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THE CAUCHY PROBLEM FOR SYMMETRIC HYPERBOLIC SYSTEMS IN L_p

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1. Introduction.

It is well known that the initial-value problem for a symmetric hyperbolic system,

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j \dfrac{\partial u}{\partial x_j} + Bu, & x \in R^n , \\ u(0,x) = u_0(x), & 0 \leqq t \leqq T , \end{cases}$$

is well posed in L_2. The purpose of this note is to prove that the problem (1) is well posed in L_p, $p \neq 2$, $1 \leqq p \leqq \infty$, if and only if the matrices A_j commute (Theorem 2). This will be proved by noticing that, a necessary and sufficient condition for (1) to be well posed in L_p is that $\exp\left(i \sum_{j=1}^{n} A_j y_j\right)$ is a multiplier on L_p which in turn will be proved to be the case if and only if the A_j commute (Theorem 1). This last statement follows by application of the technique developed by Hörmander in [1] (frequent references will be made to that paper) and a matrix theorem by Motzkin and Taussky [3].

The corresponding problem for the wave operator has been treated by Littman [2]; his result is included in ours.

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2. Multipliers on L_p.

First some notation. If $v = (v_1, \ldots, v_N)$ and $u = (u_1, \ldots, u_N)$ are complex vectors, $\langle u,v \rangle$ will denote their scalar product and $|v|$ the Euclidean norm,

$$\langle u,v \rangle = \sum_{j=1}^{N} u_j \bar{v}_j, \qquad |v| = \left(\sum_{j=1}^{N} |v_j|^2 \right)^{\frac{1}{2}} .$$

The norm $|A|$ of an $N \times N$-matrix A is defined as

$$|A| = \sup \{ |Av| ;\ v = (v_1, \ldots, v_N),\ |v| \leqq 1 \} .$$

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If $\Omega \subset R^n$ is open and $v_j \in C^\infty(\Omega)$ for $j=1,\ldots,N$, then we say that $v=(v_1,\ldots,v_N)$ belongs to $\mathscr{C}^\infty(\Omega)$. If $g \in C^\infty(R^n)$, and if

$$\sup\{|x|^m |D^k g(x)|;\ x \in R^n\} < \infty \tag{2}$$

for $m=0,1,\ldots$ and for any multi-index $k=(k_1,\ldots,k_n)$,

$$D^k = (\partial^{k_1}/\partial x_1{}^{k_1}) \ldots (\partial^{k_n}/\partial x_n{}^{k_n}) ,$$

then we say that g belongs to S. We give the linear space S the topology defined by the family (2) of semi-norms. We denote by $\mathscr{S}$ the set of functions $v=(v_1,\ldots,v_N)$ with $v_j \in S$, $j=1,\ldots,N$. The dual space S' of S is the set of tempered distributions (in the sense of Schwartz).

The convolution between a tempered distribution μ and a function $g \in S$ is denoted by $\mu * g$, and defined by $\mu(g(x-\cdot))=\mu * g(x)$. This notion also has an obvious meaning if g, say, is replaced by a vector in $\mathscr{S}$. We can then also replace μ by an $N \times N$-matrix, the elements of which are tempered distributions. The Fourier transform of a tempered distribution μ is denoted by $\hat{\mu}$, and defined by $\hat{\mu}(f)=\mu(\hat{f})$, $f \in S$,where $\hat{f}$ is the function

$$\hat{f}(y) = \int_{R^n} \exp(2\pi i\langle x,y\rangle) f(x)\, dx .$$

The Fourier transform is also defined for matrices and vectors of tempered distributions by applying the transform elementwise.

By $\mathscr{L}_p$ we means the set of functions $v=(v_1,\ldots,v_N)$ with $v_j \in L_p$, $j=1,\ldots,N$, and for $p<\infty$ we set

$$\|v\|_p = \left(\int_{R^n} |v(x)|^p\, dx\right)^{1/p}$$

and for $p=\infty$

$$\|v\|_\infty = \operatorname{ess\,sup}\{|v(x)|;\ x \in R^n\} .$$

Classically a multiplier on L_p is a function λ such that for each $f \in L_p$, $\lambda\hat{f}$ is the Fourier transform of a function in L_p. Following Hörmander [1] we formalize this as follows: We say that λ is a multiplier on L_p, $\lambda \in M_p$, if $\lambda \in S'$ and if

$$M_p(\lambda) = \sup\{\|\hat{\lambda} * f\|_p;\ f \in S,\ \|f\|_p \leqq 1\} < \infty .$$

We will need the following natural generalization to matrices: We define $\mathscr{M}_p$, the multipliers on $\mathscr{L}_p$, as the set of $N \times N$-matrices μ with elements in S' satisfying

$$\mathscr{M}_p(\mu) = \sup\{\|\hat{\mu} * v\|_p;\ v \in \mathscr{S},\ \|v\|_p \leqq 1\} < \infty .$$

Since the norms $\|v\|_p$ and $\sup_j \|v_j\|_p$ are equivalent, this definition can also be expressed by saying that $\mu = (\mu_{jk}) \in \mathscr{M}_p$ if $\mu_{jk} \in M_p, j,k = 1,\ldots,N$.

In order to get shorter statements it will be convenient to define $M_p(\lambda) = \infty$ if $\lambda \notin M_p$, and correspondingly for $\mathscr{M}_p$. Thus $\mu \in \mathscr{M}_p$ if and only if $\mathscr{M}_p(\mu) < \infty$.

We collect some facts about $\mathscr{M}_p$ in the following lemma.

LEMMA 1. *Suppose* $1 \leqq p \leqq \infty$. *Then*

(i) $\mathscr{M}_p = \mathscr{M}_q$, $1/p + 1/q = 1$, and $\mathscr{M}_1 \subset \mathscr{M}_p \subset \mathscr{M}_2$.

(ii) $\mathscr{M}_p$ *is a Banach algebra under pointwise (matrix-) multiplication and addition, with the norm* $\mathscr{M}_p(\cdot)$. *It is non-commutative for* $N > 1$.

(iii) $\mathscr{M}_2$ *is the set of essentially bounded* $N \times N$*-matrices, and* $\mathscr{M}_2(\cdot) = \operatorname{ess\,sup} |\cdot|$. $\mathscr{M}_1$ *is the set of* $N \times N$*-matrices, the elements of which are Fourier–Stieltjes transforms of bounded measures.*

(iv) *Suppose* $y_0 \in R^n$ *and* $a \in R - \{0\}$ *and let* $a^*f(y) = f(ay)$ *and* $f_{y_0}(y) = f(y + y_0)$. *Then* $\mathscr{M}_p(f) = \mathscr{M}_p(a^*f) = \mathscr{M}_p(f_{y_0})$.

(v) *Let* $\mathscr{M}_p(f_i) \leqq C$, *all* $i \in I$, *and suppose* $f_i \to f$ *in* S' *(e.g. uniformly on compact subsets of* R^n*). Then* $\mathscr{M}_p(f) \leqq C$.

(vi) *Let* $\alpha_j \in R$, $j = 0, 1, \ldots, n$, *and* $\alpha(y) = \alpha_0 + \sum_{j=1}^n \alpha_j y_j$. *Then* $M_p(\exp(i\alpha)) = 1$.

(vii) *If* $k \in S$, *then* $\mathscr{M}_p(kE) \leqq \|\hat{k}\|_1$, *where* E *is the unite matrix.*

PROOF. For the case $N = 1$ these statements are all contained in Chapter I of [1]. Most of the generalisations to $N > 1$ are obvious. Below we will just give references to the corresponding statements in [1] for those cases.

(i) For $N = 1$ this is Theorem 1.3 in [1].

(ii) Corollary 1.4 in [1].

(iii) Theorem 1.4 and 1.5 in [1].

(iv) Theorem 1.13 in [1]. See also Lemma 3(iii) below.

(v) By Hölders inequality we have $(1/p + 1/q = 1)$

$$\left| \int \langle \hat{f}_i * u(x), v(x) \rangle \, dx \right| \leqq C \|u\|_p \, \|v\|_q, \qquad u,v \in \mathscr{S}.$$

Since $f_i \to f$ in S' implies that $\hat{f}_i \to \hat{f}$ in S', we see that also $\hat{f}$ satisfies this inequality. The converse of Hölders inequality then gives

$$\|\hat{f} * u\|_p \leqq C \|u\|_p, \qquad u \in \mathscr{S},$$

that is, $\mathscr{M}_p(f) \leqq C$.

(vi) Multiplying $\hat{u} \in S$ by $\exp(i\alpha)$ corresponds to a translation of u with $(\frac{1}{2}\pi)(\alpha_1, \ldots, \alpha_n)$ followed by multiplication with $\exp(i\alpha_0)$, and hence $M_p(\exp(i\alpha)) = 1$.

(vii) This follows from the inequality

$$\|\hat{k}*u\|_p \leqq \|\hat{k}\|_1 \|u\|_p, \qquad k \in S,\ u \in \mathscr{S}\ (\text{or } k \in \mathscr{S},\ u \in S)\,, \tag{3}$$

which is proved just as in the scalar case.

We want to study functions which are locally multipliers on $\mathscr{L}_p$ and so make the following definition: Let B be an open ball in R^n (the open ball with center x and radius r will be denoted $B(x,r)$). We say that an $N \times N$-matrixfunction φ is an $\mathscr{L}_p$-multiplier on B, $\varphi \in \mathscr{M}_{p,B}$, if there is a $\mu \in \mathscr{M}_p$ such that $\varphi = \mu$ on B. If $\varphi \in \mathscr{M}_{p,B}$ and $\varphi = \mu$ on B, $\mu \in \mathscr{M}_p$, we can define

$$\mathscr{M}_{p,B}(\varphi) = \sup\{\|\hat{\mu}*\hat{u}\|_p;\ u \in \mathscr{S},\ u=0 \text{ outside } B,\ \|\hat{u}\|_p \leqq 1\}$$

since μu does not depend on the behavior of μ outside B. We note that $\mathscr{M}_{p,B}(\cdot)$ is an semi-norm, and that $\mathscr{M}_{p,R^n}(\cdot) = \mathscr{M}_p(\cdot)$. For $N=1$ we write $M_{p,B}$ and $M_{p,B}(\cdot)$.

The following well known lemma will be useful in this context.

Lemma 2. *Suppose that B is a bounded open ball in R^n and ε a positive number. Then there is a function $k \in S$ such that $k=1$ on B, k has compact support and $\|\hat{k}\|_1 \leqq 1+\varepsilon$.*

Proof. We can suppose that $B = B(0,r)$. Let $m(r)$ be the volume of $B(0,r)$. Choose ϱ so that $m(r+\varrho)+1 \leqq (1+\varepsilon)^2 m(\varrho)$, and let g be the characteristic function of $B(0,\varrho)$. Further choose $h \in S$, such that $h=1$ on $B(0,r+\varrho)$ and $h=0$ outside $B(0,r+2\varrho)$ and

$$\int |h(y)|^2\, dy \leqq m(r+\varrho)+1\,.$$

Set

$$k(y) = (m(\varrho))^{-1}\, h*g(y)\,.$$

Then $k \in C^\infty(R^n)$, $k=1$ on $B(0,r)$ and has compact support. By Schwartz' inequality and Parsevals formula

$$\|\hat{k}\|_1 \leqq (m(\varrho))^{-1} \|h\|_2 \|g\|_2 \leqq \left(\frac{m(r+\varrho)+1}{m(\varrho)}\right)^{\frac{1}{2}} \leqq 1+\varepsilon$$

and so k is the desired function.

We can now give some facts about $\mathscr{M}_{p,B}$.

Lemma 3. *Suppose $1 \leqq p \leqq \infty$. Then*

(i) *if $B \subset B'$, then $\mathscr{M}_{p,B}(\varphi) \leqq \mathscr{M}_{p,B'}(\varphi)$.*

(ii) *if $a \in R-\{0\}$ and $y_0 \in R^n$, then $\mathscr{M}_{p,B}(\varphi) = \mathscr{M}_{p,a^{-1}B}(a^*\varphi) = \mathscr{M}_{p,B-y_0}(\varphi_{y_0})$.*

(iii) $\mathcal{M}_{p,B}(\nu\varphi) \leqq \mathcal{M}_p(\nu)\mathcal{M}_{p,B}(\varphi)$ *and if* $k \in S$, *then* $\mathcal{M}_{p,B}(k\varphi) \leqq \mathcal{M}_{p,B}(\varphi)\|\hat{k}\|_1$.

(iv) *if* $\mathcal{M}_{p,B}(\varphi) \leqq C$ *for all bounded open balls in* R^n, *then* $\mathcal{M}_p(\varphi) \leqq C$.

Proof. (i) Obvious.

(ii) We note that a change of coordinates in R^n only changes the $\mathcal{L}_p$-norm and the set B. Hence $\mathcal{M}_{p,B}(\cdot)$ will just change to $\mathcal{M}_{p,a^{-1}B}(\cdot)$ under the transformation $y \to ay$. If $\varphi = \mu$, on $B, \mu \in \mathcal{M}_p$, then let $\mu_1 = \mu_{y_0}$, and so $\mu_1 = \varphi_{y_0}$ on $B - y_0$. It follows that $\hat{\mu}_1 = \exp(2\pi i\langle \cdot, y_0\rangle)\hat{\mu}$. Since multiplication with a scalar function of absolute value 1 is an isometry on $\mathcal{L}_p$, we see that $\mathcal{M}_{p,B}(\varphi) = \mathcal{M}_{p,B-y_0}(\varphi_{y_0})$.

(iii) The first assertion follows from Lemma 1(ii) and the definitions. The second is an application of this, using Lemma 1(vii).

(iv) Let $\varepsilon > 0$ be arbitrary. Choose a sequence $\{B_j\}_1^\infty$ of bounded open balls and functions k_j such that (a) $\bar{B}_j \subset B_{j+1}$ and $\bigcup_{j=1}^\infty B_j = R^n$, (b) $k_j = 1$ on B_j and $k_j = 0$ outside B_{j+1}, (c) $\|\hat{k}_j\|_1 \leqq 1 + \varepsilon$. This is possible by Lemma 2. Let $\varphi = \mu_j$ on B_{j+1}, $\mu_j \in \mathcal{M}_p$. Let $\nu_j = \mu_j k_j$. Then by (3)

$$\mathcal{M}_p(\nu_j) = \sup\{\|\hat{\mu}_j * \hat{k}_j * \hat{f}\|_p ;\ f \in \mathcal{S},\ \|f\|_p \leqq 1\} \leqq \mathcal{M}_{p,B_{j+1}}(\varphi)\|\hat{k}_j\|_1 \leqq (1+\varepsilon)C .$$

Since $\nu_j \to \varphi$ uniformly on compact subsets of R^n, Lemma 1(iv) gives that $\mathcal{M}_p(\varphi) \leqq (1+\varepsilon)C$. As $\varepsilon > 0$ was arbitrary (iv) is proved.

We will now state the main theorem of this section.

Theorem 1. *Suppose* $1 \leqq p \leqq \infty$ *and* $p \neq 2$. *Let* A_j *be Hermitian* $N \times N$-*matrices* $(j = 1, \ldots, n)$. *Then* $\exp\left(i \sum_{j=1}^n A_j y_j\right)$ *belongs to* $\mathcal{M}_p$ *if and only if the matrices* $A_1, \ldots, A_n$ *commute.*

We need some lemmas for the necessity part of the proof.

Lemma 4. *Let* $B = B(x_0, r)$, $r > 0$. *If* $v \in \mathcal{S}$ *and* $v \neq 0$ *on* B, *then there is a constant* C *and a ball* $B' = B(x_0, r')$, $0 < r' \leqq r$, *such that for each* $g \in S$ *with* $g = 0$ *outside* B', *we have*

$$\|\hat{g}\|_p \leqq C\|\hat{v} * \hat{g}\|_p . \tag{4}$$

Proof. Since v is continuous, there is a k, $1 \leqq k \leqq N$, and a ball $B' = B(x_0, r')$, $0 < r' \leqq r$, such that $v_k \neq 0$ on $\bar{B}'$. Hence there is a $w_k \in S$ such that $w_k v_k = 1$ on B'. We get for any $g \in S$ with $g = 0$ outside B'

$$g = w_k v_k g$$

and so the inequality (3) gives

$$\|\hat{g}\|_p = \|\hat{w}_k * \hat{v}_k * \hat{g}\|_p \leqq \|\hat{w}_k\|_1 \|\hat{v}_k * \hat{g}\|_p \leqq \|\hat{w}_k\|_1 \|\hat{v} * \hat{g}\|_p = C\|\hat{v} * \hat{g}\|_p .$$

Lemma 5. *Let* $p \neq 2$ *and let* B *be an open ball in* R^n. *Assume that*

$\lambda \in M_{p,B} \cap C^2(B)$, that $|\lambda| = 1$ on B, and that there is a constant C such that

$$M_{p,B}(\lambda^m) \leqq C, \qquad m = 1,2,\ldots .$$

Then there is an $x_0 \in R^n$ and a complex number c with $|c| = 1$, such that

$$\lambda(y) = c \exp(i\langle x_0, y\rangle), \qquad y \in B .$$

PROOF. If $B = R^n$ this is Theorem 1.14 in [1]. We want to prove it for bounded B. Thereby we assume that it is already known that if A is a real quadratic form and $\exp(iA) \in M_p$, $p \neq 2$, then $A = 0$ (Lemma 1.4 in [1]).

Let $\lambda = \exp(if)$, f be real and $f \in C^2(B)$. It will be sufficient to prove that the second order derivatives of f vanish in B. Thus let y_0 be an arbitrary point in B. According to Lemma 3(ii) it is no restriction of the generality to assume that $y_0 = 0$. Let

$$f(y) = f(0) + \langle x_0, y\rangle + A(y) + o(|y|^2), \quad y \to 0 ,$$

where A is a real quadratic form in y. Let

$$g(y) = f(y) - f(0) - \langle x_0, y\rangle .$$

Then, by Lemma 1(vi) and Lemma 3(iii)

$$M_{p,B}(\exp(img)) \leqq C, \qquad m = 1,2,\ldots .$$

Set $g_m(y) = mg(m^{-\frac{1}{2}}y)$. Then $g_m \to A$ uniformly on compact sets and also by Lemma 3 (ii),

$$M_{p,m^{\frac{1}{2}}B}(\exp(ig_m)) = M_{p,B}(\exp(img)) \leqq C$$

By the above we can find a $\mu_m \in M_p$ such that $\mu_m = \exp(ig_m)$ on $m^{\frac{1}{2}}B$.

Let B' be a bounded open ball. Using Lemma 2 we see that there is a bounded open ball B'' and a function $k \in S$ such that $k = 1$ on B', k is zero outside B'' and $\|\hat{k}\|_1 \leqq 2$. Choose m_0 so large that $m^{\frac{1}{2}}B \supset B''$, for $m \geqq m_0$. From Lemma 3(iii) we then get, $m > m_0$,

$$M_p(\mu_m k) \leqq M_{p,m^{\frac{1}{2}}B}(\mu_m)\, \|\hat{k}\|_1 \leqq 2M_{p,m^{\frac{1}{2}}B}(\exp(ig_m))$$

and so, from the above

$$M_p(\mu_m k) \leqq 2C .$$

Since $\mu_m k \to \exp(iA)k$ uniformly, Lemma 1(v) gives us

$$M_p(\exp(iA)k) \leqq 2C .$$

Hence

$$M_{p,B'}(\exp(iA)) = M_{p,B'}(\exp(iA)k) \leqq M_p(\exp(iA)k) \leqq 2C .$$

Since the ball B' was arbitrary, Lemma 3(iv) shows that $\exp(iA) \in M_p$ and so $A = 0$, and the lemma is proved.

LEMMA 6. *Suppose that A_j are Hermitian $N \times N$-matrices, $j = 1,\ldots,n$, such that the eigenvalues (repeated with proper multiplicities) of*

$$\sum_{j=1}^{n} A_j y_j$$

for all $y = (y_1,\ldots,y_n)$ in an open non-void ball B in R^n are of the form

$$\sum_{j=1}^{n} \alpha_{kj} y_j, \qquad k = 1,\ldots,N ,$$

where the α_{kj} are constants. Then the matrices $A_1,\ldots,A_n$ commute.

PROOF. When the conditions in Lemma 6 are satisfied for all complex y_j (instead of $(y_1,\ldots,y_n) \in B$) this is a theorem by Motzkin and Taussky (Theorem 2 in [3]). From the analyticity of the both members in the equality

$$\det\left(xE - \sum_{j=1}^{n} A_j y_j\right) = \prod_{k=1}^{N} \left(x - \sum_{j=1}^{n} \alpha_{kj} y_j\right), \quad y \in B,$$

we see that it is also satisfied for all complex y_j, and Lemma 6 follows from the theorem of Motzkin and Taussky.

PROOF OF THEOREM 1. Suppose $\mu(y) = \exp(i \sum_{j=1}^{n} A_j y_j)$ belongs to $\mathscr{M}_p$, $p \neq 2$. Since the elements of μ belong to $C^\infty(R^n)$ there is a non-void open ball B in R^n and functions $\lambda_1,\ldots,\lambda_N$ in $C^\infty(R^n)$ such that $\lambda_1(y),\ldots,\lambda_N(y)$ are the eigenvalues of $\mu(y)$, counted with proper multiplicities, for each $y \in B$, and such that for each $\lambda_j(y)$, $y \in B$, there is an eigenvector $v_j(y) \neq 0$, and $v_j \in \mathscr{C}^\infty(B)$.

Since the behavior outside B will be of no interest in the following, we can suppose that $v_j \in \mathscr{S}$ and that λ_j on B coincides with a function $f_j \in S$ (if necessary by shrinking the ball B somewhat). Let C_j be the constant associated with v_j as in Lemma 4, and we can suppose that the corresponding balls B_j' are equal, to B' say. Let g be any function in S with $g = 0$ outside B'. Lemma 4 then gives

$$\|\widehat{f_j^m * \hat{g}}\|_p \leqq C\|f_j^m * \hat{v}_j * \hat{g}\|_p . \tag{5}$$

Since $g = 0$ outside B' we have

$$f_j^m v_j g = \lambda_j^m v_j g = \mu^m v_j g .$$

The inversion theorem and the inequality (3) then shows that

$$(6)\qquad \|\widehat{f_j^m}*\hat{v}_j*\hat{g}\|_p = \|\widehat{\mu^m}*\hat{v}_j*\hat{g}\|_p \leqq \mathscr{M}_p(\mu^m)\,\|\hat{v}_j\|_1\,\|\hat{g}\|_p\,.$$

Combination of (5) and (6) shows that

$$\|\widehat{f_j^m}*\hat{g}\|_p \leqq C_j\mathscr{M}_p(\mu^m)\,\|\hat{v}_j\|_1\,\|\hat{g}\|_p\,.$$

Since $\mu^m(y)=\mu(my)=m^*\mu(y)$, Lemma 1(iv) gives that

$$\|\widehat{f_j^m}*\hat{g}\|_p \leqq C_j\mathscr{M}_p(\mu)\,\|\hat{v}_j\|_1\,\|\hat{g}\|_p = C_j'\|\hat{g}\|_p\,.$$

As μ is unitary we have $|\lambda_j|=1$, that is, $|f_j|=1$ on B'. Since $f_j \in M_p$ (Lemma 1(vii)) we also have

$$M_{p,B'}(\lambda_j^m) = M_{p,B'}(f_j^m) \leqq C_j', \qquad m=1,2,\ldots\,.$$

It follows that the conditions in Lemma 5 are satisfied, and from Lemma 6 we conclude that $A_1,\ldots,A_n$ commute.

To prove the converse we note that the Frobenius theorem shows that in this case $A_1,\ldots,A_n$ have a common diagonalization. It follows that there is a constant invertible matrix P such that

$$\exp\left(i\sum_{j=1}^{n}A_jy_j\right) = P(\exp(i\alpha_k(y))\,\delta_{kl})P^{-1}$$

where

$$\alpha_k(y) = \sum_{j=1}^{n}\alpha_{kj}y_j, \qquad k = 1,\ldots,N;\ \alpha_{kj} \text{ real constants}\,.$$

By Lemma 1(vi) we have $\exp(i\alpha_k) \in M_p$ and so $\exp(i\sum_{j=1}^{n}A_jy_j)$ belongs to $\mathscr{M}_p$ and the theorem is proved.

3. The initial value problem.

We now turn to the Cauchy problem. Let $A_j (j=1,\ldots,n)$ and B be $N\times N$-matrices and let $u=u(t,x)$ and $u_0=u_0(x)$ be N-dimensional complex vector functions. We consider the Cauchy problem

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^{n}A_j\frac{\partial u}{\partial x_j}+Bu, & x \in R^n\,, \\ u(0,x) = u_0(x), & 0\leqq t\leqq T\,. \end{cases}$$

We say that the problem (1) is well posed in L_p if for each $u_0 \in \mathscr{S}$ there is a solution $u=u(t,x)$ of (1) in $\mathscr{L}_p$-norm (by which we mean that

$$\frac{1}{h}(u(t+h,x)-u(t,x)) \to \sum_{j=1}^{n}A_j\frac{\partial u}{\partial x_j}+Bu$$

in $\mathscr{L}_p$ when $h\to 0$) depending continuously (in $\mathscr{L}_p$) on the initial value u_0, i.e. there is a constant $C(T)$ such that

$$\|u(t,\cdot)\|_p \leqq C(T)\|u_0\|_p, \qquad 0 \leqq t \leqq T. \tag{7}$$

Obviously such a solution is unique.

For $p=\infty$ this definition of well posed problems is weaker than the usual, since $\mathscr{S}$ is not dense in $\mathscr{L}_\infty$.

Our main result is Theorem 2.

THEOREM 2. *Suppose $p \neq 2$, $1 \leqq p \leqq \infty$. Then the Cauchy problem* (1), *where A_j are Hermitian $N \times N$-matrices and B is any $N \times N$-matrix, is well posed in L_p if and only if the matrices $A_1, \ldots, A_n$ commute.*

By the remarks above this gives a necessary condition also for the usual definition of well posed problems in $\mathscr{L}_\infty$.

PROOF. Assume first that (1) is well posed in L_p, $p \neq 2$. Then since $u \in \mathscr{L}_p$, we can take the Fourier transforms, in the distribution sense, of the elements of (1) with respect to x (t fixed) and get

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t}(t,y) = \left(-2\pi i \sum\limits_{j=1}^{n} A_j y_j + B\right) \hat{u}(t,y), \\ \hat{u}(0,y) = \hat{u}_0(y), \end{cases} \qquad \begin{array}{l} y \in R^n, \\ 0 \leqq t \leqq T, \end{array}$$

and so, with $\varphi_t(y) = \exp\left(t(i \sum_{j=1}^n A_j y_j + B)\right)$

$$\hat{u}(t,y) = \varphi_t(-2\pi y)\hat{u}_0(y).$$

Hence by (7) and Lemma 1(iv)

$$\mathscr{M}(\varphi_t) \leqq C(T), \qquad 0 \leqq t \leqq T. \tag{*}$$

On the other hand, suppose (*) is satisfied. If $\varphi_t(-2\pi y) = \hat{\mu}_t(y)$, then

$$u(t,x) = \mu_t * u_0(x)$$

and so $u(t,\cdot) \in \mathscr{C}^\infty(R^n)$ (the elements of μ_t are in S' and differentiation is continuous in S). From

$$\frac{\partial \hat{\mu}_t(y)}{\partial t} = \left(-2\pi i \sum_{j=1}^{n} A_j y_j + B\right) \hat{\mu}_t(y), \qquad \hat{\mu}_0(y) = E,$$

it follows that

$$\hat{\mu}_{t+h}(y) = \hat{\mu}_t(y) + \left(-2\pi i \sum_{j=1}^{n} A_j y_j + B\right) h + h^2 R_2(h;t;y),$$

where the elements in R_2 are second order polynomials in y with coefficients which are uniformly bounded in M_p, by (*) and Lemma 1(iv).

Consequently there is a constant C such that

$$\left\| \frac{1}{h}(u(t+h,\cdot)-u(t,\cdot)) - \left(\sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j}(t,\cdot)+Bu(t,\cdot)\right)\right\|_p \leqq |h| C \sum_{|\alpha|\leqq 2} \|D^\alpha u_0\|_p ,$$

and so u is a solution of (1) in $\mathscr{L}_p$-norm. By Lemma 1(iv) and (*)

$$\|u(t,\cdot)\|_p = \|\mu_t * u_0\|_p \leqq C(T)\,\|u_0\|_p .$$

Hence (1) is well posed in L_p.

By Theorem 1, the following lemma then completes the proof of Theorem 2.

LEMMA 7. *Let* A_j $(j=1,\ldots,n)$ *and* B *be* $N\times N$*-matrices and let*

$$\varphi_t(y) = \exp\left(t\left(i\sum_{j=1}^{n} A_j y_j + B\right)\right).$$

Then

$$\mathscr{M}_p(\varphi_t) \leqq C(T), \qquad 0\leqq t\leqq T , \tag{8}$$

if and only if $\exp(i\sum_{j=1}^n A_j y_j)$ *belongs to* $\mathscr{M}_p$.

PROOF. Suppose first that (8) holds. Then, by Lemma 1(iv), $\psi_t(y) = \exp(i\sum_{j=1}^n A_j y_j + tB)$ satisfies

$$\mathscr{M}_p(\psi_t) \leqq C(T), \qquad 0<t\leqq T .$$

If we let $t\to 0$, we see that $\psi_t(y)\to \exp(i\sum_{j=1}^n A_j y_j)$ uniformly on compact subsets of R^n, and so Lemma 1(v) shows that $\exp(i\sum_{j=1}^n A_j y_j)$ belongs to $\mathscr{M}_p$.

On the other hand, if $\mu(y)=\exp(i\sum_{j=1}^n A_j y_j)$ is in $\mathscr{M}_p$, then by Lemma 1(iv)

$$\mathscr{M}_p((t)^*\mu) \leqq \mathscr{M}_p(\mu), \qquad 0\leqq t\leqq T .$$

Let D be a bounded open ball in R^n. Then $\varphi_t \in \mathscr{M}_{p,D}$, $0\leqq t\leqq T$. The elements of φ_t and their derivatives are bounded on compact subsets of R^n, uniformly for $0\leqq t\leqq T$. If we multiply φ_t by a function $k_D \in C^\infty(R^n)$ with value 1 on D and with compact support, then $k_D\varphi_t$ has elements belonging to S which, together with their derivatives, are uniformly bounded in L_1 for $0\leqq t\leqq T$. Hence $\mathscr{M}_{p,D}(\varphi_t)\leqq \mathscr{M}_p(k_D\varphi_t)$ is uniformly bounded for $0\leqq t\leqq T$.

We also note that φ_t is a solution of

$$\frac{\partial \varphi_t}{\partial t}(y) = \left(i\sum A_j y_j + B\right)\varphi_t(y), \quad \varphi_0(y) = E ,$$

that is,

$$\varphi_t(y) = \mu(ty) + \int_0^t \mu((t-v)y)\, B\, \varphi_v(y)\, dv\,. \tag{9}$$

If we apply the $\mathscr{M}_{p,D}$-semi-norm on the both members of (9) we get, using Lemma 3(iii), Lemma 1(iv) and the remarks above, that

$$\mathscr{M}_{p,D}(\varphi_t) \leqq \mathscr{M}_p(\mu) + \mathscr{M}_p(\mu) \int_0^t |B|\, \mathscr{M}_{p,D}(\varphi_v)\, dv\,,$$

and so, since the integral is bounded by the remarks about φ_t above, Gronwall's lemma applies:

$$\mathscr{M}_{p,D}(\varphi_t) \leqq \mathscr{M}_p(\mu) \exp\Big(\mathscr{M}_p(\mu) \int_0^t |B|\, dv\Big) \leqq \mathscr{M}_p(\mu) \exp\big(T|B|\mathscr{M}_p(\mu)\big), \quad 0 \leqq t \leqq T\,.$$

By Lemma 3(iv) then

$$\mathscr{M}_p(\varphi_t) \leqq \mathscr{M}_p(\mu) \exp\big(T|B|\mathscr{M}_p(\mu)\big) = C(T), \qquad 0 \leqq t \leqq T\,,$$

and Lemma 7 is proved.

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