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A REMARK ON THE CLOSED GRAPH THEOREM IN LOCALLY CONVEX VECTOR SPACES

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Introduction.

Let E be a separated locally convex (real or complex) vector space and E' its dual space. Then E is said to be *polar*, if a linear subspace L of E' is weakly closed whenever $L \cap U^\circ$ is weakly closed in E' for every neighbourhood U of the origin in E. It is called *weakly polar* if this holds for weakly dense subspaces of E'. (In the terminology of Pták [5] this is B-complete and B_r-complete, respectively. The present notion, which is more suitable for our purposes, was introduced by L. Gårding in an unpublished manuscript on open mapping and closed range theorems.) Weakly polar spaces are in particular complete [5, 5.7]. In the theory of locally convex vector spaces, one of the most general forms of the closed graph theorem known so far is due to V. Pták [5, 4.9] and A. P. and W. Robertson [6] and reads as follows:

(A) *Every closed linear mapping of a separated barrelled space E into a weakly polar space F is continuous.*

It is impossible to weaken the condition on E in proposition (A). In fact, Mahowald [3] proved that, if the closed graph theorem holds for mappings of a separated locally convex vector space E into any Banach space F, then E is necessarily barrelled. As for the space F, only a partial converse is known: If we suppose that the closed graph theorem holds for mappings of any barrelled space E into every quotient of F, then F must be polar, provided that F itself is barrelled. This result is contained implicitly in the investigations of Pták [5] (cf. also [7, ch. 6 supplement]). Hence there might exist further generalizations of proposition (A), and it is the purpose of the present note to show that this is actually the case.

In the first section we introduce a new class of spaces, the (*weakly*) *t-polar* spaces, which are obtained by replacing, in the definition of a (weakly) polar space, the neighbourhood U by a barrel T. Every (weakly) polar space is obviously (weakly) t-polar. On the contrary, we show by

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an example that there are t-polar Mackey spaces which are not polar and, as a matter of fact, not even complete. However, for barrelled spaces the two notions coincide.

The second section contains the main theorem, which states that (A) still holds true if we assume that F is weakly t-polar instead of weakly polar. According to the above-mentioned example this is a proper extension of (A). In the case F is t-polar we also establish a variant of the main result for closed linear relations. The formulation in terms of relations is very convenient when, for example, turning closed graph theorems into statements about open mappings, and it was systematically employed by L. Gårding in the manuscript mentioned before.

In conclusion we discuss in which sense our results are best possible.

t-polar spaces.

We shall assume that the reader is familiar with the standard concepts and notations in the theory of locally convex vector spaces. All such spaces are supposed to be separated.

DEFINITION. *A locally convex vector space E is said to be t-polar if a linear subspace L of E' is weakly closed whenever $L \cap T^{\circ}$ is weakly closed for every barrel T in E. It is called weakly t-polar if this property holds for weakly dense subspace L of E'.*

Note that a subset B of E' is the polar set of a barrel in E if and only if B is absolutely convex, weakly closed and weakly bounded. Observe also that the notion of a (weakly) t-polar space does not depend on the original topology of E but only on the dual pair $\langle E,E' \rangle$. A barrelled and (weakly) t-polar space is obviously (weakly) polar. It follows also that every (weakly) polar space is (weakly) t-polar. We shall prove that the converse is not true by constructing a counter example.

Let G denote the space of all (real or complex) sequences with only a finite number of coordinates different from zero, equipped with the norm $(a_n) \to \sup_n |a_n|$. Then G' equals the space F of all absolutely summable sequences. The latter space equipped with the weak topology $\sigma(F,G)$ will provide us with the desired counter example. In the first place, F is t-polar. For the weakly bounded subsets of $F' = G$ are exactly those which are bounded in the norm of G. Furthermore, if L is a linear subspace of G whose intersection with the unit sphere is weakly closed, hence closed, then L itself is closed, and therefore also closed in the weak topology $\sigma(G,F) = \sigma(G,G')$. This proves the assertion. On the other hand, F is not polar. Indeed, the equicontinuous subsets of $G = F'$ are all of finite dimension, and hence any linear subspace L of G satisfies the condition that $L \cap U^\circ$ is weakly closed for all neighbourhoods U of the origin in F. Since there are for example hyperplanes in G which are not closed for $\sigma(G,F)$, this proves even more than was desired, namely that F is non-complete (cf. Pták [4], [5, 5.6] and Collins [2]).

It is important in this context to make the obvious remark that proposition (A) still holds true if we assume merely that F is weakly polar in its Mackey topology. Nevertheless, (A) is not applicable to the space F constructed above; for according to Example 3 in [1, ch. IV, § 2] the weak topology $\sigma(F,G)$ is equal to the Mackey topology $\tau(F,G)$.

We shall need the following property of t-polar spaces:

LEMMA 1. *If H is a closed subspace of a t-polar space E, then E/H, in its quotient topology, is t-polar.*

PROOF. The dual of E/H can be identified with the weakly closed subspace H° of E'. Since in addition every polar in H° of a barrel in E/H is of the form $H^\circ \cap T^\circ$, where T is a barrel in E, the lemma follows immediately from the definition of a t-polar space.

The closed graph theorem.

We shall now prove the main result of the paper.

THEOREM 1. *If A is a closed linear mapping of a barrelled space E into a weakly t-polar space F, then A is continuous.*

PROOF. As usual, the adjoint mapping A' of A is defined as the set of all pairs $(y',x') \in F' \times E'$ for which

$$\langle Ax,y' \rangle = \langle x,x' \rangle, \qquad x \in E .$$

Since A is closed, the domain $D(A')$ of A' is weakly dense in F', and A' is continuous if $D(A')$ is provided with the topology induced by $\sigma(F',F)$ and E' with $\sigma(E',E)$. We claim that the theorem follows if we can prove that

(1) $$D(A') \cap T^\circ \text{ is weakly closed}$$

for every barrel $T \subset F$. Indeed, since F is weakly t-polar, we then conclude that $D(A')=F'$. Therefore, A' maps weakly relatively compact and in particular equicontinuous subsets of F' into weakly relatively compact subsets of E'. But $\sigma(E',E)$-compact subsets of E' are equicontinuous, and hence it follows that, given a neighbourhood V of 0 in F, there exists a neighbourhood U of 0 in E such that $A(U) \subset V$, that is, A is continuous.

To prove (1), let y' belong to the weak closure of $D(A') \cap T^\circ$ and let y_α' be a net in $D(A') \cap T^\circ$ converging weakly to y'. Then, by the weak continuity of A', its image $A'y_\alpha'$ is a bounded Cauchy net for $\sigma(E',E)$. But weakly bounded subsets of E' are weakly relatively compact, and hence $A'y_\alpha'$ converges to some point $x' \in E'$. Since, by its very definition, A' is weakly closed in $F' \times E'$, this proves that $y' \in D(A') \cap T^\circ$, and hence (1) follows. The proof of the theorem is complete.

We shall now consider the more general case when A is a closed linear *relation* from E into F, that is, A is a closed linear subspace of $E \times F$. We write

$$A^{-1} = \{(y,x) : (x,y) \in A\}, \qquad Ax = \{y : (x,y) \in A\},$$
$$D(A) = \{x : (x,y) \in A \text{ for some } y\}, \qquad R(A) = D(A^{-1}).$$

Furthermore, A is said to be *continuous* if $A^{-1}(V) = \{x : Ax \cap V \neq \emptyset\}$ is open in $D(A)$ for each open set $V \subset F$, and A is said to be *open*, if A^{-1} is continuous. It is easy to check that these definitions coincide with the usual ones in the case A is a function.

THEOREM 2. *If A is a closed linear relation from all of a barrelled space E into a t-polar space F, then A is continuous.*

PROOF. Since A is closed, $A(0)$ is a closed subspace of F. Let K be the canonical mapping of F onto $G = F/A(0)$. Then

$$(KA)x = K(Ax), \qquad x \in E,$$

defines KA as a linear function of E into G. One realizes easily that A is continuous if and only if KA is continuous and hence, taking Lemma 1 and Theorem 1 into account, we are through if we can prove that KA is closed. However, this follows immediately from the fact that KA is the image of A under the canonical quotient mapping of $E \times F$ onto $E \times F/\{0\} \times A(0) = E \times G$. The proof is finished.

It is quite obvious that Theorem 2 has an equivalent formulation in terms of open relations:

THEOREM 2′. *If A is a closed linear relation from a t-polar space E onto a barrelled space F, then A is open.*

The converse of Theorem 1.

We shall finally discuss in which sense Theorem 1 and Theorem 2 are best possible. As was remarked in the introduction it is impossible to weaken the conditions on the space E. In the general case we have not been able to prove (nor to disprove) that weak t-polarity is the weakest possible condition on F in Theorem 1. However, we have the following partial converse:

THEOREM 3. *Suppose F satisfies the following requirement: There is a weakly dense subspace G of F such that*

(2) *$B \subset F'$ is weakly bounded if and only if B is $\sigma(F',G)$-bounded;*

(3) *weakly bounded subsets of F' are relatively compact for $\sigma(F',G)$.*

Then, if the closed graph theorem holds for mappings of any barrelled space E into F, the space F is necessarily weakly t-polar.

PROOF. Let L be a weakly dense subspace of F' such that $L \cap T^{\circ}$ is weakly closed for every barrel T in F. Consider G in the topology of uniform convergence on sets of the form $L \cap T^{\circ}$. It follows from (2) and (3) that G is barrelled and $G' = L$. If A is the natural injection of G into F, then clearly $D(A') = L$. Since A is continuous and hence closed in the topologies $\sigma(G,L)$ and $\sigma(F,L)$, we conclude that A is closed also in the stronger original topologies of G and F. Consequently A is continuous, and in particular $D(A') = F'$. Hence $L = F'$, which completes the proof.

The reader will have no difficulty in proving the corresponding converse of Theorem 2:

THEOREM 4. *Suppose F satisfies* (2) *and* (3). *Then, if Theorem* 2 *holds for relations from any barrelled space E into F, the space F is necessarily t-polar.*

In particular, if F itself is barrelled, then (2) and (3) are fulfilled with $G = F$, and we get back a more convenient formulation of the partial converse of proposition (A) mentioned in the introduction.

ADDED IN PROOF: Our example of a non-complete t-polar space was also considered by T. Husain and M. Mahowald, *Barrelled spaces and the open mapping theorem*, Proc. Amer. Math. Soc. 13 (1962), 423–424, in a slightly different context.

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