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MINIMUM-STABLE WEDGES OF SEMICONTINUOUS FUNCTIONS

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1. Introduction.

Mokobodzki [13] has shown that every lower semicontinuous concave function from a compact convex subset K of a locally convex separated real topological vector space into $\mathsf{R} \cup \{\infty\}$ can be approximated from below by an increasing filtering family of continuous real concave functions on K; he has also proved in [13] a similar result for upper semicontinuous functions. Theorems 1 and 2 of the present paper generalize these two results of Mokobodzki.

Next, theorem 1 is applied in § 5 to extend the scope of the argument used in [10] to characterize Choquet simplexes. The main result here is the separation theorem, theorem 3. Some consequences are indicated, including a proposition of Mr E. B. Davies [7] that implies that every closed G_δ face of a Choquet simplex is exposed.

A simple application of the preceding theory to classical potential theory is described in § 6. This application rests on condition (S) of § 5. Boboc and Cornea [4] have indicated that more delicate arguments reveal other situations in potential theory that meet the condition (S): these are not considered here.

2. Preliminaries.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of all real continuous functions on X. We shall denote by $M(X)$, $M_+(X)$, and $P(X)$ respectively the Radon, the positive Radon, and the probability Radon measures on X. If $f: X \to \mathsf{R} \cup \{\infty\}$ is a Borel measurable function bounded below and $\mu \in M_+(X)$, we shall denote by $\mu(f)$ the extended real number $\int^* f\, d\mu$; $\mu(-f)$ will then mean $-\mu(f)$. We recall that $M(X)$ is the Banach dual of $C(X)$ for the pairing $(\mu, h) \to \mu(h)$ and that $P(X)$ is a vaguely (i.e. weak*) compact subset of $M(X)$.

We consider a wedge $\mathscr{W}$ in $C(X)$ that contains the constant functions.

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To each point $x \in X$ we assign the set of measures

$$R_x \equiv R_x(\mathscr{W}) = \{\mu \in M_+(X) : \mu(f) \leqq f(x) \ \forall f \in \mathscr{W}\}.$$

Note that, since $\mathscr{W}$ contains the constant functions, $R_x \subseteqq P(X)$; it follows that R_x is vaguely compact.

By a *$\mathscr{W}$-concave* function we shall mean any semibounded Borel measurable extended real-valued function f on X such that $\mu(f) \leqq f(x)$ whenever $x \in X$ and $\mu \in R_x$. $\mathscr{W}$-convex functions are defined analogously. Except where the contrary is indicated, we shall assume that $\mathscr{W}$ is *minimum-stable* (min-stable) in the sense that

$$\min(f,g) \in \mathscr{W} \quad \text{whenever} \quad f,g \in \mathscr{W}.$$

The main theorems of §§ 3 and 4 below (theorems 1 and 2) characterize certain lower semicontinuous, and certain upper semicontinuous, $\mathscr{W}$-concave functions in terms of monotone approximation by elements of $\mathscr{W}$.

The following construction will be used. For each upper semicontinuous function $f: X \to \mathsf{R} \cup \{-\infty\}$ and point $x \in X$ write

$$\hat{f}(x) = \inf\{g(x) : g \in \mathscr{W}, f \leqq g\},$$

so that $\hat{f}: X \to \mathsf{R} \cup \{-\infty\}$ is upper semicontinuous and

$$f(x) \leqq \hat{f}(x) \leqq \max_{y \in X} f(y).$$

For fixed x the restriction to $C(X)$ of the map $f \to \hat{f}(x)$ is real-valued and linear. This fact makes it easy to prove, by a Hahn–Banach argument, the following theorem.

PROPOSITION 1. *For each function $f \in C(X)$ and point $x \in X$,*

$$\hat{f}(x) = \max\{\mu(f) : \mu \in R_x\}.$$

We can now characterize the $\mathscr{W}$-concave continuous functions:

COROLLARY. *For each $f \in C(X)$ the following assertions are equivalent:*

(i) $f \in \overline{\mathscr{W}}$;
(ii) *f is $\mathscr{W}$-concave;*
(iii) $f = \hat{f}$.

This is a trivial extension of *Satz* 7 of [2]; the equivalence (i) $\Leftrightarrow$ (ii) is a special case of *théorème* 1 of [6]. That (i) implies (ii) is obvious. Proposition 1 supplies the step (ii) $\Rightarrow$ (iii). Finally, by Dini's theorem, the minimum-stability of $\mathscr{W}$ implies that (i) follows from (iii).

For the remainder of this section we may drop the assumption that $\mathscr{W}$ is min-stable.

By a $\mathscr{W}$*-affine* function will be meant one that is $\mathscr{W}$-concave and also $\mathscr{W}$-convex. (Thus the $\mathscr{W}$-affine continuous functions are just those in $\mathscr{A} = \overline{\mathscr{W}} \cap (-\overline{\mathscr{W}})$ when $\mathscr{W}$ *is* min-stable.)

A function defined merely on a non-empty closed subset E of X is called, by a convenient abuse of language, $\mathscr{W}$*-concave* ($\mathscr{W}$-convex etc.) if it is $\mathscr{W}_E$-concave ($\mathscr{W}_E$-convex etc.) with respect to the set of restrictions

$$\mathscr{W}_E \equiv \{f|E : f \in \mathscr{W}\} .$$

Thus to say that a function g on E is $\mathscr{W}$-concave means that g is a semi-bounded extended real-valued Borel measurable function such that $\mu(g) \leqq g(x)$ whenever $x \in E$ and $\mu \in R_x(\mathscr{W})$ with $\operatorname{supp}\mu$ (the support of μ) a subset of E (so that $\mu(g)$ has a clear meaning).

A non-empty closed subset E of X is, by definition, a $\mathscr{W}$*-extreme* subset of X if for each $x \in E$ and $\mu \in R_x(\mathscr{W})$ we have $\operatorname{supp}\mu \subseteqq E$. The following construction is useful. Suppose that E is a $\mathscr{W}$-extreme set, that

$$f : X \to \mathsf{R} \cup \{\infty\}, \qquad g : E \to \mathsf{R} \cup \{\infty\}$$

are lower semicontinuous and $\mathscr{W}$-concave, and that $g \leqq f|E$. Define $f_1 : X \to \mathsf{R} \cup \{\infty\}$ by

$$f_1(x) = \begin{cases} g(x) & \text{for} \quad x \in E , \\ f(x) & \text{for} \quad x \in X \setminus E . \end{cases}$$

Then f_1 is lower semicontinuous and $\mathscr{W}$-concave.

Now suppose (for convenience' sake) that $\mathscr{W}$ separates the points of X. Then we recall that the *Choquet boundary* $\partial_{\mathscr{W}} X$ of X relative to $\mathscr{W}$ is then defined as the set of all one-point $\mathscr{W}$-extreme subsets of X (see [1], [2]).

3. Lower semicontinuous $\mathscr{W}$-concave functions.

The main theorem here generalizes *proposition* 2 of [13]; it also describes a situation that satisfies the approximation condition (A) of [9], though we shall not use this fact here.

THEOREM 1. *A function* $f : X \to \mathsf{R} \cup \{\infty\}$ *is lower semicontinuous and* $\mathscr{W}$*-concave if and only if it is the pointwise limit of a non-empty increasing filtering family of elements of* $\mathscr{W}$.

Approximation from below without the filtering condition is dealt with by the following more elementary statement, analogous to *Lemma* 2.4.2 of [3].

PROPOSITION 2. *A function $f: X \to \mathsf{R} \cup \{\infty\}$ satisfies the equation*

$$(1) \qquad f(x) = \sup\{g(x) : g \in \mathscr{W},\ g \leqq f\}$$

for all x in X if and only if it is lower semicontinuous and such that, for each point (x,y) of X^2 for which $f(x) < f(y)$, we can find a $g \in \mathscr{W}$ such that $g(x) < g(y)$.

PROOF. The necessity of the conditions in proposition 2 is clear. To prove their sufficiency take first the case of a bounded f. The condition on pairs of points implies that for each $(x,y) \in X^2$ and each $\varepsilon > 0$ we can find $g_{x,y} \in \mathscr{W}$ such that

$$g_{x,y}(x) = f(x) - \varepsilon, \qquad g_{x,y}(y) = f(y) - \varepsilon .$$

The lower semicontinuity of f implies now that $g_{x,y}(z) < f(z)$ for all z in an open neighbourhood U_y of y. But X is compact, so we can choose finitely many points $y_1, y_2, \ldots, y_n$ in X such that

$$X = U_{y_1} \cup U_{y_2} \cup \ldots \cup U_{y_n} .$$

Writing

$$g_x = \min(g_{x,y_1}, g_{x,y_2}, \ldots, g_{x,y_n}) ,$$

we have

$$g_x \in \mathscr{W}, \qquad g_x < f, \qquad g_x(x) = f(x) - \varepsilon ,$$

which yields equation (1). For a general f we consider the functions

$$f_n \equiv \min(f, n)$$

and apply the above reasoning.

PROOF OF THEOREM 1. To prove theorem 1 we have to do somewhat more. Suppose that $f: X \to \mathsf{R} \cup \{\infty\}$ is lower semicontinuous and $\mathscr{W}$-concave. To prove the stated approximation property it is enough to show that for each $u \in C(X)$ with $u < f$ we can find $g \in \mathscr{W}$ such that $u \leqq g < f$.

The lower semicontinuity of f allows us to choose $\varepsilon > 0$ that $u + \varepsilon \leqq f$. Taking $x \in X$ and $\mu \in R_x$ we have

$$\mu(u) + \varepsilon \leqq \mu(f) \leqq f(x) .$$

Hence, by proposition 1, $\hat{u} + \varepsilon \leqq f$. For each x in X we can accordingly find a function $h_x \in \mathscr{W}$ such that

$$u \leqq h_x, \qquad h_x(x) < f(x) ,$$

and then an open neighbourhood V_x of x such that $h_x(z) < f(z)$ for all $z \in V_x$. Choosing a finite covering

$$X = V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_k}$$

and taking

$$h = \min(h_{x_1}, h_{x_2}, \ldots, h_{x_k}),$$

we have $h \in \mathscr{W}$ and $u \leqq h < f$, as desired.

If, conversely, f satisfies the approximation condition then it is obviously lower semicontinuous and $\mathscr{W}$-concave.

4. Upper semicontinuous $\mathscr{W}$-concave functions.

The following result generalizes *proposition* 1 of Mokobodzki's paper [13].

THEOREM 2. *A function $f: X \to \mathsf{R} \cup \{-\infty\}$ is upper semicontinuous and $\mathscr{W}$-concave if and only if it is the infimum of a non-empty family of elements of $\mathscr{W}$.*

PROOF. Since $\mathscr{W}$ is min-stable the functions that satisfy this infimum condition are actually pointwise limits of non-empty decreasing *filtering* families of elements of $\mathscr{W}$ and hence are $\mathscr{W}$-concave as well as upper semicontinuous.

Suppose, conversely, that $f: X \to \mathsf{R} \cup \{-\infty\}$ is upper semicontinuous and $\mathscr{W}$-concave. It suffices to show that $f = \hat{f}$. Choose $x \in X$, $\varepsilon > 0$, and take first the case $f(x) > -\infty$. Then, for each $\mu \in R_x$,

$$\mu(f) \leqq f(x) < f(x) + \varepsilon .$$

Since f is upper semicontinuous we can find for each $\mu \in R_x$ a function v_μ in $C(X)$ such that

$$f \leqq v_\mu, \qquad \mu(v_\mu) < f(x) + \varepsilon .$$

It follows that there is a relative vague neighbourhood O_μ of μ in R_x such that

$$\nu(v_\mu) < f(x) + \varepsilon, \quad \nu \in O_\mu .$$

Recalling that R_x is vaguely compact, we find a finite covering

$$R_x = O_{\mu_1} \cup O_{\mu_2} \cup \ldots \cup O_{\mu_n} .$$

Defining now

$$v = \min(v_{\mu_1}, v_{\mu_2}, \ldots, v_{\mu_m})$$

we have $v \in C(X)$, $v \geqq f$, and, for all $\nu \in R_x$,

$$\nu(v) \leqq \min_{1 \leqq r \leqq m} \nu(v_{\nu_r}) < f(x) + \varepsilon .$$

By proposition 1 this implies that

$$\hat{v}(x) < f(x)+\varepsilon .$$

We can therefore find a function g in $\mathscr{W}$ such that $g \geqq v \geqq f$ and

$$g(x) < f(x)+\varepsilon ,$$

which shows that $\hat{f}(x)=f(x)$ at each point x for which $f(x) > -\infty$.

If $f(x) = -\infty$ one shows by a similar argument that for each natural number n there is a function $g \in \mathscr{W}$ such that $g \geqq f$ and $g(x) < -n$, from which it follows that $\hat{f}(x) = -\infty$.

COROLLARY. *An upper semicontinuous function* $f:X \to \mathrm{R} \cup \{-\infty\}$ *is* $\mathscr{W}$*-concave if and only if* $f=\hat{f}$.

5. A separation property.

In this section we suppose that $\mathscr{W}$ satisfies the separation condition (S): *whenever* $-f$, $g \in \mathscr{W}$ *with* $f<g$ *we can find a* $\mathscr{W}$*-affine continuous function* h *such that* $f<h<g$.

This comes very close to saying that $\mathscr{W}$ is a "geometrical simplex" in the sense of Boboc and Cornea [4].

The wedge of all continuous concave functions on a Choquet simplex has property (S) [4]; we shall consider a second example in § 6.

The results of §§ 3 and 4 make it natural to enquire when semicontinuous $\mathscr{W}$-affine functions can be approximated by filtering families of elements of $\mathscr{A}$. A partial answer is given by

PROPOSITION 3. *Suppose that* $\mathscr{W}$ *has the property* (S) *and that* $g:X \to \mathrm{R} \cup \{\infty\}$ *is a lower semicontinuous* $\mathscr{W}$*-affine function. Then* g *is the supremum of an increasing filtering family of elements of* $\mathscr{A}$.

For ordinary affine functions Mokobodzki has a better result (*corollaire* to *proposition* 2 of [13]).

PROOF. Let $f \in C(X)$ satisfy $f<g$. We seek an h in $\mathscr{A}$ to satisfy $f<h<g$. By theorem 2 we can find u in $-\mathscr{W}$ such that $f<u<g$. Then by theorem 1 there is a $v \in \mathscr{W}$ such that $u<v<g$. By property (S) we can now find $h \in \mathscr{A}$ to satisfy $u<h<v$, and this concludes the proof.

With the help of theorem 1 we now show that property (S) implies a similar property for the semicontinuous functions of that theorem. The following theorem generalizes the main result of [10].

THEOREM 3. *Suppose that $\mathscr{W}$ has the property* (S) *and that*

$$-f, g \colon X \to \mathsf{R}\cup\{\infty\}$$

are $\mathscr{W}$-concave lower semicontinuous functions such that $f \leqq g$. Then there is a $\mathscr{W}$-affine real continuous function h such that $f \leqq h \leqq g$.

PROOF. Suppose first that $f<g$. Then we can find a function $w \in C(X)$ such that $f<w<g$, by a well known theorem of topology. Next we can choose, by theorem 1, functions $-u, v \in \mathscr{W}$ such that

$$f < u < w < v < g .$$

Finally by condition (S) we can find $h \in \mathscr{A}$ satisfying $u<h<v$, which completes the proof for this case.

To complete the proof one argues from the above special case by using a device of Dieudonné [8] (already used for concave functions on a Choquet simplex in [10]).

First define

$$\alpha = \inf\{g(x) : x \in X\}, \qquad \beta = \sup\{f(x) : x \in X\} .$$

Dismissing the trivial cases $\alpha=\infty$, $\beta=-\infty$, we can suppose α, β real. In this case it suffices to consider the two functions $\max(\alpha, f)$ and $\min(\beta, g)$. This assertion is trivial when $\alpha \geqq \beta$. When $\alpha<\beta$ we have

$$f \leqq \max(\alpha, f) \leqq \min(\beta, g) \leqq g ,$$

and $-\max(\alpha, f)$, $\min(\beta, g)$ are bounded real lower semicontinuous $\mathscr{W}$-concave functions. We can therefore take it that f, g are bounded real functions: suppose this. (These remarks clarify a passage in [10].)

One now defines by recurrence three sequences $\{f_m\}$, $\{g_m\}$, $\{h_m\}$ of real-valued functions on X such that:

(a) $-f_m, g_m$ are lower semicontinuous and $\mathscr{W}$-concave;
(b) $h_m \in \mathscr{A}$;
(c) for each $m \geqq 0$,

$$f-2^{-m} \leqq f_m < h_m < g_m \leqq g+2^{-m} . \tag{2}$$

To construct such sequences take $f_0=f-1$, $g_0=g+1$ and choose, using the first part of the proof, $h_0 \in \mathscr{A}$ so that $f_0<h_0<g_0$. At the nth step define

$$f_{n+1} = \max(f-2^{-(n+1)}, h_n-2^{-(n+1)}) ,$$
$$g_{n+1} = \min(g+2^{-(n+1)}, h_n+2^{-(n+1)}) .$$

Then $-f_{n+1}, g_{n+1} \colon X \to \mathsf{R}$ are $\mathscr{W}$-concave and $f_{n+1}<g_{n+1}$. We can therefore, by the first part of the proof, take $h_{n+1} \in \mathscr{A}$ so that $f_{n+1}<h_{n+1}<g_{n+1}$, which yields (2) for $m=n+1$, and also

$$h_n - 2^{-(n+1)} < h_{n+1} < h_n + 2^{-(n+1)},$$

so that

$$\|h_{n+1} - h_n\| \leqq 2^{-(n+1)}.$$

Hence $h = \lim_{n\to\infty} h_n$ exists in $\mathscr{A}$ and satisfies $f \leqq h \leqq g$.

COROLLARY. *Suppose that* $\mathscr{W}$ *has the property* (S) *and that*

$$-f, g : X \to \mathsf{R} \cup \{\infty\}$$

are $\mathscr{W}$*-concave lower semicontinuous functions such* $f \leqq g$. *Suppose further that* E *is a* $\mathscr{W}$*-extreme subset of* X *and that* $h : E \to \mathsf{R}$ *is a* $\mathscr{W}$*-affine continuous function such that*

$$f|E \leqq h \leqq g|E.$$

Then there exists a function $\bar{h}$ *in* $\mathscr{A}$ *that extends* h *and satisfies* $f \leqq \bar{h} \leqq g$.

PROOF. To prove this let

$$g_1(x) = \begin{cases} g(x) & \text{for} \quad x \in X \setminus E, \\ h(x) & \text{for} \quad x \in E, \end{cases}$$

$$f_1(x) = \begin{cases} f(x) & \text{for} \quad x \in X \setminus E, \\ h(x) & \text{for} \quad x \in E. \end{cases}$$

By a remark of § 2 the functions $-f_1, g_1$ are $\mathscr{W}$-concave and lower semicontinuous, and, obviously, $f_1 \leqq g_1$. So by theorem 3 we can choose $\bar{h} \in \mathscr{A}$ such that $f_1 \leqq \bar{h} \leqq g_1$. This $\bar{h}$ clearly meets our requirements.

Effros [11] has used a particular case of this corollary (see his theorem 2.4) in a study of the facial structure of Choquet simplexes.

We shall call a subset A of X a $\mathscr{W}$*-peak set* if there exists a $g \in \mathscr{W}$ such that

$$A = \{x \in X : g(x) = \max_{y \in x} g(y)\}.$$

Davies [7] has deduced from the corollary to theorem 3 the following result.

PROPOSITION 4. *Suppose that* $\mathscr{W}$ *has the property* (S) *and that* E *is a subset of* X *such that* (a) E *is a* G_δ *set,* (b) E *is* $\mathscr{W}$*-extreme,* (c) E *is an intersection of* $\mathscr{W}$*-peak sets. Then there exists a function* $h \in \mathscr{A}$ *such that*

$$\begin{aligned} h(x) &= 1 \quad \textit{for} \quad x \in E, \\ h(x) &< 1 \quad \textit{for} \quad x \in X \setminus E. \end{aligned}$$

It follows immediately that *if* Q *is a* G_δ *face of a Choquet simplex* K *then there exists a continuous affine function* $h : K \to \mathsf{R}$ *such that* $h(x) = 1$ *on* Q, *and* $h(x) < 1$ *on* $X \setminus Q$. (That is, Q is *exposed*.) For the case of the extreme points of a metrizable Choquet simplex this was conjectured by Bauer (p. 121 of [2]) and has been announced as a result by Boboc and Cornea [4].

6. Superharmonic functions.

We consider here the application of the preceding theory to a simple situation in classical potential theory.

Let Ω be a bounded domain in R^n for some $n \geqq 2$ and take X to be $\bar{\Omega}$. If a function $f: X \to \mathsf{R} \cup \{\infty\}$ can be extended to a function defined and superharmonic on some neighbourhood (depending on f) of X we say that f belongs to the class $\mathscr{S}$. If an extension of the above type to a *continuous* superharmonic function is possible we say that f belongs to the class $\mathscr{W}$. Obviously $\mathscr{W} \subseteqq \mathscr{S}$ and, by the trivial half of theorem 1 and a result of classical potential theory (see § 6 of *Chapitre* II of [5]) all the functions in $\mathscr{S}$ are $\mathscr{W}$-concave. Finally we say that $f \in \mathscr{H}$ if f admits an extension to a function defined and harmonic in a neighbourhood (depending on f) of X. Obviously $\mathscr{H} \subseteqq \mathscr{W} \cap (-\mathscr{W})$, and the functions in $\bar{\mathscr{H}}$ are all $\mathscr{W}$-affine.

The wedge $\mathscr{W}$ has the property (S). In fact one can prove directly, without using theorem 3, the analogous separation property for $\mathscr{S}$:

PROPOSITION 5. *Suppose that* $-f, g \in \mathscr{S}$ *with* $f < g$. *Then there is a function* $h \in \mathscr{H}$ *such that* $f < h < g$.

PROOF. We can find bounded open sets G, G_1 such that

$$X \subseteqq G \subseteqq \bar{G} \subseteqq G_1$$

with $-f, g$ extensible to superharmonic functions (denoted by the same symbols $-f, g$) on G_1. Now choose $k \in C(G^*)$ so that

$$f|G^* < k < g|G^* ,$$

and solve the Dirichlet problem for G with boundary data k. The solution function $h: G \to \mathsf{R}$ clearly satisfies

$$f|G < h < g|G .$$

On restricting h to X we obtain the desired element of $\mathscr{H}$.

COROLLARY 1. *The set* $\mathscr{A} \equiv \bar{\mathscr{W}} \cap (-\bar{\mathscr{W}})$ *of all* $\mathscr{W}$*-affine continuous real functions coincides with* $\bar{\mathscr{H}}$.

PROOF. We have remarked that $\bar{\mathscr{H}} \subseteqq \mathscr{A}$. Suppose conversely that

$f \in \mathscr{A}$. Then f is both $\mathscr{W}$-concave and $\mathscr{W}$-convex and so, by the corollary to proposition 1, we can choose $-u, v \in \mathscr{W}$ so that

$$f-\varepsilon < u < f < v < f+\varepsilon .$$

By proposition 5 we can find $g \in \mathscr{H}$ so that $u<g<v$. This proves that $f \in \overline{\mathscr{H}}$.

Corollary 2. *Suppose that* $-f, g : X \to \mathsf{R} \cup \{\infty\}$ *are* $\mathscr{W}$*-concave lower semicontinuous functions (e.g. members of* $\mathscr{S}$*) and that* $f \leqq g$. *Then there is a function* $h \in \overline{\mathscr{H}}$ *such that* $f \leqq h \leqq g$.

This is now immediate, by theorem 3.

Now consider the $\overline{\mathscr{H}}$-peak sets. A single-point $\overline{\mathscr{H}}$-peak set is called an $\overline{\mathscr{H}}$-*peak point*. (By the maximum principle every such point is in the topological boundary Ω^* of Ω.) We can now give a very short proof of the following result of Gamelin and Rossi [12].

Proposition 6. *The Choquet boundary* $\partial_{\overline{\mathscr{H}}} X$ *of* X *relative to* $\overline{\mathscr{H}}$ *is precisely the set of all* $\overline{\mathscr{H}}$*-peak points of* X.

Proof. That $\overline{\mathscr{H}}$-peak points are in the Choquet boundary is clear. Conversely if $x_0 \in \partial_{\overline{\mathscr{H}}} X$ then *a fortiori* $x_0 \in \partial_{\mathscr{W}} X$. The set $\{x_0\}$ therefore satisfies conditions (a) and (b) of proposition 4; that it also satisfies (c) is clear from, for instance, the fact that ordinary affine functions are everywhere harmonic. Using proposition 5 and its corollary 1 we see that proposition 4 can be applied to show that x_0 is an $\overline{\mathscr{H}}$-peak point of X.

Proposition 6 makes it possible to apply the corollary to theorem 3 to the present situation to obtain the following result.

Proposition 7. *Let* $-f, g : X \to \mathsf{R} \cup \{\infty\}$ *be* $\mathscr{W}$*-concave lower semicontinuous functions (e.g. members of* $\mathscr{S}$*) such that* $f \leqq g$. *Suppose further that* E *is a closed non-empty set of* $\overline{\mathscr{H}}$*-peak points of* X *and that* $h : E \in \mathsf{R}$ *is a continuous function such that*

$$f|E \leqq h \leqq g|E .$$

Then there exists a function $\bar{h} \in \overline{\mathscr{H}}$ *that extends* h *and satisfies* $f \leqq \bar{h} \leqq g$.

Proof. By proposition 6 the set E is $\overline{\mathscr{H}}$-extreme and hence also $\mathscr{W}$-extreme. Moreover, since

$$E \subseteq \partial_{\overline{\mathscr{H}}} X \subseteq \partial_{\mathscr{W}} X ,$$

the function $h: E \to \mathsf{R}$ is trivially $\mathscr{W}$-affine. By the corollary to theorem 3 and corollary 1 of proposition 5, the result now follows.

The weaker statement that

$$\overline{\mathscr{H}}|E = C(E)$$

whenever E is as in proposition 7 complements a result (theorem 1.3) of Gamelin and Rossi [12].

It is clear from a remark of Boboc and Cornea [4] about condition (S) that the foregoing results about $\overline{\mathscr{H}}$-peak points have their counterparts for the regular points for the Dirichlet problem in Ω (see also *Satz* 16 of Bauer [2]).

Note added in proof, 1 July, 1966.

Mr. E. B. Davies has pointed out to me that the approximation technique used above to prove theorem 3 can also be applied, *mutatis mutandis*, to theorem 1 (and that there is a somewhat similar use of the technique in another connection in L. Nachbin's book *Order and Topology* (van Nostrand, 1965)). This remark yields immediately the following statement.

THEOREM 1′. *Let* $f\colon X \to \mathsf{R} \cup \{\infty\}$ *be a lower semicontinuous* $\mathscr{W}$*-concave function and let* $u \in C(X)$ *be such that* $u \leqq f$. *Then there exists a* $\mathscr{W}$*-concave funtion* $v \in C(X)$ *such that* $u \leqq v \leqq f$.

It is now a simple exercise to prove the following

COROLLARY. *Let* E *be a* $\mathscr{W}$*-extreme subset of* X *and suppose that* K *is a compact subset of* X *with* $K \cap E = \emptyset$. *Then there is a* $\mathscr{W}$*-concave function* $v \in C(X)$ *such that*

(a) $0 \leqq v \leqq 1$,
(b) $v(x) = 0$ *for all* $x \in E$,
(c) $v(x) = 1$ *for all* $x \in K$.

If also E *is a* G_δ *set then we can find a* $\mathscr{W}$*-concave function* $v \in C(X)$ *satisfying* (a), (b), *and such that* $v(x) > 0$ *for all* $x \in X \setminus E$.

Mr. A. Y. Lazar has kindly communicated to me a number of theorems about Choquet simplexes that he has obtained independently by methods rather different from those used above. In particular he has proved substantial portions of proposition 4 and the corollary to theorem 3; his methods also yields further results, to be published.

REFERENCES

1. H. Bauer, *Minimalstellen von Funktionen und Extremalpunkte* II, Arch. Math. 11 (1960), 200–205.
2. H. Bauer, *Šilovscher Rand und Dirichletsches Problem,* Ann. Institut Fourier (Grenoble) 11 (1961), 89–136.
3. H. Bauer, *Konvexität in topologischen Vektorräumen,* Vorlesung an der Universität Hamburg, Hamburg, 1965, pp. 109.
4. N. Boboc and A. Cornea, *Cônes des fonctions continues sur un espace compact,* C. R. Acad. Sci. Paris 261 (1965), 2564–2567.
5. M. Brelot, *Élements de la théorie classique du potentiel,* Les Cours de la Sorbonne, Paris, 1959, pp. 191.
6. G. Choquet and J. Deny, *Ensembles semi-réticulés de fonctions continues,* J. Math. Pures Appl. 36 (1957), 179–190.
7. E. B. Davies, *A generalized theory of convexity.* (To appear.)
8. J. Dieudonné, *Une généralisation des espaces compacts,* J. Math. Pures Appl. 23 (1944), 65–76.
9. D. A. Edwards, *Choquet boundary theory for certain spaces of lower semicontinuous functions,* Proceedings of the Tulane Symposium on function algebras 1965. (To appear.)
10. D. A. Edwards, *Séparation des fonctions réelles définies sur un simplexe de Choquet,* C. R. Acad. Sci. Paris 261 (1965), 2798–2800.
11. E. G. Effros, *Structure in simplexes,* Mimeographed notes, Mathematical Institute, Aarhus University, Aarhus 1965, pp. 31.
12. T. W. Gamelin and H. Rossi, *Jensen measures and algebras of analytic functions,* Mimeographed notes, M.I.T., 1965, pp. 54.
13. G. Mokobodzki, *Quelques propriétés des fonctions numériques convexes sur un ensemble convexe compact,* Séminaire Brelot–Choquet–Deny de la théorie du potentiel, 6e année (1962), exposé no 9, pp. 3.

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