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EXTREME POSITIVE OPERATORS ON ALGEBRAS OF FUNCTIONS

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This paper is developed around the following general theme: If J is an appropriate convex set of linear operators from one algebra to another, and if T is an extreme point of J, then T is an algebra homomorphism. This assertion is true in a surprisingly large number of cases. A result of this general type which motivated the present paper is the following [8]: Let A and B be algebras (under the usual pointwise operations) of real-valued functions on the sets X and Y, respectively, and suppose that $1 \in A$. Let $K_0'(A,B)$ be the convex set of all linear operators T from A to B which satisfy $T \geqq 0$ (that is, $Tf \geqq 0$ whenever $f \geqq 0$) and $T1 \leqq 1$. Suppose, moreover, that the functions in A are bounded. Then T is an extreme point of $K_0'(A,B)$ if and only if T is multiplicative. This theorem was itself motivated by A. and C. Ionescu Tulcea [5], who proved essentially the same thing for the case $A=C(X)$, $B=C(Y)$, X and Y compact Hausdorff spaces. (It follows from the results of the present paper that this last result is valid for X and Y arbitrary topological spaces.) As in [8], the methods of this paper are quite algebraic and elementary.

Our general set-up, then, is this. We let A and B be (nontrivial) algebras of real-valued functions on the sets X and Y, respectively, and we define several convex sets of linear operators from A to B, as follows:

$K_1(A,B)$ consists of those T such that $T \geqq 0$ and $T1=1$.

Naturally, this presupposes the existence of constant functions in A and B. In order to avoid this, we consider a closely related set (which is identical with the set $K_0'(A,B)$ introduced above in case $1 \in A$):

$K_0(A,B)$ consists of those $T \geqq 0$ such that $Tf \leqq 1$ whenever $0 \leqq f \leqq 1$.

We also look at the convex cone $K(A,B)$ of *all* positive operators:

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$K(A,B)$ consists of all $T \geqq 0$.

In the case of this last set, of course, we must consider extreme *rays*, rather than extreme points. We will use the terminology

"T is extreme in $K(A,B)$"

to mean that

T lies on an extreme ray ,

i.e. if $S \in K(A,B)$ and $T-S \in K(A,B)$, then there exists a constant $\lambda \geqq 0$ such that $S=\lambda T$. Equivalently, if $T=T_1+T_2, T_i \in K(A,B)$, then there exist constants $\lambda_i \geqq 0$ such that $T_i=\lambda_i T$.

In case $B=R$, the real numbers, then each of the above sets becomes a convex set of linear functionals on A, which we denote by $K_1(A)$, $K_0(A)$ and $K(A)$, respectively.

The main idea which allows us to prove results of the above type is to consider algebras A which satisfy the following hypothesis:

HYPOTHESIS (a): *$f(1+f)^{-1} \in A$ whenever $f \in A$, $f \geqq 0$.*

If $1 \in A$, then it is easily seen that hypothesis (a) is equivalent to the assertion that $f^{-1} \in A$ whenever $f \in A$ and $f \geqq \delta$ for some constant $\delta > 0$.

There are several natural classes of algebras which arise in analysis and which satisfy hypothesis (a); we list some of these at the end of the paper.

In order to simplify the statements of some of the results, we make use of an additional hypothesis:

HYPOTHESIS (b): *Every function in A is bounded.*

It is readily seen that neither of these hypotheses implies the other.

We will denote the subalgebra of all bounded functions in A by A_b. Note that hypothesis (a) implies that $A_b \neq \{0\}$.

By a *positive* function we mean one which is nowhere negative; the set of positive functions in A or B is denoted by A_+ or B_+. Similarly, an operator T is said to be positive if $T \geqq 0$.

We restrict ourselves throughout to algebras of real-valued functions. It is an easy task to extend the results to algebras of complex-valued functions, if the algebra A is assumed to be self-adjoint. As was shown in [8], if $K_1(A,B)$ contains an extreme element, then it follows easily that A is self-adjoint. The same observation may be made concerning $K_0(A,B)$ and $K(A,B)$.

The rest of this paper is divided into four main parts.

In the first, we consider the sets $K_0(A,B)$ and $K_1(A,B)$, as well as some closely related sets. Under hypothesis (a) or (b) on A, the extreme points of these sets *are* multiplicative. (The algebra P of all real polynomials satisfies neither (a) nor (b), and it is shown, in fact, that $K_1(P,P)$ contains an extreme operator which is not multiplicative.)

In the second part we look at the cone $K(A,B)$ of positive operators. The results here vary widely, depending on different hypotheses on A and B. Generally speaking, the extreme rays of $K(A,B)$ are *not* generated by multiplicative operators.

In the third part we consider the converse problem of determining when every multiplicative operator is extreme.

In the final part we look at some related results and some open questions.

2. Normalized positive operators.

The following lemma extends the essential idea from [8] to cover the case when $1 \notin A$.

LEMMA 1. *Suppose that $T \in K_0(A,B)$ and that $g \in A$. Define $U_g: A \to B$ by*

$$U_g(f) = Tfg - TfTg, \quad f \in A .$$

If $0 \leqq g \leqq 1$, then $T \pm U_g \in K_0(A,B)$. If $1 \in A$ and $T \in K_1(A,B)$, then $T \pm U_g \in K_1(A,B)$.

PROOF. If $f \geqq 0$, then

$$(T+U_g)f = Tf(1-Tg)+Tfg \geqq 0 ,$$

since $1-Tg \geqq 0$. Also, $f-fg \geqq 0$ and hence

$$(T-U_g)f = T(f-fg)+TfTg \geqq 0 .$$

If $0 \leqq f \leqq 1$, then $0 \leqq fg \leqq g$ and $Tf \leqq 1$, so

$$\begin{aligned}(T+U_g)f &\leqq Tf(1-Tg)+Tg \\ &\leqq (1-Tg)+Tg = 1 .\end{aligned}$$

Finally, we have

$$\begin{aligned}0 \leqq f-fg+g &= f(1-g)+g \\ &\leqq (1-g)+g = 1 ,\end{aligned}$$

so

$$\begin{aligned}(T-U_g)f &= Tf-Tfg+TfTg \\ &\leqq Tf-Tfg+Tg \\ &= T(f-fg+g) \leqq 1 .\end{aligned}$$

If $1 \in A$ and $T1=1$, then $U_g1=0$, so the second assertion is obvious.

The next result shows that in the present context, the only algebras A of interest are those which are generated by A_+.

LEMMA 2. *If* $K_0(A,B)$ *or* $K(A,B)$ *contains an extreme element, then* $A = A_+ - A_+$. *If* $K_0(A,B)$ *contains a nontrivial extreme point then* $A_b \neq \{0\}$; *in fact,* $A_b = \{0\}$ *if and only if* $K_0(A,B) = K(A,B)$.

PROOF. Suppose there exists f in A, $f \notin A_+ - A_+$. Choose a linear subspace $N \supset A_+ - A_+$ such that $f \notin N$ and $A = N + Rf$. Choose $g \neq 0$ in B and define $U: A \to B$ by $U(h+rf) = rg$ whenever $h \in N$, $r \in R$. Then $U \neq 0$ but $U = 0$ on A_+. It follows that the line RU generated by U is contained in $K_0(A,B)$ and in $K(A,B)$, so neither of these sets contains an extreme element.

It is obvious that if $A_b = \{0\}$, then $K_0(A,B) = K(A,B)$. On the other hand, if $A_b \neq \{0\}$ we can find h in A and x in X such that $0 \leqq h \leqq 1$ and $h(x) \neq 0$. If we choose g in B_+, $g \neq 0$ and define $T: A \to B$ by $Tf = f(x)g$, then $\lambda T \in K(A,B)$ for all $\lambda > 0$, but $\lambda T \notin K_0(A,B)$ for sufficiently large λ. Since the only extreme point of $K(A,B)$ is the zero operator, it is clear that $A_b = \{0\}$ implies that $K_0(A,B)$ contains no nontrivial extreme point.

THEOREM 3. *If* T *is an extreme point of* $K_0(A,B)$ *and if* A *satisfies hypothesis* (a) *or hypothesis* (b), *then* T *is multiplicative.*

PROOF. It is immediate from Lemma 1 that if $g \in A$, $0 \leqq g \leqq 1$, then $U_g = 0$, that is, $Tfg = TfTg$ for all f in A. Consider, first, hypothesis (b). If $g \in A_+$, then $0 \leqq ag \leqq 1$ for some $a > 0$; it follows that $Tfg = TfTg$ for f in A, g in A_+. By Lemma 2, $A = A_+ - A_+$ and we conclude that T is multiplicative. Consider, now hypothesis (a). If $h \in A_+$, then $g = h(1+h)^{-1} \in A$ and $0 \leqq g \leqq 1$, so $Tfg = TfTg$ for all f in A. Since $h = g + gh$, we have (for all f in A)

$$Tfh = Tg(f+fh) = TgT(f+fh) = TfTg + TfhTg$$

and $Th = Tg + TgTh$. From this it follows that

$$\begin{aligned} TfTh + TfhTh &= (Tf + Tfh)Th \\ &= (Tf + Tfh)(Tg + TgTh) \\ &= (TfTg + TfhTg) + TfTgTh + TfhTgTh \\ &= Tfh + TfhTh, \end{aligned}$$

so $TfTh = Tfh$ whenever $f \in A$, $h \in A_+$. Since $A = A_+ - A_+$ the proof is complete.

COROLLARY 4. *If* A *and* B *contain the constants and* A *satisfies hypothesis* (a) *or* (b), *then every extreme point* T *of* $K_1(A,B)$ *is multiplicative.*

PROOF. Observe that not only do we have $K_1(A,B) \subset K_0(A,B)$, but (after a simple verification) every extreme point of $K_1(A,B)$ is an extreme point of $K_0(A,B)$, so Theorem 3 applies.

The next result was motivated by a question by L. Dubins and G. Schwartz. The sequences of subalgebras arise in certain questions in probability theory, where A and B consist of all bounded functions which are measurable with respect to a σ-ring and the subalgebras are the functions which are measurable with respect to (an increasing sequence of) sub-σ-rings.

PROPOSITION 5. *Suppose that A_n and B_n are sequences of subalgebras of A and B respectively, with*

$$1 \in A_1 \subset \ldots \subset A_n \subset A_{n+1} \subset \ldots \subset A\,,$$
$$1 \in B_1 \subset \ldots \subset B_n \subset B_{n+1} \subset \ldots \subset B\,.$$

Let J be the set of all positive operators T such that $T1=1$ and

$$TA_n \subset B_n, \qquad n=1,2,3,\ldots\,.$$

If the functions in A are bounded and if T is an extreme point of J, then T is multiplicative.

PROOF. We carry out an induction as follows: Suppose that $n \geqq 1$ and that $Tfg=TfTg$ whenever $f,g \in A$ and f or g is in A_{n-1}. (We let A_0 be the algebra of constants, so the above supposition is valid for $n=1$). We will show that this holds if f or g is in A_n. Suppose, then, that $g \in A_n$, with $0 \leqq g \leqq 1$, and let U_g be the operator defined in Lemma 1. Since $g \in A_n \subset A_{n+1} \subset \ldots$ we have $U_g A_k \subset B_k$ for $k \geqq n$. If $f \in A_k$, $1 \leqq k \leqq n-1$, then (by the induction hypothesis) $U_g f=0$. Thus, $U_g A_k \subset B_k$ for all k, and hence (by Lemma 1), $T \pm U_g \in J$. Since T is extreme, $U_g=0$, that is, $Tfg=TfTg$ if $g \in A_n$, $0 \leqq g \leqq 1$, $f \in A$. Since the functions in A_n are bounded and $1 \in A_n$, we have $Tfg=TfTg$ if $g \in A_n$, $f \in A$. It follows by induction that the same is true for $g \in \bigcup A_n$, $f \in A$. From this it follows that if $g \in A$, then $U_g=0$ on $\bigcup A_n$. Thus, if $g \in A$ and $0 \leqq g \leqq 1$, then $T \pm U_g \in J$ and the same arguments show that T is multiplicative on A.

It is easily seen that the same induction argument will lead to the same conclusion if J is taken to be the operators T in $K_0(A,B)$ for which $TA_n \subset B_n$, provided each A_n is generated by its positive cone and satisfies hypothesis (a).

The next result generalizes (and gives a simple proof of) a result of Lloyd [7] concerning the extreme points of the set of positive projections from $C(X)$ (X compact Hausdorff) onto a subalgebra E of $C(X)$ which contains 1.

PROPOSITION 6. *Suppose that C and D are subalgebras of A and B respectively, with $1 \in C$ and $1 \in D$. Suppose that T_0 is a multiplicative operator in $K_1(C,D)$ and let J_1 be the set of all operators T in $K_1(A,B)$ whose restriction to C is T_0. If the functions in A are bounded, and if T is an extreme point of J_1, then T is multiplicative.*

PROOF. If $g \in C$, $0 \leqq g \leqq 1$, then U_g vanishes on C (since $T = T_0$ is multiplicative on C), hence $T \pm U_g \in J_1$ and we conclude that $U_g = 0$ on A. This conclusion is valid for arbitrary g in C, since $1 \in C$ and the functions in C are bounded. Thus, $Tfg = TfTg$ if f or g is in C. Thus if $g \in A$, $0 \leqq g \leqq 1$, then (again) $U_g = 0$ on C and $T \pm U_g \in J_1$; it is easily concluded that T is multiplicative.

Lloyd's theorem follows by taking $E = B = D = C$ and T_0 to be the identity map on E. It is also easy to modify Proposition 6 in the manner suggested after Proposition 5.

The next proposition was suggested by the problem of simultaneous extensions of continuous functions. Suppose that X and Y are topological spaces, with $X \subset Y$, and let $A = C(X)$, $B = C(Y)$ be the continuous functions on X and Y respectively. Let $S: B \to A$ denote the restriction operator. If a linear mapping $T: A \to B$ is such that ST is the identity on A, then T is called a simultaneous extension operator; if $T \geqq 0$, it is called a *positive* simultaneous extension operator. If X and Y are compact Hausdorff spaces and X is metrizable, then there exist such operators for which $T1 = 1$. (See the expository paper [9] for references.) It follows easily from known descriptions of the multiplicative operators in this case ([1], [8]) (together with the following result) that an operator T is an extreme positive extension operator with $T1 = 1$ if and only if $Tf = f \circ \varphi$, $f \in A$, where φ is a retract of Y onto X.

PROPOSITION 7. *Suppose that $S: B \to A$ is a multiplicative operator in $K_1(B,A)$ such that $SB = A$. Let J_2 be the convex set of all T in $K_1(A,B)$ such that ST is the identity operator on A. If the functions in A are bounded, then any extreme point T of J_2 is multiplicative.*

PROOF. Suppose that $g \in A$, $0 \leqq g \leqq 1$ and define U_g as in Lemma 1 Then $T \pm U_g \in K_1(A,B)$ and

$$(SU_g)(f) = S(Tfg - TfTg) = fg - (ST)f(ST)g = 0\,,$$

so $S(T \pm U_g) = ST$; this implies that $T \pm U_g \in J_2$, so $U_g = 0$. As before, then, T is multiplicative.

It is easily seen that the above result is valid if, in place of hypothesis (b), we require that A satisfies $A = A_+ - A_+$ and hypothesis (a).

Proposition 7 may also be applied to prove a result of A. and C. Ionescu Tulcea [5] concerning liftings. Simply let $B = M^\infty$, the algebra of all bounded measurable functions (on a given measure space), let A be an algebra of functions isomorphic to L^∞, the quotient algebra obtained by identifying functions in B which agree almost everywhere, and let S be the quotient map. It is also possible to give an abstract formulation of another Ionescu Tulcea result (on "strong" liftings) by combining the ideas of Propositions 6 and 7, but the statement becomes a bit complicated.

The next result shows that something like hypothesis (a) or (b) is needed in order to prove results like Theorem 3.

THEOREM 8. *Let P denote the algebra of all polynomials. The operator T defined for p in P, x real, by*

$$(Tp)(x) = \frac{1}{x^2+1} p(x^2+1) + \frac{x^2}{x^2+1} p[(x^2+1)(x+1)]$$

is extreme in $K_1(P,P)$, but is not multiplicative.

PROOF. It is easily checked that $T(P) \subset P$, that $T \in K_1(P,P)$ and that T is not multiplicative. (For instance, if i denotes the polynomial $i(x) = x$, then $T(i^2) \neq (Ti)^2$.) It remains then, to show that T is extreme, i.e. if $U: P \to P$ is linear and $T \pm U \in K_1(P,P)$, then $U = 0$. For each real x, let L_x denote the functional $p \to (Tp)(x)$ and let N_x denote $p \to (Up)(x)$. For fixed x, it is clear that L_x has the following form, where $x_1 = x^2 + 1$, $x_2 = (x^2+1)(x+1)$:

$$L_x(p) = \alpha p(x_1) + (1-\alpha)p(x_2), \qquad 0 \leqq \alpha \leqq 1, \quad x_i \text{ real}.$$

Furthermore, $T \pm U \geqq 0$ implies that if $p \geqq 0$, then $|N_x(p)| \leqq L_x(p)$. This implies that N_x is a linear combination of the functionals $p \to p(x_1)$, $p \to p(x_2)$. Indeed, to show this it suffices that $N_x(p) = 0$ whenever $p \in P$ and $p(x_1) = 0 = p(x_2)$. If p satisfies the latter condition, then so does p^2 (and $p^2 \geqq 0$), hence $|N_x(p^2)| \leqq L_x(p^2) = 0$. Clearly, $T \pm U \in K_1(P,P)$ implies $N_x(1) = 0$, so for any integer $n > 1$,

$$1 = L_x[(np+1)^2] \geqq |N_x(n^2p^2 + 2np + 1)| = |2nN_x(p)|;$$

it follows that $N_x(p) = 0$. Thus, for any real x there exist real numbers $a(x)$ and $b(x)$ such that

$$N_x(p) = a(x)\, p(x^2+1) + b(x)\, p[(x^2+1)(x+1)]\,.$$

Since $N_x(1) = 0$, we have $b(x) = -a(x)$ for all x. Thus, U must have the following form:

$$(1) \qquad (Up)(x) = a(x)\{p(x^2+1) - p[(x^2+1)(x+1)]\}, \qquad p \in P,\ x \text{ real}\,.$$

It is clear from this that $(Up)(0) = 0$ for all p. If $x \neq 0$, we can choose positive polynomials which vanish at one of the points x^2+1 or $(x^2+1)(x+1)$ but not at the other. Since $T \pm U \geqq 0$, this procedure yields the inequalities

$$0 \leqq \frac{1}{x^2+1} \pm a(x) \quad \text{and} \quad 0 \leqq \frac{x^2}{x^2+1} \pm a(x), \qquad x \neq 0\,,$$

and hence

$$(2) \qquad |(x^2+1)a(x)| \leqq 1 \quad \text{and} \quad |(x^2+1)a(x)| \leqq x^2, \qquad x \neq 0\,.$$

If we apply (1) to the polynomial $i(x) = x$ we see that

$$(Ui)(x) = -a(x)\, x(x^2+1)$$

is a polynomial q, with the properties (from (2))

$$|q(x)| \leqq |x| \quad \text{and} \quad |q(x)| \leqq |x|^3, \qquad x \text{ real}\,.$$

An examination of these inequalities shows that $q = 0$, and hence $a(x) = 0$ for $x \neq 0$; it follows that $U = 0$, and the proof is complete.

As noted in [8], the multiplicative elements of $K_1(P,P)$ are extreme, and they are all of the form $p \to p \circ q$ for a fixed polynomial q.

3. Positive operators.

We now turn our attention to the cone $K(A,B)$ of positive operators from A to B. As noted in Lemma 2, the existence of an extreme ray in $K(A,B)$ implies that $A = A_+ - A_+$. It also implies (under additional hypotheses on A) that B_+ contains extreme rays; this fact is a consequence of the next theorem.

THEOREM 9. *Suppose that $T \neq 0$ lies on an extreme ray of $K(A,B)$ and suppose that A satisfies either hypothesis* (a) *or hypothesis* (*b*). *Then either*

(1) *The operator T satisfies $Tfg = 0$ for all f,g in A*

or

(2) *There exists a positive multiplicative functional M on A and a function h on an extreme ray of B_+ such that $Tf = hMf$ for all f in A.*

If $1 \in A$, then (clearly) case (1) *cannot hold.*

PROOF. Suppose that (b) holds, and choose $g \in A$, $0 \leqq g \leqq 1$. For any f in A we have $Tf = Tfg + T(f - fg)$. Since this expresses T as the sum of two positive operators, there exists a constant $\lambda(g)$ with $0 \leqq \lambda(g) \leqq 1$ such that $Tfg = \lambda(g)Tf$ for all f in A. It is easily verified that the function λ defined in this way is positive and affine on the set of all g with $0 \leqq g \leqq 1$; furthermore, $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$ for any such g_1, g_2. Since the functions in A are bounded (and since $A = A_+ - A_+$) we can extend λ linearly to a positive multiplicative functional M on A which satisfies

$$(*) \qquad Tfg = M(g)Tf \qquad \text{for all } f, g \text{ in } A .$$

Using commutativity, we have $M(g)Tf = M(f)Tg$ for all f, g in A. If $M(g) = 0$ for every g, then (*) shows that $Tfg = 0$ for all f, g in A. Otherwise, $M(g) \neq 0$ for some g in A, hence $M(g) > 0$ for some g in A_+, and consequently

$$Tf = \frac{Tg}{M(g)} M(f) \qquad \text{for all } f \text{ in } A .$$

Letting $h = Tg/M(g)$, we have $h \in B_+$ and we want to show that h lies on an extreme ray of B_+. Suppose, then, that $h_0 \in B_+$, $0 \leqq h_0 \leqq h$. Define T_0 by $T_0 = h_0 M$. It is clear that $T_0 \in K(A,B)$ and $T_0 \leqq T$, hence $T_0 = \lambda T$ for some constant λ. Choosing f such that $M(f) \neq 0$, this implies that $h_0 M(f) = \lambda h M(f)$, so $h_0 = \lambda h$ and hence h is extreme.

We now turn to hypothesis (a). For any f in A and $g \in A_+$ we have

$$\frac{f}{1+g} = f - f\frac{g}{1+g} \in A ,$$

and hence the identity

$$Tf = T\left(\frac{f}{1+g}\right) + T\left(\frac{fg}{1+g}\right)$$

expresses T as the sum of two positive operators. It follows that there exists a constant $\mu(g)$ with $0 \leqq \mu(g) \leqq 1$ such that

$$T\left(\frac{f}{1+g}\right) = \mu(g)Tf \quad \text{for all } f \text{ in } A .$$

Writing $f + fg$ in place of f we see that $Tf = \mu(g)T(f + fg)$ for all f in A, g in A_+. Since $T \neq 0$, $\mu(g) > 0$ for every g in A_+ and hence

$$Tfg = \frac{1-\mu(g)}{\mu(g)} Tf \quad \text{for all } f \text{ in } A, g \text{ in } A_+.$$

It is easily verified that $M(g) = (1-\mu(g))/\mu(g)$ defines a positive, multiplicative, additive and positive-homogeneous function on A_+, and its linear extension to A satisfies (*). Since the first half of the proof following the identity (*) did not use hypothesis (b), it applies again to the present case to complete the proof.

Since the sum of two operators which satisfy conclusion (1) of the above theorem are of the same form, it is clear that not all operators satisfying (1) are extreme. [At the end of Section 4 we show that extreme operators exist which satisfy (1).] For operators which satisfy (2), however, the situation is better—they are always extreme (Theorem 13 and Lemma 17). It is interesting to note that (unless $h=1$ in case (2)), the extreme positive operators are *not* multiplicative. This assumes, of course, either hypothesis (a) or (b); as noted in the previous section, there exist many positive multiplicative operators on the polynomials, and Proposition 18 will show that they are extreme in $K(P,P)$.

The conclusions to the next theorem are similar to those in the above result. The fact that we are dealing with *functionals* (rather than operators) makes it possible to eliminate all hypotheses on A. This result was first proved, in an entirely different manner, by G. Choquet in 1964 (unpublished).

THEOREM 10. *Let A be an algebra of real-valued functions and let $K(A)$ denote the convex cone of positive linear functionals on A. If L lies on an extreme ray of $K(A)$, then either*

(1) $L(fg) = 0$ *for all* f, g *in* A

or

(2) *There exist a constant $\lambda \geqq 0$ and a multiplicative functional M such that $L = \lambda M$.*

If $1 \in A$, *then case* (1) *implies that* $L = 0$.

The proof of this theorem consists of two lemmas, the first of which may be known, but does not (to the best of our knowledge) appear in the literature.

LEMMA 11. *Suppose that E' is a partially ordered linear space, with positive cone E_+', and suppose that E is a linear subspace of E' such that $E' = E + E_+'$. If L is a linear functional on E which lies on an extreme ray of the cone of positive functionals on E, then there is an extension L' of L to*

E' such that L' lies on an extreme ray of the cone of the positive functionals on E'.

PROOF. The proof proceeds precisely as the proof of the usual "monotone extension theorem" (see, for example, [3]) with two additional precautions. As a first precaution, one applies Zorn's lemma to the set of all *extreme* extensions of L (rather than to the set of all extensions). The second precaution involves the "induction step" which shows the possibility of extending to a subspace of one more dimension. The usual proof involves choosing a real number from a certain finite closed interval; one observes that an *extreme* extension may be obtained by choosing an endpoint of this interval.

LEMMA 12. *If A is any algebra of real-valued functions, with $A = A_+ - A_+$, then the set A' of the functions of the form*

$$f(1+g)^{-1}, \qquad f \in A,\ g \in A_+ ,$$

is an algebra containing A, and $A' = A + A_+'$. Furthermore, if $f \in A'$, $g \in A_+'$, then $f(1+g)^{-1} \in A'$.

PROOF. It is straightforward to verify that A' is an algebra containing A. If $f(1+g)^{-1}$ is an element of A', then $f = f_1 - f_2$ where $f_i \in A_+$, hence

$$f(1+g)^{-1} = -f_2 + (f_1 + f_2 g)(1+g)^{-1} \in A + A_+' .$$

Finally, the last assertion is also easily verified.

The proof of the first part of the theorem is now completed as follows: If L lies on an extreme ray of $K(A)$, $L \neq 0$, then $A = A_+ - A_+$ and the algebra A' contains A and satisfies the hypotheses of Lemma 11. Hence there exists a functional L' which lies on an extreme ray of $K(A')$ and extends L. The algebra A' satisfies hypothesis (a) of Theorem 9, so the desired conclusions follow.

In order to exhibit extreme functions which satisfy conclusion (1) of Theorem 10, we give an example which is analogous to one suggested by Choquet.

EXAMPLE. *Let A denote the algebra of all polynomials with a double zero at 0, that is*

$$p \in A \quad \textit{if and only if} \quad p(x) = x^2 q(x) \textit{ for some polynomial } q .$$

Then the functional defined by $L_0(p) = q(0)$ is extreme in $K(A)$ and satisfies $L_0(p_1 p_2) = 0$ for all p_1, p_2 in A.

PROOF. The functional L_0 is well-defined, since q is uniquely determined by p. In fact, the map

$$p(x) = x^2 q_p(x) \to q_p(x)$$

is a well-defined positive linear operator which is one-to-one from A onto the algebra P of *all* polynomials. Its adjoint, which we denote by T, is one-to-one from $K(P)$ onto $K(A)$ and hence carries extreme rays of $K(P)$ onto extreme rays of $K(A)$. Since evaluation at 0 is multiplicative and positive, Theorem 13 shows that it lies on an extreme ray of $K(P)$.

Since T is given explicitly by $(TL)(p) = L(q_p)$, $L \in K(P)$, $p \in A$, the desired result follows.

4. The converse problem.

In order to prove that a positive multiplicative operator T is extreme (in the appropriate set) we first prove the result when $B = R$, and then apply this to the functionals defined by $f \to (Tf)(y)$ (f in A, y in Y). The first result (below) is for the cone $K(A)$; it requires the following two simple but useful inequalities. Suppose that T is a positive functional (or operator); then

(1) $(Tfg)^2 \leqq T(f^2)T(g^2)$ for all f, g in A
(2) $[T(f^2)]^2 \leqq TfT(f^3)$ for all f in A, $f \geqq 0$.

The first inequality is proved by considering the discriminant of

$$0 \leqq T(f-\lambda g)^2 = T(f^2) - 2\lambda Tfg + \lambda^2 T(g^2), \qquad \lambda \text{ real},$$

while the second inequality is proved in a similar manner by expanding $0 \leqq T[f(f-\lambda)^2]$. (Note that neither inequality requires that $1 \in A$.)

THEOREM 13. *If M is a nontrivial positive multiplicative linear functional on A and if $A = A_+ - A_+$, then M lies on an extreme ray of the cone $K(A)$.*

PROOF. The proof is immediate from the following: If

$$M = L_1 + L_2, \qquad L_1, L_2 \geqq 0,$$

then there exists a constant $\lambda \geqq 0$ such that $L_1 = \lambda M$ on A_+. To see that this assertion is true, one verifies first that if $\alpha, \beta \in [0,1]$, then $[\alpha\beta]^{\frac{1}{2}} + [(1-\alpha)(1-\beta)]^{\frac{1}{2}} \leqq 1$, with equality holding if and only if $\alpha = \beta$. We now want to show that $L_1 = \lambda M$ on A_+, for some $\lambda \geqq 0$. Since $L_1 f = 0$ whenever $f \in A_+$ and $Mf = 0$, we need only show that L_1/M is constant on those f in A_+ such that $Mf > 0$. By homogeneity, this means that we want to show that $L_1 f = L_1 g$ whenever $f, g \in A_+$ and $Mf = 1 = Mg$. Consider, first, such a function f. We have

$$Mf = 1 = L_1 f + L_2 f,$$

so if we let $\alpha = L_1 f$, $\beta = L_1 f^3$, then

$$L_2 f = 1-\alpha, \qquad L_2 f^3 = 1-\beta .$$

From inequality (2) above we conclude that

$$Mf^2 = 1 = L_1 f^2 + L_2 f^2 \leqq [\alpha\beta]^{\frac{1}{2}} + [(1-\alpha)(1-\beta)]^{\frac{1}{2}} \leqq 1 ,$$

and hence $\alpha = \beta$ and $L_1 f^2 = [\alpha\beta]^{\frac{1}{2}} = \alpha = L_1 f$. Similarly, $L_1 g^2 = L_1 g$. Now, $(f-g)^2 \geqq 0$ and $M(f-g)^2 = 0$, so

$$0 = L_1(f-g)^2 L_1 g^2 \geqq [L_1(fg-g^2)]^2 ,$$

by inequality (1) above. Thus, $L_1(fg) = L_1 g^2$ and (similarly) $L_1(fg) = L_1 f^2$. This shows that $L_1 f = L_1 g$ and completes the proof.

We know from Lemma 2 that if $K(A)$ contains an extreme ray, then $A = A_+ - A_+$, so this hypothesis does not restrict the generality of Theorem 13. One cannot expect to show that every multiplicative functional on A lies on an extreme ray of $K(A)$, since there exist multiplicative functionals which are not positive. For instance, if X is a subset of the real line and P_X is the algebra of restrictions of polynomials to X, then it is easily verified that the multiplicative functionals are those of the form $p \to p(\alpha)$ (for some real α), and that such a functional is positive if and only if α is in the closure of X. Of course, if M is a multiplicative linear functional on A and if every element of A_+ is a finite sum of squares, then $M \geqq 0$. (Thus, every positive function in P_X is a sum of squares if and only if X is dense in R.)

It is evident from Theorem 13 and the fact that $K_1(A) \subset K(A)$ that *any multiplicative functional in* $K_1(A)$ *is an extreme point of that set*; we will make subsequent use of this fact.

THEOREM 14. *Suppose that* $A = A_+ - A_+$ *and that* M *is a multiplicative functional in* $K_0(A)$. *If* $M(A_b) \neq \{0\}$, *then* M *is an extreme point of* $K_0(A)$. *If* $M(A_b) = \{0\}$, *then* M *lies on an extreme ray of* $K_0(A)$.

PROOF. Since M is multiplicative and $K_0(A) \subset K(A)$, Theorem 13 implies that M lies on an extreme ray of $K(A)$. If $M(A_b) = \{0\}$, then $R_+ M \subset K_0(A)$ and M lies on an extreme ray of $K_0(A)$. We assume, then, that $M(A_b) \neq \{0\}$. If $M = \frac{1}{2}L_1 + \frac{1}{2}L_2$, with L_1, L_2 in $K_0(A)$, then $L_i = \lambda_i M$ for some $\lambda_i \geqq 0$. Suppose that the following condition held:

(*) Given $\varepsilon > 0$, there exists f in A, $0 \leqq f \leqq 1$, with $Mf > 1-\varepsilon$.

It would follow (for each such f) that

$$1 \geqq L_i f = \lambda_i M f > \lambda_i(1-\varepsilon) ,$$

so that $\lambda_i \leqq 1$. Since $2M = \lambda_1 M + \lambda_2 M$ and $M \neq 0$, we have $\lambda_1 + \lambda_2 = 2$, and hence we would obtain $\lambda_1 = \lambda_2 = 1$, completing the proof. It remains, then, to prove property (*). Since $M(A_b) \neq \{0\}$, we can choose $f \in A_b$, $f \geqq 0$, such that $Mf > 0$. (Take the square of any element in A_b on which M does not vanish, and use the fact that M is multiplicative.) By taking a large positive multiple of f, if necessary, we can assume that $Mf(1+Mf)^{-1} > 1 - \frac{1}{2}\varepsilon$. Since f is bounded, we can choose constants λ and μ such that $0 \leqq f \leqq \mu < \lambda$. Let s_n in A be defined by

$$s_n = \sum_{k=0}^{n} \frac{f}{1+\lambda}\left(\frac{\lambda - f}{1+\lambda}\right)^k .$$

By expanding both sides of the following equation (and using the fact that M is multiplicative) it is easily verified that

$$M[f(\lambda - f)^k] = Mf(\lambda - Mf)^k .$$

Thus, if we let $\alpha = (1+\lambda)^{-1}(\lambda - Mf)$, then $0 < \alpha \leqq \lambda(1+\lambda)^{-1} < 1$ and

$$Ms_n = \frac{Mf}{1+\lambda}\sum_0^n \alpha^k = \frac{Mf}{1+\lambda}\,\frac{1-\alpha^{n+1}}{1-\alpha} = \frac{Mf}{1+Mf}(1-\alpha^{n+1}) .$$

Since $\alpha^{n+1} \to 0$, we see that $Ms_n > 1 - \varepsilon$ for sufficiently large n. Furthermore,

$$0 \leqq s_n \leqq (1+f)^{-1}f \leqq 1 ,$$

so the proof is complete.

We proceed next to consider the converse problem for operators. The case $K_1(A,B)$ is first (and easiest); we then proceed to $K_0(A,B)$ and $K(A,B)$.

PROPOSITION 15. *Any multiplicative operator T in $K_1(A,B)$ is an extreme point of $K_1(A,B)$.*

PROOF. For each y in Y, the functional defined on A by $f \to (Tf)(y)$ is multiplicative and in $K_1(A)$. As noted above, this implies that it is extreme in $K_1(A)$. It follows easily that T is extreme in $K_1(A,B)$.

It is *not* true that every multiplicative operator T in $K_0(A,B)$ is extreme, even if $T(A_b) \neq \{0\}$. Consider, for instance, the following example: Let $X = Y = R$, and let B be the algebra of all functions on R. Choose a bounded function $h \neq 0$ in B such that $h(1) = 0$, and let A be the algebra generated by h and those polynomials p which vanish at 0. Let T be the identity map which embeds A in B. Then T is certainly in $K_0(A,B)$

and is multiplicative. Define functions α_1 and α_2 in B by $\alpha_1(x)=\alpha_2(x)=1$ if $x\neq 1$, while $\alpha_1(1)=0$, $\alpha_2(1)=2$. If we let $T_i=\alpha_i T$, then

$$T_i \in K_0(A,B), \quad T_i \neq T, \qquad i = 1,2\,,$$

and $T=\frac{1}{2}T_1+\frac{1}{2}T_2$, so T is not extreme.

This example shows the need of *some* hypotheses in order that a converse theorem be valid for $K_0(A,B)$. The next result illustrates several of the possibilities. As noted in Lemma 2, the hypothesis that $A=A_+-A_+$ is no loss in generality.

PROPOSITION 16. *Suppose that $A=A_+-A_+$ and that T is a multiplicative operator in $K_0(A,B)$. If A satisfies hypothesis* (a) *or* (b), *or if $1\in A$, then T is an extreme point of $K_0(A,B)$.*

PROOF. For each y in Y, let M_y be the multiplicative functional in $K_0(A)$ defined by $M_y f=(Tf)(y)$. If M_y were an extreme point of $K_0(A)$ for each y in Y, then it would follow easily that T is extreme in $K_0(A,B)$. There are two ways for M_y to be extreme in $K_0(A)$; either $M_y(A_b)\neq\{0\}$ (so that Theorem 14 applies) or $M_y=0$. But if $1\in A$, then $M_y 1=1$ (hence $M_y(A_b)\neq\{0\}$) or $M_y 1=0$ (hence $M_y=0$). The same two cases arise if hypothesis (b) holds, i.e. if $A_b=A$. Finally, if hypothesis (a) holds and $M_y\neq 0$, then $M_y f\neq 0$ for some $f\in A_+$. It follows that

$$h = f(1+f)^{-1}\in A_b \quad \text{and} \quad h+hf = f\,,$$

so that $Mh+MhMf=Mf$. Thus, $Mh\neq 0$, that is $M(A_b)\neq\{0\}$, and the proof is complete.

We conclude this section with several results concerning $K(A,B)$. The next lemma will help us prove a partial converse to Theorem 9.

LEMMA 17. *Suppose that $T=hL$, where L is an extreme positive functional on A and h lies on an extreme ray of B_+. Then T is extreme in $K(A,B)$.*

PROOF. Suppose that $U\colon A\to B$ and that $0\leqq U\leqq T$. For fixed f in A_+, we have $0\leqq Uf\leqq hLf$; since Lf is a constant and h is extreme, we conclude that $Uf=\lambda(f)h$ for some constant $\lambda(f)\geqq 0$. It is easily verified that λ is an additive, positive homogeneous, positive functional on A_+, hence can be extended to a positive linear functional L_1 on A. (The existence of L implies that $A=A_+-A_+$, by Lemma 2.) Furthermore, L_1 satisfies $U=hL_1$. From this it follows that $0\leqq L_1\leqq L$; since L is extreme, we conclude that L_1 is a positive multiple of L and hence U is a positive multiple of T, so T is extreme.

In case the functional L is multiplicative, this yields a partial converse to Theorem 9. If L is extreme and satisfies $L(fg)=0$ for all f,g (as in the example at the end of the previous section), this shows how to construct (one-dimensional) extreme operators which satisfy conclusion (1) of Theorem 9. (See the example after Proposition 19 for an algebra B such that B_+ has nontrivial extreme rays.) An infinite dimensional operator satisfying conclusion (1) of Theorem 9 is exhibited at the end of this section.

PROPOSITION 18. *Suppose that* $1 \in A$ *and that every bounded function in* B_+ *is constant. If* $T \neq 0$ *is a multiplicative operator in* $K(A,B)$, *then* T *lies on an extreme ray of* $K(A,B)$.

PROOF. Suppose that $U: A \to B$ and $0 \leqq U \leqq T$. For each y in Y, the functionals on A defined by

$$M_y(f) = (Tf)(y) \quad \text{and} \quad L_y(f) = (Uf)(y), \qquad f \in A\,,$$

satisfy $0 \leqq L_y \leqq M_y$. Since $A = A_+ - A_+$ and M_y is multiplicative, Theorem 14 implies that there exists a number $h(y)$ with $0 \leqq h(y) \leqq 1$ and $L_y = h(y)M_y$. This means that $U = hT$, for a certain function h on Y, $0 \leqq h \leqq 1$. We can partition Y into two sets Y_0 and Y_1 as follows: Since T is multiplicative, $T1$ takes on the values 0 and 1 only; let

$$Y_0 = \{y\colon (T1)(y)=0\}, \qquad Y_1 = \{y\colon (T1)(y)=1\}\,.$$

It is immediate that $Tf = T1\,Tf = 0$ on Y_0 for *all* f in A, hence the same is true for U. Since $U1 = hT1 = h$ on Y_1, we find that $U = U1\,T$, and $0 \leqq U1 \leqq T1 \leqq 1$. Since $U1 \in B_+$, it is constant (by hypothesis) and the proof is complete.

It is interesting to note that the converse to this result is not valid. Indeed, if we let $A = B = P$ (the polynomials), then certainly $1 \in A$ and every bounded function in B_+ is constant, but the following example shows that there exists an extreme ray in $K(P,P)$ which is not generated by a multiplicative operator.

EXAMPLE. *There exists an operator* T *which lies on an extreme ray of* $K(P,P)$, *but which is not (a constant multiple of) a multiplicative operator.*

We use the same operator T which was defined in Theorem 8. Since $T1=1$, if $T=\lambda M$ for some multiplicative operator M and $\lambda \geqq 0$, then we would necessarily have $M1=1$ and $\lambda=1$, so $T=M$, contradicting the fact that T is not multiplicative. Suppose, now, that $T = \frac{1}{2}T_1 + \frac{1}{2}T_2$, with T_1, T_2 in $K(P,P)$. Since

$$2 = 2T1 = T_1 1 + T_2 1 \geqq T_1 1 \geqq 0\,,$$

the polynomial $T_1 1$ is constant (and so is $T_2 1$). If neither of these constants is zero, then we can write

$$T = (\tfrac{1}{2}T_1 1)\frac{T_1}{T_1 1} + (\tfrac{1}{2}T_2 1)\frac{T_2}{T_2 1}, \qquad \tfrac{1}{2}T_1 1 + \tfrac{1}{2}T_2 1 = 1 .$$

This expresses T as a convex combination of operators in $K_1(P,P)$; from Theorem 8 we can conclude that T_1 and T_2 are constant multiples of T. If one of the constants equals zero, say $T_1 l = 0$, then let F_x denote the functional $p \to (T_1 p)(x)$. For each real x we have $F_x l = 0$ and (using the notation in the proof of Theorem 8)

$$0 \leqq F_x p \leqq 2L_x p \quad \text{whenever} \quad p \geqq 0 .$$

By applying to F_x essentially the same argument as was used for N_x in the proof of Theorem 8, we can show that T_1 has the same form (1) as did U. Choosing (for each $x \neq 0$) positive polynomials which vanish at one of the points x^2+1 or $(x^2+1)(x+1)$ but not at the other, we see that in this case we have immediately that $0 \leqq a(x)$ and $0 \leqq -a(x)$ for each $x \neq 0$. Thus, $T_1 = 0$ (and hence $T_2 = 2T$), so T lies on an extreme ray of $K(P,P)$.

PROPOSITION 19. *Suppose that $1 \in A$ and that T is a positive multiplicative operator from A to B. If h is extreme in B_+, then $S = hT$ is extreme in $K(A,B)$.*

The proof is quite similar to that of Proposition 18: If $0 \leqq U \leqq S$, consider the functional on A defined by $f \to h(y)^{-1}(Uf)(y)$, for each y such that $h(y) > 0$, and apply Theorem 13. We omit the details.

Concrete examples of operators such as described in the above proposition may be found if $A = B = P$. Indeed, as was noted earlier, the multiplicative operators from P into P are given by composition [8]: Choose any q in P and let $(Tp)(x) = p(q(x))$, $p \in P$, x real. It is clear that such operators are positive, so it remains to identify the extreme elements h of P_+. This description is doubtless a well-known result, but we sketch a proof

A polynomial p lies on an extreme ray of P_+ if and only if p is a positive constant or p is of the form

$$p(x) = c^2 \textstyle\prod_{k=1}^{n} (x - a_k)^2 .$$

In fact, since every positive polynomial is a sum of squares of polynomials, it follows that an extreme positive polynomial must itself be a square, that is $p=q^2$. If q had an irreducible quadratic factor f, with $f\geqq 0$ say, we could let $0<m=\min f$ and hence (writing $q=rf$)

$$p = q^2 = r^2f(f-m)+r^2fm .$$

It follows from this that any zeros of q are real, and hence p has the required form. On the other hand, if p has the above form and $p=p_1+p_2$, $p_i\geqq 0$, then every zero of p is a zero of p_i. Upon factoring these common zeros, we obtain two positive polynomials r_1 and r_2 such that $r_1+r_2=1$, which implies that the r_i are constant and the p_i are constant multiples of p.

We conclude this section with an example which is relevant to Theorems 9 and 10. It exhibits an algebra A of bounded functions and an extreme operator T in $K(A,B)$ which has infinite dimensional range and which satisfies $Tfg=0$ for all f,g in A.

EXAMPLE. Let $X=R$ and let $Y\subset R$ denote the integers. Let $B=P_{|Y}$ (the polynomials restricted to the integers). Choose a real-valued continuous function f on R such that $0\leqq f(x)\leqq e^{-x^2}$, $f(x)=0$ if and only if $x\in Y$ and

$$(*) \qquad p_1f+p_2f^2+\ldots+p_nf^n = 0 \quad \text{implies} \quad p_1=p_2=\ldots=p_n=0$$

whenever $p_1,p_2,\ldots,p_n$ are polynomials. Such a function is easily constructed. The last property will obtain if $f(x)=e^{x-2}$ on $[\frac{1}{3}, \frac{2}{3}]$, say.

We define A to be the algebra generated by f and the function x, so that A consists of all functions of the form $p_1f+\ldots+p_nf^n$. Since $f(x)\leqq e^{-x^2}$, these functions are bounded. Property (*) guarantees that

$$T(p_1f+p_2f^2+\ldots+p_nf^n) = (p_1)_{|Y}$$

defines a function from A into B, and it is clear that T is linear. Furthermore, if $\sum p_kf^k\geqq 0$, then for $x\notin Y$ we have $f(x)>0$, so

$$p_1(x)+p_2(x)f(x)+\ldots+p_n(x)f^{n-1}(x) \geqq 0 .$$

By continuity (and the fact that $f(Y)=0$) we see that $p_1(y)\geqq 0$ for y in Y, so that $T\in K(A,B)$. It is clear that $TA=B$ and hence the range of T is infinite dimensional. It is also clear that $Tgh=0$ for g,h in A. It remains to prove that T lies on an extreme ray of $K(A,B)$.

Suppose that $U\in K(A,B)$ and that $0\leqq U\leqq T$. Let E be the supspace of A consisting of all functions of the form pf, with $p\in P$. The map $T_1: E\to P$ defined by $T_1pf=p$ is a one-to-one linear and positive map of E onto P, and T_1^{-1} is positive. The map from P onto B defined by

$T_2p = p_{|Y}$ is positive and multiplicative, and from Proposition 18 we know that T_2 is extreme in $K(P,B)$. Now,

$$T_{|E} = T_2T_1 \quad \text{and} \quad 0 \leqq U_{|E} \leqq T_2T_1 ,$$

so

$$0 \leqq U_{|E}T_1^{-1} \leqq T_2 ,$$

hence there exists $0 \leqq \lambda \leqq 1$ with $U_{|E}T_1^{-1} = \lambda T_2$ and therefore $U_{|E} = \lambda T_{|E}$. If $p \geqq 0$ and $k \geqq 2$, then $0 \leqq Upf^k = 0$. Since every polynomial p is the difference of positive polynomials, we see that $Upf^k = Tpf^k = 0$ for $p \in P$ and $k \geqq 2$. This implies that $U = \lambda T$, which completes the proof.

5. Remarks and open questions.

One of the most interesting questions related to the foregoing results concerns a different convex set of operators: If A and B are (real or complex) algebras of bounded functions with supremum norm, each containing the constants, say, let $U(A,B)$ denote the unit ball of all bounded linear operators $T: A \to B$ with $\|T\| \leqq 1$. It is known [1] that if $A = C(X)$, $B = C(Y)$ (real continuous functions on the compact Hausdorff spaces X and Y) and if X is metrizable, then T is extreme in $U(A,B)$ if and only if

$$\text{(M)} \qquad T1\,Tfg = Tf\,Tg \quad \text{and} \quad |T1| = 1, \qquad f,g \in A .$$

The proof of this result is not at all of an algebraic nature, but uses a selection theorem for certain set-valued functions. It has been shown in [2] that (M) characterizes the extreme points of $U(C(X),C(Y))$ for certain nonmetrizable X and metrizable Y. The problem remains open, however, for arbitrary X and Y, and is completely open (even for metrizable X) in the case of complex $C(X)$ and $C(Y)$. (The question (raised in [8]) was even open in the complex case for $U(A,B) \cap \{T: T1 = 1\}$ with A and B both equal to the disc algebra of all continuous functions on $|z| \leqq 1$ which are analytic in $|z| < 1$. J. Ryff has suggested an operator, however, which can be shown to be extreme in this set, but not multiplicative.) The purely algebraic methods which are used in the present paper for positive operators will not work for the set $U(A,B)$; there exist A and B (neither complete in the sup-norm) and an operator T which is extreme in $U(A,B)$ but which fails to satisfy either condition in (M). The operator T is *not* extreme in $U(A,B^\wedge)$, where $B^\wedge$ is the completion of B. This shows (in view of the uniqueness of extensions to $A^\wedge$) that there exist extreme operators defined between dense subalgebras of $C(X)$ and $C(Y)$ (for certain compact X and Y) which do not admit extreme extensions between $C(X)$ and $C(Y)$. This latter fact casts doubt on the assertion in [8] that the characterization of the extreme points of $K_1(A,B)$ could also be deduced from the Ionescu Tulcea result for $C(X)$ and $C(Y)$. (It was assumed that extreme operators on dense subalgebras admitted extreme extensions. Of course, this is true for the extreme operators in $K_1(A,B)$, but the only proof we know is to show first that they are multiplicative.) The examples mentioned above will appear in a paper which is devoted to questions concerning $U(A,B)$.

The aim of this paper has been to *characterize* the extreme elements of certain convex sets of operators, and nothing has been said so far about the *existence* of such extreme points. In specific examples (for example continuous functions on a compact Hausdorff space, or the polynomials on the line) it is usually possible to give fairly explicit descriptions of the *multiplicative* elements in terms of operators of composition: $Tf = f \circ \varphi$ for an appropriate map φ. Thus, to the extent that the multiplicative operators describe the extreme operators, the existence problem is reduced to considering maps between the underlying sets. Some interesting examples (relative to Proposition 6) are given in [7].

One can also approach the existence problem by attempting to find a locally convex topology on the space of operators from A to B under which the appropriate convex set is compact; an application of the Krein–Milman theorem then yields extreme points. This can always be done, for instance, if B is the dual of a Banach space, since an application of the Tyhonov product theorem shows that the sets $K_1(A,B)$ or $K_0(A,B)$ are compact in the "weak*-operator" topology. (See, for example, [6] for details.) This approach is particularly relevant to Proposition 5, where we can take B to be L^∞ (over a σ-finite measure space), hence the dual of L^1.

In the case of the cone of all positive operators, the extreme elements (under hypotheses (a) or (b), at least) appear to be one-dimensional operators of a particularly simple type, and the existence question reduces to finding multiplicative functionals on A (for example, point evaluations) and extreme rays in B_+. It should be noted that if $1 \in B$ and 1 is not on an extreme ray of B_+, then $K(A,B)$ might have *no* extreme rays: If h is a nonconstant function in B, with $0 \leqq h \leqq 1$, then for any T in $K(A,B)$, $U = hT$ defines an operator with $0 \leqq U \leqq T$. For specific examples (for example, $B = C(Y)$) this construction yields $U \neq 0$ such that U is not a constant multiple of T.

It is obvious that one can investigate questions of the type considered in this paper for abstract partially-ordered algebras. We feel that there is not much interest in doing so, however, because the algebras for which the present techniques work will almost invariably be isomorphic to algebras of functions. There is a natural class of partially ordered algebras for which much remains to be done, however; this is the case of abstract algebras with involution. In this context, to say that $T: A \to B$ is *positive* would mean that $T(x^*x)$ has the form y^*y for each x in A. If the algebras contain an identity, then one can define, for instance, the obvious analogue to the set $K_1(A,B)$ and pose the same question: Is every extreme point multiplicative? An affirmative answer is possible, of course, if the algebras are commutative and satisfy sufficient additional hypotheses to guarantee that they are isomorphic (as algebras and as partially ordered spaces) to algebras of functions. In the noncommutative case, however, the problem is exceedingly difficult. The situation has been studied for C^*-algebras A and B by Størmer [10]. He has shown that the extreme operators in $K_1(A,B)$ are "approximately" homomorphisms, provided A is commutative and B is finite dimensional. It is *not* true that every extreme operator is a homomorphism, even when A and B are the algebras of all 2×2 complex matrices. In this latter case, the extreme operators (themselves matrices) can be concretely identified; this has been done (independently) in [4], [10] and [11].

We conclude by listing some of the algebras A which satisfy *hypothesis* (*a*). The most obvious is the algebra $C(X)$ of all continuous functions on the topological space X. A slightly less obvious example (actually, a *class* of examples) is obtained by taking A to be any subalgebra (with or without 1) of $C(X)$ which is closed under the topology of uniform convergence on compact subsets of X. For this, it suffices to show that the function $z(1+z)^{-1}$ can be approximated uniformly on compact subsets of $\operatorname{Re} z \geqq 0$ by a sequence of complex polynomials $p_n(z)$ whlch vanish at 0 and are real for z real. [This can be done, for example, by taking p_n to be a partial sum of the power series expansion of $z(1+z)^{-1}$ about n, with, say, $|z(1+z)^{-1} - p_n(z)| < 1/n$ for $|z-n| \leqq n+\frac{1}{2}$].

Another class of examples consists of the algebras of all m-times differentiable (or continuously differentiable) functions ($1 \leqq m \leqq \infty$) on a region in R^n, $n = 1, 2, \ldots$.

If $1 \in A$ and A satisfies hypothesis (a), then any ideal A' in A satisfies (a): If $f \in A_+'$, then $(1+f)^{-1} \in A$ and hence $f(1+f)^{-1} \in A'$.

Finally, an algebra which obviously does *not* satisfy hypothesis (a) is given by the algebra P of all polynomials. Of course, the algebra of all rational functions with positive denominator satisfies hypothesis (a). This algebra contains P, and illustrates (for the case $A = P$) the kind of construction carried out in Lemma 12.

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