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TAUBERIAN PROBLEMS FOR THE n-DIMENSIONAL LAPLACE TRANSFORM I

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Introduction.

In this paper we shall consider a Tauberian problem for the n-dimensional Laplace transform corresponding to the one announced by Ganelius in [5] for $n=1$.

Our results as formulated in Theorems 2 and 3 and specialized to $n=1$ will give essentially the same estimation as that stated in [5]. Thus we can solve the original problem by using pure Fourier methods, hence avoiding the approximation technique used earlier. Since the method works in several dimensions we also ge results for this case.

The proof will be in two steps. In section 1 we attack a general Tauberian problem for the convolution transform holding for a whole class of kernels which includes the Laplace kernel and for example also the convolution kernel associated with the Meijer transform. In section 2 this general result will be applied to the Laplace transform. The method of attacking the general problem will follow a method applied by Ganelius ([6, p. 9] and [7, p. 214]), which was also used in another formulation by myself in [4].

We will use the following notation:

If $x=(x_1,x_2,\ldots,x_n) \in R^n$ and $y=(y_1,y_2,\ldots,y_n) \in R^n$, then

$$x\cdot y = \sum_{\nu=1}^{n} x_\nu y_\nu, \qquad x^y = \prod_{\nu=1}^{n} x_\nu^{y_\nu}, \qquad |x| = \sum_{\nu=1}^{n} |x_\nu| .$$

By $x \leqq y$ $(x<y)$ we mean that $x_\nu \leqq y_\nu$ $(x_\nu < y_\nu)$, $\nu=1,2,\ldots,n,$ and by $x \to +0$ and $x \to +\infty$ we mean that $x_\nu \to +0$ and $x_\nu \to +\infty$, respectively, for $\nu=1,2,\ldots,n$. Furthermore we let $R_+{}^n$ be all $x \in R^n$ such that $x \geqq 0$, and call a function H from R^n to R non-decreasing if $x \leqq y$ implies $H(x) \leqq H(y)$ and non-increasing if it implies that $H(y) \leqq H(x)$. We use standard notations for convolutions

$$K * \varphi(x) = \int K(x-t)\, \varphi(t)\, dt$$

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and for the Fourier transform

$$\hat{K}(x) = \int \exp(-ix\cdot t)\, K(t)\, dt\,,$$

where we always let an unspecified region of integration be R^n.

If α is a measure on R_+^n, then

$$\int_{R_+^n} \exp(-s\cdot t)\, d\alpha(t)$$

stands for

$$\lim_{x\to+\infty} \int_{0\leqq t\leqq x} \exp\,(-s\cdot t)\, d\alpha(t)\,.$$

We say that

$$\int_{R_+^n} \exp(-s\cdot t)\, d\alpha(t)$$

is boundedly convergent if this limit exists and there also exists a constant C, which may depend on s, such that

$$\left|\int_{0\leqq t\leqq x} \exp(-s\cdot t) d\alpha(t)\right| \leqq C \qquad \text{for all} \quad x \in R_+^n\,.$$

For convenience we use the abbreviations:

$$\begin{aligned} \exp x &= (\exp x_1, \exp x_2, \ldots, \exp x_n)\,, \\ \log x &= (\log x_1, \log x_2, \ldots, \log x_n)\,, \\ \mathbf{1} &= (1, 1, \ldots, 1)\,, \\ \max(x,y) &= (\max(x_1,y_1), \max(x_2,y_2), \ldots, \max(x_n,y_n))\,, \end{aligned}$$

and, if t is a complex vector,

$$\operatorname{Im} t = (\operatorname{Im} t_1, \operatorname{Im} t_2, \ldots, \operatorname{Im} t_n)\,.$$

We let C stand for positive absolute constants not necessarily the same each time.

1. A general Tauberian theorem.

Let us first define the class E of kernels, which we will consider in this section.

DEFINITION. By E we denote the set of all functions $K \in L(R^n)$ which satisfy the following three conditions:

1° $\hat{K}(t) \neq 0$ for all $t \in R^n$.

2° The function g defined by

$$g(t) = \hat{K}(t)^{-1}$$

can be analytically continued in a region $\operatorname{Im} t > -\varrho$, with $\varrho > 0$.

3° This function g satisfies the inequality

$$|g(t)| \leqq C \exp\big(m|x| + y \cdot \log(1+y)\big) \tag{1.1}$$

for some $m > 0$ and all $t = x + iy$ with $y = \operatorname{Im} t > \max(-1, -\varrho)$.

If now H is a continuous function from R_+^n to R which is strictly positive and non-decreasing and such that $H(x) \to +\infty$ when $x \to +\infty$, then we introduce H_1 by putting

$$H_1(x) = H\big(q(x)\big)^{-1},$$

where q is the inverse of the transformation T, defined by

$$T(x) = xH(x).$$

Although it does not follow from this construction that q is non-decreasing in the several-dimensional case, it is easy to see that H_1 is non-increasing and also satisfies

$$H_1(x) \leqq rH_1(rx) \quad \text{for} \quad r \geqq 1. \tag{1.2}$$

To show this we see that if $q = (q_1, q_2, \ldots, q_n)$ then

$$H\big(q(x)\big) = x_\nu \big(q_\nu(x)\big)^{-1} \quad \text{for all } \nu \text{ such that} \quad x_\nu > 0.$$

If now $x \leqq x'$, then either $q_\nu(x) \leqq q_\nu(x')$ for all $\nu = 1, 2, \ldots, n$ or $q_\nu(x') < q_\nu(x)$ for some ν. In the first case it is clear that

$$H\big(q(x)\big) \leqq H\big(q(x')\big)$$

and in the second case we also see that

$$H\big(q(x)\big) = x_\nu \big(q_\nu(x)\big)^{-1} \leqq x_\nu' \big(q_\nu(x')\big)^{-1} = H\big(q(x')\big),$$

which proves that H_1 is non-increasing.

If $r \geqq 1$, then either $q_\nu(rx) \geqq q_\nu(x)$ for some ν or $H\big(q(rx)\big) = H\big(q(x)\big)$, since H_1 is non-increasing. Thus in both cases whenever $x_\nu > 0$ for some ν

$$H\big(q(rx)\big) = rx_\nu \big(q_\nu(rx)\big)^{-1} \leqq rx_\nu \big(q_\nu(x)\big)^{-1} = rH\big(q(x)\big),$$

which implies (1.2).

We can now formulate our general result.

Theorem 1. *If $K \in E$ and φ is a bounded and measurable function from R^n to R, then*

$$K * \varphi(x) = O\big(\exp(-H(\exp x))\big), \quad x \to +\infty, \tag{1.3}$$

and

$$\inf_{x\leqq t\leqq x+1} {}_{H_1(\exp x)} (\varphi(t)-\varphi(x)) = O(H_1(\exp x)), \quad x\to+\infty, \tag{1.4}$$

implies

$$\varphi(x) = O(H_1(\exp x)), \quad x\to+\infty .$$

PROOF. The proof will depend on the following property: there exists a constant C such that if $u\in L(R^n)$ then

$$\sup_{x\in R^n}|u(x)| \leqq C\left\{-\inf_{x\leqq t\leqq x+h}(u(t)-u(x)) + \int_{-V\leqq\xi\leqq V}|\hat{u}(\xi)|\,d\xi\right\} \tag{1.5}$$

for all $V=(V_1,V_2,\ldots,V_n)$ and $h=(h_1,h_2,\ldots,h_n)$ with $h_\nu=V_\nu^{-1}$, $\nu=1,2,\ldots,n$. This inequality can be proved in a way analogous to the variant in one dimension earlier used by myself in [4, p. 78].

If now $y\in R_+^n$ and $\omega\in R_+$ are arbitrary with $\omega>n$ then we apply (1.5) to the function u, where

$$u(x) = \exp\left(-\tfrac{1}{2}(x-y)^2\omega^2\right)\varphi(x) .$$

It is easy to derive that

$$\hat{u}(\xi) = \exp(-i\xi\cdot y)\int\psi(y-x)\,Q(x)\,dx , \tag{1.6}$$

where

$$\psi(x) = K*\varphi(x)$$

and

$$Q(x) = (2\pi)^{-\frac{1}{2}n}\int\omega^{-n}\exp\left(ix\cdot v-\tfrac{1}{2}(v-\xi)^2\omega^{-2}\right)g(v)\,dv . \tag{1.7}$$

The arguments are similar to those used in one dimension (cf. [4, p. 81]).

In (1.7) we make the substitution $v=t+\xi+is$ with $s>\max(-1,-\varrho)$ and thus after a translation of the region of integration we obtain

$$|Q(x)| \leqq C\exp\left(-s\cdot x+\tfrac{1}{2}s^2\omega^{-2}\right)\int\omega^{-n}\exp\left(-\tfrac{1}{2}t^2\omega^{-2}\right)g(t+\xi+is)\,dt .$$

Here and in the following C denotes some constant independent of y and ω. By use of (1.1) and after an estimation of the remaining integral we have

$$|Q(x)| \leqq C\exp\left(-s\cdot x+\tfrac{1}{2}s^2\omega^{-2}+s\cdot\log(1+s)+\tfrac{1}{2}m^2n\omega^2+m|\xi|\right). \tag{1.8}$$

Here s may be chosen suitably e.g. according to the following: if

$$s=(s_1,s_2,\ldots,s_n), \qquad x=(x_1,x_2,\ldots,x_n), \qquad \varrho=(\varrho_1,\varrho_2,\ldots,\varrho_n)$$

and $z=(z_1,z_2,\ldots,z_n)$, where z will be fuıther specified later but always satisfies $0<z<y$, we let for each ν

$$s_\nu = \begin{cases} -\frac{1}{2}\gamma_\nu & \text{for} \quad x_\nu < 0 \\ \frac{1}{2}\gamma_\nu & \text{for} \quad 0 \leqq x_\nu < z_\nu \\ s_\nu(x_\nu) & \text{for} \quad z_\nu \leqq x_\nu \end{cases}$$

with $\gamma_\nu = \frac{1}{2}\min(1,\varrho_\nu)$ and where $s_\nu(x_\nu)$ is defined by the relation

$$s_\nu + \omega^2 \log(1+s_\nu) = \omega^2 x_\nu .$$

By this choice of s_ν we see that

$$-s_\nu x_\nu + \tfrac{1}{2}s_\nu{}^2\omega^{-2} + s_\nu \log(1+s_\nu) = -\tfrac{1}{2}s_\nu{}^2\omega^{-2} \quad \text{when} \quad x_\nu \geqq z_\nu .$$

If y is large enough then by (1.3) we may choose z such that

$$|\psi(x)| \leqq C\exp(-H(\exp x)) \quad \text{for} \quad x \geqq y-z .$$

Since φ is bounded we see that also ψ is bounded and hence by (1.6) we have

$$|\hat{u}(\xi)| \leqq C\left\{ \int_D |Q(x)|\,|\psi(y-x)|\,dx + \int_{R^n-D} |Q(x)|\,dx \right\} = C\{I_1+I_2\} .$$

If here $D=\{x\colon\ x\in R^n \text{ and } x\leqq z\}$ then by use of (1.3) and (1.8) with s specified as above we see that

$$I_1 \leqq C\exp\left(\tfrac{1}{2}m^2 n\omega^2 + m|\xi| - H(\exp(y-z))\right) .$$

When estimating I_2 we first observe that

$$s_\nu(x_\nu) \geqq \tfrac{1}{2}x_\nu \quad \text{for} \quad x_\nu \geqq z_\nu$$

and that certainly

$$s_\nu(x_\nu) \geqq \tfrac{1}{2}x_\nu\omega \quad \text{for} \quad x_\nu \geqq 4\omega^2 .$$

Now we see that

$$I_2 \leqq \sum_{\nu=1}^{n} \left\{ \int_{\substack{z_\nu < x_\nu < 4\omega^2 \\ x\in R^n}} |Q(x)|\,dx + \int_{\substack{x_\nu \geqq 4\omega^2 \\ x\in R^n}} |Q(x)|\,dx \right\},$$

where the first integral vanishes for all ν such that $4\omega^2 \leqq z_\nu$, and hence

$$I_2\exp(-\tfrac{1}{2}m^2 n\omega^2 - m|\xi|) \leqq C\left\{ \sum_{\nu=1}^{n} \exp(-\tfrac{1}{8}s_\nu(z_\nu)^2\omega^{-2} + 2\log\omega) + \exp(-2\omega^4)\right\}\omega^{n-1} .$$

Thus it is true that

$$(1.9)\qquad |\hat{u}(\xi)| \exp(-\tfrac{1}{2}m^2 n\omega^2 - m|\xi|)$$

$$\leqq C\left\{\exp\bigl(-H(\exp(y-z))\bigr) + \sum_{v=1}^{n} \exp(-\tfrac{1}{8}s_\nu(z_\nu)^2\omega^{-2} + 2\log\omega) + \exp(-2\omega^4)\right\}.$$

Here we choose ω and z depending on each other and y according to

$$(1.10)\qquad H(\exp(y-z)) = \lambda^2\omega^2$$

and

$$(1.11)\qquad s_\nu(z_\nu) = \lambda\omega^2 \quad \text{for } \nu=1,2,\ldots,n \text{ with } \lambda^2 \geqq m^2n+3\,.$$

Hence we see that

$$(1.12)\qquad \int\limits_{-V\leqq\xi\leqq V} |\hat{u}(\xi)|\,d\xi \leqq C\exp(m|V| - \tfrac{1}{2}\omega^2)\,.$$

When estimating the first term of the right hand member in (1.5) we write

$$u(t)-u(x) = \exp(-\tfrac{1}{2}(x-y)^2\omega^2)\,(\varphi(t)-\varphi(x)) +$$
$$+\,\varphi(t)\{\exp(-\tfrac{1}{2}(t-y)^2\omega^2) - \exp(-\tfrac{1}{2}(x-y)^2\omega^2)\}$$

and then using estimations corresponding to those in [7, p. 216], it follows that if $0<h<1$ then

$$(1.13)\qquad \inf_{x\leqq t\leqq x+h} (u(t)-u(x))$$

$$\geqq -\inf_{\substack{x\leqq t\leqq x+h\\ -1\leqq x-y\leqq 1}} (\varphi(t)-\varphi(x)) - C\exp(-\tfrac{1}{2}\omega^2) - C\omega \sup_{-1\leqq\frac{1}{2}(t-y)\leqq 1} |\varphi(t)| \sum_{\nu=1}^{n} h_\nu - C\sum_{\nu=1}^{n} h_\nu\,.$$

We are now ready to apply (1.5). Since

$$|\varphi(y)| = |u(y)| \leqq \sup_{x\in R^n} |u(x)|$$

we obtain from (1.12) and (1.13) with

$$mnV = \tfrac{1}{4}\omega^2\mathbf{1} = (h_1^{-1}, h_2^{-1},\ldots,h_n^{-1})$$

that, if y is large enough, then

$$|\varphi(y)| \leqq C\left\{ -\inf_{\substack{x\leqq t\leqq x+h\\ -1\leqq x-y\leqq 1}} (\varphi(t)-\varphi(x)) + \omega^{-1} \sup_{-1\leqq\frac{1}{2}(t-y)\leqq 1} |\varphi(t)| + \omega^{-2}\right\}.$$

By (1.10) and (1.11) we see that

$$\exp y = \exp(y-z+z) \leqq C\exp(y-z)\,H(\exp(y-z))\,,$$

and hence using this inequality combined with (1.10) and (1.2) we have

$$\omega^{-2} \leqq CH(\exp(y-z))^{-1} = CH_1\big(\exp(y-z)\,H(\exp(y-z))\big) \leqq CH_1(\exp y)\,.$$

Now this last inequality and the Tauberian condition (1.4) together with (1.2) give us

$$|\varphi(y)| \leqq C\left\{H_1(\exp y)+H_1(\exp y)^{\frac{1}{2}} \sup_{-1\leqq\frac{1}{2}(t-y)\leqq 1} |\varphi(t)|\right\}$$

for large values of y. Iterating this inequality we see that

$$\varphi(y) = O(H_1(\exp y)), \quad y\to\infty\,,$$

which was to be proved.

Remark. It is not necessary that φ should be bounded in the theorem. It still remains true if for example there exists a $\mu \in R_+{}^n$ such that

$$|\varphi(x)| \leqq C\big(1+\exp(-\mu\cdot x)\big), \qquad |\psi(x)| \leqq C\big(1+\exp(-\mu\cdot x)\big)\,,$$

both holding for all $x\in R^n$.

From the beginning we suppose that ω is large. A difference in argument comes in first when considering (1.9) where we now have to multiply the right hand member with $\exp(4\omega^2|\mu|)$. The inequality (1.12) is still true if λ is chosen sufficiently large, e.g. such that

$$\lambda^2 \geqq m^2n+3+8|\mu|\,.$$

Checking (1.13) we see that this inequality still holds and hence the theorem is true even under these weaker conditions.

2. Results for the n-dimensional Laplace transform.

Let H_0 be a continuous function from $R_+{}^n$ to R which is strictly positive, non-increasing and such that $H_0(s) \to +\infty$ when $s\to +0$. We define

$$H(s_1,s_2,\ldots,s_n) = H_0(s_1{}^{-1},s_2{}^{-1},\ldots,s_n{}^{-1})\,.$$

Thus H is of the same type as in section 1 and we can introduce H_1 in the same way as in Theorem 1. We also introduce regions $\Omega_{s,x}$ defined by

$$\Omega_{s,x} = \{t\colon\ t\in R_+{}^n \text{ and } t\leqq x\} - \{t\colon\ t\in R_+{}^n \text{ and } t\leqq s\}\,.$$

Finally let α be a measure on $R_+{}^n$ and let $\mu\in R_+{}^n$; then we may state our main result.

Theorem 2. *If*

$$(2.1)\quad F(s) = \int_{R_+{}^n} \exp(-s\cdot t)\,d\alpha(t) = O\big(s^{-\mu}\exp(-H_0(s))\big), \quad s\to +0\,,$$

where the integral is boundedly convergent for all $s > 0$, *and*

$$|F(s)| \leqq C(1+s^{-\mu}) \quad \text{for all} \quad s>0\,, \tag{2.2}$$

and also

$$\inf_{s \leqq x \leqq s \exp H_1(s)} \left(\int_{\Omega_{s,x}} d\alpha(t) \right) = O((1+s^{\mu})H_1(s)), \quad |s| \to +\infty \; (s \geqq 0)\,, \tag{2.3}$$

then

$$\int_{0 \leqq t \leqq s} d\alpha(t) = O(s^{\mu} H_1(s)), \quad s \to +\infty\,.$$

PROOF. First observe that, if β is defined by

$$\beta(t) = \begin{cases} \int_{0 \leqq s \leqq t} d\alpha(s) & \text{when } t>0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

then β is of bounded variation both in sense of Vitali and Hardy–Krause (cf. [1, p. 825]) in any region of form $0 \leqq t \leqq a$, $a>0$. We also see that

$$F(s) = \int_{R_+^n} \exp(-s \cdot t)\, d\beta(t)\,,$$

where we still have bounded convergence.

Now we make a partial integration in the last integral and obtain

$$F(s) = s^1 \int_{R_+^n} \exp(-s \cdot t)\, \beta(t)\, dt\,, \tag{2.4}$$

where the right hand member is absolutely convergent (cf. [2, p. 469 and p. 474]). In (2.4) we make the substitutions

$$s = \exp(-x) \quad \text{and} \quad t = \exp v$$

and thus obtain

$$K * \varphi(x) = O\big(\exp(-H(\exp x))\big), \quad x \to +\infty\,,$$

with

$$K(x) = \exp(-|\exp(-x)| - (1+\mu)\cdot x)$$

and

$$\varphi(x) = \exp(-\mu \cdot x)\, \beta(\exp x)\,,$$

which is equivalent to (2.1).

We want to apply Theorem 1 and have to prove that the conditions of this theorem are satisfied.

First we prove that $K \in E$. We see that $K \in L(R^n)$ and that

$$\hat{K}(t) = \prod_{\nu=1}^{n} \Gamma(1+\mu_\nu+it_\nu)$$

if

$$t = (t_1,t_2,\ldots,t_n) \qquad \text{and} \qquad \mu = (\mu_1,\mu_2,\ldots,\mu_n)\,.$$

Since $\Gamma(\cdot)^{-1}$ is analytic in the whole complex plane, by Stirlings formula and the well-known formula

$$\Gamma(1-z)^{-1} = \pi^{-1}\sin(\pi z)\,\Gamma(z)$$

we see that

$$|\Gamma(1+\mu_\nu+iz_\nu)^{-1}| \leqq C\exp\big(m|x_\nu|+y_\nu\log(1+y_\nu)\big)$$

if $z_\nu = x_\nu+iy_\nu$ with $y_\nu \geqq -\frac{1}{2}$ and $m \geqq 1+\pi$. The conclusion that $K \in E$ follows from this.

To prove that the Tauberian condition (1.4) is satisfied we observe that

$$\varphi(t)-\varphi(x) = \exp(-\mu\cdot t)(\beta(\exp t)-\beta(\exp x)) + \\ + \exp(-\mu\cdot x)\,\beta(\exp x)\big(\exp(-\mu\cdot(t-x))-1\big)$$

and hence (1.4) follows from (2.3) if for example

$$(2.5) \qquad |\beta(s)| \leqq C(1+s^\mu) \qquad \text{for} \quad s \in R_+{}^n\,.$$

If we can prove (2.5), then Theorem 2 follows from the remark to Theorem 1.

To prove (2.5) we let $0 \leqq s \leqq x \leqq 2s$ and write

$$\beta(x)-\beta(s) = \sum_{k=1}^{m} \big(\beta(s^{(k)})-\beta(s^{(k-1)})\big)\,,$$

where all $s^{(k)}$ lie on the line segment from s to x, with

$$s = s^{(0)} < s^{(1)} < \ldots < s^{(m)} = x$$

and chosen so that

$$s^{(k-1)} \leqq s^{(k)} \leqq s^{(k-1)}\exp H_1(2s)\,.$$

Thus by (2.3), (1.2) and the fact that β is locally bounded we can conclude that

$$\beta(x)-\beta(s) \geqq -C(1+s^\mu) \quad \text{for all } s \text{ and } x,\ 0 \leqq s \leqq x \leqq 2s\,.$$

Using a simple variant of a method, which was introduced by Karamata [9] and also used by Delange [3], it is now easy to see that (2.5) follows.

If $0<s\leqq x\leqq\frac{4}{3}s$ and

$$F_0(s_1,s_2,\ldots,s_n) = F(s_1^{-1},s_2^{-1},\ldots,s_n^{-1}),$$

then

$$F_0(3x)-F_0(2s) = \int_{R_+^n} \exp(-u\cdot\mathbf{1})\,(\beta(3x_1u_1,3x_2u_2,\ldots,3x_nu_n)- -\beta(2s_1u_1,2s_2u_2,\ldots,2s_nu_n))\,du = I_1+I_2,$$

where I_1 is the integral evaluated over the region $\frac{1}{3}\mathbf{1}\leqq u\leqq\frac{1}{2}\mathbf{1}$, and I_2 is the integral evaluated over the rest of R_+^n. Now

$$\begin{aligned}&\beta(3x_1u_1,3x_2u_2,\ldots,3x_nu_n)-\beta(2s_1u_1,2s_2u_2,\ldots,2s_nu_n)\\ &= (\beta(3x_1u_1,3x_2u_2,\ldots,3x_nu_n)-\beta(2s_1u_1,2s_2u_2,\ldots,2s_nu_n))(1+s^\mu u^\mu)^{-1}(1+s^\mu u^\mu)\\ &\geqq -C(1+s^\mu)(1+u^\mu)\end{aligned}$$

and hence

$$I_2 \geqq -C(1+s^\mu),$$

and by (2.2)

$$I_1 = F_0(3x)-F_0(2s)-I_2 \leqq C(1+s^\mu)$$

Since

$$\begin{aligned}&\beta(3x_1u_1,3x_2u_2,\ldots,3x_nu_n)-\beta(2s_1u_1,2s_2u_2,\ldots,2s_nu_n)\\ &\quad= \beta(3x_1u_1,3x_2u_2,\ldots,3x_nu_n)-\beta(x)+\beta(s)-\beta(2s_1u_1,2s_2u_2,\ldots,2s_nu_n)+\\ &\qquad\qquad+\beta(x)-\beta(s)\\ &\quad\geqq -C(1+s^\mu)+\beta(x)-\beta(s)\end{aligned}$$

if $\frac{1}{3}\mathbf{1}\leqq u\leqq\frac{1}{2}\mathbf{1}$, we see that

$$CI_1 \geqq -C(1+s^\mu)+\beta(x)-\beta(s)$$

and hence

$$\beta(x)-\beta(s) \leqq C(1+s^\mu).$$

Now we can easily derive that

$$|\beta(x)-\beta(s)| \leqq C(1+s^\mu) \quad \text{for all } s \text{ and } x,\ 0\leqq s\leqq x\leqq 2s.$$

This inequality implies that, if $u>0$ then

$$|\beta(u_1s_1,u_2s_2,\ldots,u_ns_n)-\beta(s)| \leqq C(1+s^\mu)(1+u^\mu)(1+|\log u|).$$

To see this write

$$\beta(u_1s_1,s_2,\ldots,s_n)-\beta(s) = \sum_{k=1}^{m}(\beta(u_1^{(k)}s_1,s_2,\ldots,s_n)-\beta(u_1^{(k-1)}s_1,s_2,\ldots,s_n))$$

with $u_1^{(0)}=1<u_1^{(1)}<\ldots<u_1^{(m)}=u_1$ and chosen so that

$$u_1^{(k-1)} \leqq u_1^{(k)} \leqq u_1^{(k-1)}2, \qquad k=1,2,\ldots,m.$$

We conclude that

$$|\beta(u_1 s_1, s_2, \ldots, s_n) - \beta(s)| \leqq C(1 + u_1^{\mu_1} s^{\mu}) m ,$$

where m can be chosen so that $m \leqq 2 + 2 \log u_1$, and, because of the symmetry, we also see that

$$|\beta(u_1 s_1, s_2, \ldots, s_n) - \beta(s)| \leqq C(1 + s^{\mu})(1 + |\log u_1|), \quad \text{when} \quad 0 < u_1 < 1 .$$

Thus the required inequality follows from similar arguments for the other coordinates.

Finally, since

$$F_0(s) - \beta(s) = \int_{R_+^n} \exp(-u \cdot \mathbf{1}) \left(\beta(u_1 s_1, u_2 s_2, \ldots, u_n s_n) - \beta(s)\right) du$$

and thus

$$|F_0(s) - \beta(s)| \leqq C(1 + s^{\mu}) ,$$

(2.5) follows from (2.2). Hence we have proved Theorem 2.

We also state the following slightly different result.

THEOREM 3. *Let α, μ, H and H_1 be defined as above, and suppose that $H(s) \to +\infty$ when $|s| \to +\infty$. If for $s \in R_+^n$*

$$(2.6) \quad F_0(s) = \int_{R_+^n} \exp\left(-\sum_{\nu=1}^{n} \frac{t_\nu}{s_\nu}\right) d\alpha(t) = O(1 + s^{\mu}) \exp(-H(s)), \quad |s| \to +\infty ,$$

where the integral is boundedly convergent for all $s > 0$, and

$$(2.7) \qquad \inf_{s \leqq x \leqq s \exp(H_1(s))} \left(\int_{\Omega_{s,x}} d\alpha(t) \right) = O((1 + s^{\mu}) H_1(s)), \quad |s| \to +\infty ,$$

then

$$\int_{0 \leqq t \leqq s} d\alpha(t) = O((1 + s^{\mu}) H_1(s)), \quad |s| \to +\infty \ (s \geqq 0) .$$

PROOF. Theorem 1 can be modified to prove this theorem if, instead of (1.3), we have

$$K * \varphi(x) = O\big((1 + \exp(-\mu \cdot x)) \exp(-H(\exp x))\big), \quad x \to +\infty ,$$

and, instead of (1.4), we have

$$\inf_{x \leqq t \leqq x + \mathbf{1} H_1(\exp x)} (\varphi(t) - \varphi(x)) = O\big((1 + \exp(-\mu \cdot x)) H_1(\exp x)\big), \quad x \to \infty ,$$

and where $x \to +\infty$ means that $|\exp x| \to +\infty$. I leave the details to the reader.

REMARK 1. Condition (2.2) is not necessary in the one-dimensional case, since it is only used to prove (2.5), which then follows from (2.1) and (2.3).

REMARK 2. Bounded convergence follows if we know that $|\beta(x)| \leqq C\exp(mx)$ for all $m>0$. (Cf. [2, p. 469].) It also follows if we add the following condition: for each $\nu=1,2,\ldots,n$ there exists an $\varepsilon_\nu>0$ such that

$$\beta(x_1,x_2,\ldots,x_\nu,\ldots,x_n)-\beta(x_1,x_2,\ldots,x_\nu',\ldots,x_n) \geqq -C(1+x^\mu)$$

for all $x=(x_1,x_2,\ldots,x_n)\in R_+^n$ such that $0<x_\nu'\leqq x_\nu\leqq\varepsilon_\nu$ (cf. [3, p. 80]). When α is the measure associated with a Dirichlet series this is trivially true. It also may be noted that bounded convergence are the same as ordinary convergence in the one-dimensional case.

REMARK 3. It is no restriction to suppose that in (2.1) $H_0(s)=1$ when $s_\nu\geqq 1$ for some ν, and hence that in (2.3) we have $H_1(s)=1$ when $s_\nu\leqq 1$ for some ν. Sometimes this can be useful.

3. Some examples and comments.

To show how the construction of H_1 from H_0 in Theorem 2 works, we shall give some examples.

If

$$H_0(s) = -\mathbf{1}\cdot\log s \tag{3.1}$$

then

$$H_1(s) = O((\mathbf{1}\cdot\log s)^{-1}), \quad s\to+\infty,$$

and if

$$H_0(s_1,s_2,\ldots,s_n) = \left(\sum_{\nu=1}^n s_\nu^{-1}\right)^\varepsilon, \quad \varepsilon>0, \tag{3.2}$$

then

$$H_1(s) = |s|^{-\varepsilon/(1+\varepsilon)}.$$

If furthermore

$$H_0(s) = |s|^{-\varepsilon}, \quad \varepsilon>0, \tag{3.3}$$

then we have

$$H_1(s) = \left(\sum_{\nu=1}^n s_\nu^{-1}\right)^{\varepsilon/(1+\varepsilon)}.$$

More generally we can see that, for all H_0, such that the function R defined by

$$R(x) = H(\exp x) = H_0(\exp(-x))$$

is sub-additive, that is

$$R(x+y) \leqq R(x)+R(y) \quad \text{for all} \quad x\in R^n \text{ and } y\in R^n;$$

then we have

$$H_1(s) = O(H(s)^{-1}), \quad s \to +\infty .$$

For all such H_0 the result is in fact best possible (cf. [8]). This case includes (3.1).

It is also known that in case (3.2) and (3.3) the result is best possible in one dimension (cf. e.g. [10]). By a combination of this fact and formula (30) on p. 165 in [11], which can be generalized to higher dimensions, it can be seen that in case (3.3) the result is essentially best possible.

It also may be noted that, according to known results (cf. e.g. [10]), in one dimension

$$\overline{\lim_{s\to+0}}\, sH(s) = +\infty$$

implies

$$\int_{0\leq t\leq s} d\alpha(t) = 0 .$$

In a forthcoming paper I intend to continue my studies on Tauberian problems for the n-dimensional Laplace transform.

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