponding to* F (*see* (4.5)).

PROOF. $\sum \lambda_j^2 = |F| < 1$ implies that $\lambda_j > -1$, $j = 1, 2, \ldots$, and henced that $d(-1;F) \neq 0$. If we use Theorem 1 to determine a K satisfying (2.2), then [3, pp. 172–173] $d(-1;F) = [d(-1;K)]^2$ so that $d(-1;K) \neq 0$. Moreover as is easily checked,

$$F^{-1}(s,t) = 2K^{-1}(s,t) + \int_0^b K^{-1}(s,u)K^{-1}(t,u)\, du\,.$$

Thus by Theorem 5,

(5.4)
$$E\{H(Tx)\} = [d(-1;F)^{-\frac{1}{2}}\, E\left\{H(x) \exp\left[-\frac{1}{2\sigma^2}\int_0^b\!\!\int_0^b F^{-1}(s,t)\, dx(s)dx(t)\right]\right\},$$

where

$$(Tx)(t) = x(t) + \int_0^b\!\!\int_0^t K(u,s)\, du dx(s)\,.$$

But by Theorem 3, the transformation $y(t) = (Tx)(t)$ determines a Gaussian process with covariance function $r(s,t)$. This in turn means that

(5.5) $$E^r\{H(y)\} = E^{w_\sigma}\{H(Tx)\}\,,$$

where for clarity we have used the notation $E^r\{\ldots\}$ to denote expectation on the process with covariance function r. Formula (5.5) together with (5.4) yield the fact that P_r is absolutely continuous with respect to P_{w_σ} and formula (5.3). That P_{w_σ} is also absolutely continuous with respect to P_r is a consequence of the positivity of dP_r/dP_{w_σ}.

We may weaken the conditions on r but as might be expected there is some loss in the elegance of the formula for the derivative dP_r/dP_{w_σ} as we see in

THEOREM 8. *Let* $r(s,t)$ *be a covariance function satisfying*

(5.6) $$r(0,t) = 0 \quad \textit{for} \quad t \in I \equiv [0,b]\,,$$

(5.7) $\quad F(s,t) \equiv \sigma^{-2}\, \partial^2[r(s,t) - w_\sigma(s,t)]/\partial t \partial s$ *exists on* $I \times I$,

(5.8) $\quad F \in L^2$ *and* $\lambda_1^- \geqq -1$ (*see* (2.7)) .

Then the Gaussian measures P_r *and* P_{w_σ} *are equivalent and*

(5.9)
$$dP_r/dP_{w_\sigma} = [\delta(-1; F^{-1})]^{\frac{1}{2}} \exp\left[\frac{1}{2\sigma^2}\sum_j \frac{\lambda_j}{1+\lambda_j}\left(\left(\int_0^b \varnothing_j(s)\, dx(s)\right)^2 - \sigma^2\right)\right]$$

where $\{\varphi_j, \lambda_j\}$ *is an eigen system for* F *and*

$$F^{-1}(s,t) = -\sum_j \frac{\lambda_j}{1+\lambda_j}\varphi_j(s)\varphi_j(t) .$$

PROOF. We use Theorem 2 to determine a kernel K satisfying (2.2) As in that theorem, let $\mu_j = -1 + (1+\lambda_j)^{\frac{1}{2}}$ so that $\lambda_1^- > -1$ implies that $\mu_j > -1$ and therefore that $\delta(-1; K) > 0$. One may easily check the following calculation.

$$\delta(-1; K^{-1})\Psi(x; K^{-1})$$
$$= \prod_j \left[1 - \frac{\mu_j}{1+\mu_j}\exp\left(\frac{\mu_j}{1+\mu_j}\right)\right] \exp\left\{-\frac{1}{2\sigma^2}\sum_j\left[\frac{-2\mu_j}{1+\mu_j}(N_j^2(x) - \sigma^2) + \right.\right.$$
$$\left.\left. + \frac{\mu_j^2}{(1+\mu_j)^2}N_j^2(x)\right]\right\}$$
$$= \exp\left\{\frac{1}{2}\sum_j\left[\log\frac{1}{1+\lambda_j} + \frac{\lambda_j}{1+\lambda_j} + \frac{\lambda_j}{\sigma^2(1+\lambda_j)}(N_j^2(x) - \sigma^2)\right]\right\}$$
$$= \left[\prod_j\left(1 - \frac{\lambda_j}{1+\lambda_j}\right)\exp\left(\frac{\lambda_j}{1+\lambda_j}\right)\right]^{\frac{1}{2}} \exp\left[\frac{1}{2\sigma^2}\sum_j\frac{\lambda_j}{1+\lambda_j}(N_j^2(x) - \sigma^2)\right]$$

which is precisely the expression given for dP_r/dP_w in (5.9). Thus if we apply Theorem 6, we obtain

$$E\{H(Tx)\} = [\delta(-1; K^{-1})]^{\frac{1}{2}}\, E\left\{H(x)\exp\left[\frac{1}{2\sigma^2}\sum_j\frac{\lambda_j}{1+\lambda_j}(N_j^2(x) - \sigma^2)\right]\right\}.$$

The proof is now completed exactly as in Theorem 7 with Theorem 4 playing an auxiliary role.

Adding two independent Gaussian processes produces another Gaussian process. One suspects that adding a suitable Gaussian process $y(t)$ to the Wiener process $x(t)$ should yield a process $z(t)$ whose measure is equivalent to Wiener measure. A criterion for a process $y(t)$ with covariance function $p(s,t)$ to be suitable is obtained by noting that

$$r(s,t) = E\{z(s)z(t)\} = E\{x(s)x(t)\} + E\{y(s)y(t)\}$$
$$= w_\sigma(s,t) + p(s,t)$$

and applying the following corollary to Theorem 8.

COROLLARY. If $r(s,t) = w_\sigma(s,t) + p(s,t)$, where $p(s,t)$ is a covariance function, $p(0,t) = 0$ and $\partial^2 p(s,t)/\partial t \partial s$ exists and is in L^2, then P_r is equivalent to P_{w_σ}.

The proof reduces to the simple observation that since $p(s,t)$ is a covariance function, $\partial^2 p(s,t)/\partial t \partial s$ is positive semidefinite and therefore $\lambda_1^- \geqq 0$.

We remark that this corollary is closely related to results of Parzen [5, especially pp. 164–165].

We conclude this section with a theorem which connects the results of the present paper with some earlier ones of the author [11].

THEOREM 9. *Let $r(s,t)$ be continuous covariance function with uniformly bounded second derivatives on $I \times I$ except on the diagonal $s = t$. If P_r is equivalent to P_{w_σ}, then*

$$r(0,t) = 0 \quad \text{on } I, \tag{5.10}$$

$$\lim_{s \to t-} \frac{r(t,t) - r(s,t)}{t-s} - \lim_{s \to t+} \frac{r(t,t) - r(s,t)}{t-s} = \sigma^2 \tag{5.11}$$

on I.

Conversely if (5.10) *and* (5.11) *hold, then there exists $T > 0$ such that P_r is equivalent to P_{w_σ} on $\{C_0(0,T), B\}$, that is, P_r and P_{w_σ} are initially equivalent.*

PROOF. Suppose that P_r is equivalent to P_{w_σ}. Then

$$P_{w_\sigma}\{x : x(0) = 0\} = 1 = P_r\{x : x(0) = 0\}$$

from which it follows that $r(0,t) = E^r\{x(0)x(t)\} = 0$. Formula (5.11) is proved in [11, § 2].

Conversely suppose that (5.10) and (5.11) hold. Then, noting that $w_\sigma(s,t)$ also satisfies (5.11), we may check that $\partial[r(s,t) - w(s,t)]/\partial s$ exists on the diagonal $s = t$. The conditions on the second derivatives of r imply that this first partial derivative exists and is continuous on $I \times I$ and that $\partial^2[r(s,t) - w_\sigma(s,t)]/\partial t \partial s$ exists except possibly on the diagonal $s = t$. This together with $r(0,t) = 0$ insures that

$$(5.12)\quad r(s,t) = w_\sigma(s,t) + \int_0^s\int_0^t \{\partial^2[r(u,v) - w_\sigma(u,v)]/\partial v\,\partial u\}\,dv\,du$$

$$= w_\sigma(s,t) + \sigma^2 \int_0^s\int_0^t F(u,v)\,dv\,du\,.$$

Thus the conclusion of Theorem 4 holds even though $F(s,t)$ may not be defined at $s=t$ and we may apply Theorem 8 provided of course that $\lambda_1^- > -1$. But for the interval $[0,T]$, this is clearly true if T is sufficiently small (see (2.7)).

6. Examples.

We illustrate our theorems with a covariance function r which has arisen in many statistical problems. Let

$$w(s,t) = \min(s,t) \qquad \text{and} \qquad r(s,t) = w(s,t) - st\,.$$

Then $F(s,t) = -1$ so the hypotheses of Theorem 3 are satisfied if $0 < b \leqq 1$. It is a simple exercise to check that $K(s,t) = (-1 + (1-b)^{\frac{1}{2}})/b$. Hence r determines a Gaussian process $y(t)$ with representation in terms of the Wiener process $x(t)$,

$$y(t) = x(t) + \int_0^b\int_0^t K(u,s)\,du\,dx(s)$$

$$= x(t) + tx(b)[(-1 + (1-b)^{\frac{1}{2}})/b]\,.$$

Moreover if $b < 1$,

$$K^{-1}(s,t) = (1 - (1-b)^{\frac{1}{2}})/(b(1-b)^{\frac{1}{2}})$$

and $d(-1; F) = 1-b$. Hence by Theorem 7, P_r is equivalent to P_w and

$$dP_r/dP_w = (1-b)^{-\frac{1}{2}} \exp[-x^2(b)/2(1-b)]\,.$$

This example also illustrates the necessity of condition (5.8) of Theorem 8. For in the present instance, $\lambda_1^- = -b$ and hence if $\lambda_1^- = -1$, then $b=1$. But we know that P_r is not equivalent to P_w on $\{C_0(0,1), B\}$ since $P_r\{x: x(1)=0\} = 1$ while $P_w\{x: x(1)=0\} = 0$.

In conclusion, we mention that our results permit the calculation of the Wiener integral (expectation) of certain quadratic type functionals. In particular if $F \in \mathrm{BVH}$ and is symmetric and if $|\lambda| < 1/|F|$, then

$$(6.0)\qquad E^w\left\{\exp\left[\tfrac{1}{2}\lambda \int_0^b\int_0^b F(s,t)\,dx(s)\,dx(t)\right]\right\} = [d(\lambda; F)]^{-\frac{1}{2}}\,.$$

To prove this, note first that $|\lambda| < 1/|F|$ implies that $d(\lambda; F) \neq 0$. Next, use Theorem 1 to solve the equation

$$2K(s,t) = -\lambda F(s,t) - \int_0^b K(s,u)K(t,u)\,du$$

for K. By [3, pp. 172–173],

$$d(\lambda; F) = [d(-1; K)]^2$$

and hence, applying Theorem 5 and formula (4.2) with $\sigma = 1$ and $G = 1$, we get (6.0).

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