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MILMAN'S THEOREM FOR CONVEX FUNCTIONS

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1. Introduction.

Let K be a convex compact subset of a locally convex Hausdorff topological vector space, and let M be a subset of K. Then K is the closed convex hull of M if and only if the closure of M contains the extreme points of K, (e.g. [4, p. 335]). The *if* part of this fundamental result is the Krein–Milman theorem, the *only if* part is Milman's theorem. Recently, J.-C. Aggeri [1] extended the Krein–Milman theorem to convex functions. It is the purpose of the present note to give an analogous extension of Milman's theorem (Theorem 1). We also establish a dual result (Theorem 2).

2. Preliminaries.

Let E be a locally convex Hausdorff topological vector space over R, and let f be a function on E with values in $]-\infty, +\infty]$, not identically $+\infty$. Such a function is said to be *convex* if for all $x,y \in E$, $t \in]0,1[$,

$$f((1-t)x+ty) \leqq (1-t)f(x)+tf(y) .$$

Convexity of the function f is equivalent to convexity of the supergraph of f, that is, the set

$$[f] = \{(x,a) \in E \times \mathsf{R} \mid f(x) \leqq a\} .$$

The supergraph $[f]$ is closed (in the product topology on $E \times \mathsf{R}$) if and only if f is lower semi-continuous (l.s.c.) in the usual sense. Hence, f is l.s.c. and convex if and only if $[f]$ is a closed convex set. A function which is the supremum of its affine continuous minorants is clearly l.s.c. and convex. Conversely, if f is l.s.c. and convex, then f is the supremum of its affine continuous minorants.

Any function f which is minorized by some l.s.c. function has a greatest l.s.c. minorant f_{cl}. Clearly,

$$[f_{\mathrm{cl}}] = \mathrm{cl}[f] .$$

And any function f which is minorized by some l.s.c. convex function

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has a greatest l.s.c. convex minorant f''. The function f'' is the supremum of the affine continuous minorants of f. One has

$$[f''] = \operatorname{cl}\operatorname{conv}[f] .$$

(For proofs of the results quoted above, see e.g. [2]. In the terminology of conjugate convex functions [2], [3], [5], f'' is the second conjugate of f).

A function f is said to be *subdifferentiable* at a point x if some affine continuous minorant φ of f takes the value $f(x)$ at x, that is, if the graph of φ supports $[f]$ at $(x,f(x))$. An affine continuous minorant φ of f with this property is said to be *exact* at x. The set of points x such that f is subdifferentiable at x is denoted by $\operatorname{dom}\partial f$.

A function f is said to be *inf-compact* if for all $\lambda \in \mathsf{R}$ the set

$$\{x \in E \mid f(x) \leqq \lambda\}$$

is compact (or empty). A function f is said to be *inf-compact in the direction* $\xi \in E'$ (the dual of E), if the function

$$z \to f(z) - \langle \xi, z\rangle, \quad z \in E ,$$

is inf-compact. Clearly, a function which is inf-compact in some direction is l.s.c.

Following J.-C. Aggeri, we shall say that a point $x \in E$ is *extreme with respect to the function* f if $f(x) < \infty$ and f is not affine on any relatively open segment containing x. Hence, x is extreme with respect to f if and only if $(x,f(x))$ is an extreme point of $[f]$.

The *indicator function* ψ_C of a non-empty subset C of E is defined by

$$\psi_C(x) = \begin{cases} 0 & \text{for} \quad x \in C, \\ +\infty & \text{for} \quad x \in E \setminus C . \end{cases}$$

Now, denoting by f_{ext} the function $f + \psi_D$, where D is the set of points which are extreme with respect to f, the theorem of J.-C. Aggeri [1] asserts:

(I) *If f is convex and inf-compact in all directions, then f is the supremum of the affine continuous minorants of* $f_{\text{ext}} + \psi_{\operatorname{dom}\partial f}$.

Note that the conclusion in (I) is equivalent to

$$[f] = \operatorname{cl}\operatorname{conv}[f_{\text{ext}} + \psi_{\operatorname{dom}\partial f}] ,$$

which in turn implies

$$[f] = \operatorname{cl}\operatorname{conv}[f_{\text{ext}}] .$$

It is clear that application of (I) to the indicator function ψ_C of a compact convex set C yields the Krein–Milman theorem.

3. Main result.

We shall prove the following converse of (I):

THEOREM 1. *If f is convex and inf-compact in some direction, and g is such that f is the supremum of the affine continuous minorants of g, that is,*

$$[f] = \mathrm{cl}\,\mathrm{conv}\,[g]\,,$$

then g_{cl} is a minorant of f_{ext}, that is,

$$[f_{\mathrm{ext}}] \subset [g_{\mathrm{cl}}] = \mathrm{cl}\,[g]\,.$$

PROOF. Let $(y,f(y)) \in [f_{\mathrm{ext}}]$; we shall prove that then $(y,f(y)) \in \mathrm{cl}\,[g]$. Let $\xi \in E'$ be a direction of inf-compactness for f, and let

$$H = \{(x,a) \in E \times \mathsf{R} \mid a = \langle \xi, x-y\rangle + f(y) + 1\}\,,$$
$$K = \{(x,a) \in E \times \mathsf{R} \mid a \leqq \langle \xi, x-y\rangle + f(y) + 1\}\,,$$

and

$$M = ([f] \cap H) \cup ([g] \cap K)\,.$$

We claim that

(*) $$\mathrm{cl}\,\mathrm{conv}\,M = [f] \cap K\,.$$

Clearly, $\mathrm{cl}\,\mathrm{conv}\,M$ is contained in $[f] \cap K$. Suppose (z,c) is in $[f] \cap K$, and not in $\mathrm{cl}\,\mathrm{conv}\,M$. Then, by a standard separation theorem, there exists a closed hyperplane J in $E \times \mathsf{R}$ which strictly separates $\mathrm{cl}\,\mathrm{conv}\,M$ and $\{(z,c)\}$. Evidently, J is non-vertical, and the set $\mathrm{cl}\,\mathrm{conv}\,M$ is in the upper half space J^+ determined by J. Since $[f]$ is the closed convex hull of $[g]$, and (z,c) is in the lower half space J^- determined by J, there is a point $u \in E$ such that $(u,g(u))$ is in J^-. And since $\mathrm{cl}\,\mathrm{conv}\,M$ is in J^+, it follows that $(u,g(u))$ is not in K. Consequently, the segment

$$I = [(u,g(u)),\,(z,c)]$$

contains a point p which is in H. From the convexity of $[f]$ it follows that p is in $[f]$. So p is in M, and therefore in J^+. This, however, contradicts that p belongs to the segment I which is contained in J^-. Hence, we have proved (*).

To complete the proof of the theorem, note that since $(y,f(y))$ is an extreme point of $[f]$, it is an extreme point of $[f] \cap K$. And this set is compact, f being inf-compact in the direction ξ. Hence, by (*) and Milman's theorem, $(y,f(y))$ is in $\mathrm{cl}\,M$, and therefore in $\mathrm{cl}\,[g]$.

An immediate consequence of (I) and theorem 1 is the following corollary.

COROLLARY. *If f is convex and inf-compact in all directions, then* $(f_{\text{ext}}+\psi_{\text{dom}\,\partial f})_{\text{cl}}$ *is a minorant of* f_{ext}, *that is,*

$$[f_{\text{ext}}] \subset \text{cl}[f_{\text{ext}}+\psi_{\text{dom}\,\partial f}]\,.$$

It is evident that application of theorem 1 to the appropriate indicator functions yields Milman's theorem.

4. A dual result.

Consider the following two problems:

(A) *Find functions g which have the same affine continuous minorants as a given l.s.c. convex function f.*

(B) *Find sets Φ of affine continuous minorants of a given l.s.c. convex function f such that f is the supremum of the functions in Φ.*

The problems (A) and (B) are dual in a sense which may be described in terms of conjugate convex functions ([2], [3], [5]): If f is a function on a real locally convex Hausdorff topological vector space E which is minorized by some affine continuous function, then the conjugate f' of f is defined by

$$f'(\xi) = \sup_{\xi\in E}(\langle\xi,x\rangle - f(x)), \qquad \xi\in E'\,.$$

It is a l.s.c. convex function on E'. The conjugate of f', that is, the function f'' on E defined by

$$f''(x) = \sup_{\xi\in E'}(\langle\xi,x\rangle - f'(\xi)), \qquad x\in E\,,$$

is the greatest l.s.c. convex minorant of f. Therefore, if f is l.s.c. and convex, then there is a complete duality between f and f'. If so, then there is a one-to-one correspondance between the points (x,a) in $[f]$ and the affine continuous minorants $\xi \to \langle\xi,x\rangle - a$ of f', — and similarly between the points (ξ,α) in $[f']$ and the affine continuous minorants $x \to \langle\xi,x\rangle - \alpha$ of f. Hence, the problems (A) and (B) are dual in the sense that a solution of (A) for f yields a solution of (B) for f', and conversely. (In order to make the duality complete, one should require that the set Φ be closed with respect to subtraction of positive constants, and with respect to performing limits of increasing sequences of functions that differ from a fixed one by a constant.) Note that the affine continuous minorants of f of the form $x \to \langle\xi,x\rangle - f'(\xi)$, where $f'(\xi) < +\infty$, are the maximal ones, and similarly in E'. Also, note that the function $x \to \langle\xi_0,x\rangle - f'(\xi_0)$ on E

is exact at a point x_0 if and only if the function $\xi \to \langle \xi, x_0 \rangle - f(x_0)$ on E' is exact at ξ_0.

Let us say that an affine continuous minorant φ of a function f is *extreme* if for no non-zero affine continuous function ω the function $\varphi + t\omega$ is a minorant of f for all $t \in]-1,1[$. Equivalently, $x \to \langle \xi, x \rangle - \alpha$ is extreme if (ξ, α) is an extreme point of $[f']$. In [1], J.-C. Aggeri proved:

(II) *If f is convex and everywhere finite and continuous, then f is the supremum of its extreme exact affine continuous minorants.*

In fact, using the duality explained above, (II) follows immediately from (I) and the following theorem of J. J. Moreau [6]:

(III) *Let f be a l.s.c. convex function on E. Then f is finite and $\tau(E,E')$ continuous at a point x if and only if f' on E' is $\sigma(E',E)$ inf-compact in the direction x.*

Here $\tau(E,E')$ is the Mackey topology on E, and $\sigma(E',E)$ is the weak (weak*) topology on E'.

By a similar approach, we shall deduce from theorem 1:

THEOREM 2. *Let f be a l.s.c. convex function on E which is finite and continuous at some point. Let Φ be a set of affine continuous minorants of f such that f is the supremum of the functions in Φ. Then every extreme affine continuous minorant of f is a pointwise limit of functions in Φ.*

PROOF. Let A be the set of all affine continuous functions on E, equipped with the topology of pointwise convergence on E. Let $\mathrm{cl}\,\Phi$ denote the closure of Φ in A. Let

$$\Phi_1 = \{(\xi,\alpha) \in E' \times \mathsf{R} \mid x \to \langle \xi, x \rangle - \alpha \text{ is in } \mathrm{cl}\,\Phi\}$$

and let

$$\Phi_2 = \Phi_1 + \{(o,\beta) \in E' \times \mathsf{R} \mid 0 \leqq \beta\},$$

where o is the zero element of E'. Let E' be equipped with the weak topology $\sigma(E',E)$.

Now, it is easy to verify that Φ_1 is closed in $E' \times \mathsf{R}$. Therefore, Φ_2 is the supergraph of a function g on E' with $g' = f$. Hence, by theorem 1 and (III), to complete the proof of theorem it suffices to prove that g is l.s.c., that is, Φ_2 is closed. By (III), f' is inf-compact in some direction $x_0 \in E$. The affine continuous function

$$\xi \to \langle \xi, x_0 \rangle - f(x_0), \qquad \xi \in E',$$

is a maximal minorant of f'. Let $\gamma < f(x_0)$, and let K_γ be the closed non-vertical half space

$$\{(\xi,\alpha) \in E' \times \mathrm{R} \mid \alpha \leqq \langle \xi, x_0 \rangle - \gamma\} .$$

It follows from the inf-conpactness that $[f'] \cap K_\gamma$ is compact, and since Φ_1 is a closed subset of $[f']$, the set $\Phi_1 \cap K_\gamma$ is compact. This implies that the set

$$S = \Phi_1 \cap K_\gamma + \{(o,\alpha) \in E' \times \mathrm{R} \mid 0 \leqq \alpha \leqq f(x_0) - \gamma\}$$

is compact. But evidently

$$S \cap K_\gamma = \Phi_2 \cap K_\gamma ,$$

and consequently $\Phi_2 \cap K_\gamma$ is compact. Since $\gamma < f(x_0)$ is arbitrary, this proves that Φ_2 is closed.

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