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A GENERALIZATION OF A THEOREM OF WIENHOLTZ CONCERNING ESSENTIAL SELFADJOINTNESS OF SINGULAR ELLIPTIC OPERATORS

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In [4] Wienholtz studied differential expressions of the form

$$Su = \frac{1}{k(x)}\left\{-\sum_{j,k=1}^{n}\frac{\partial}{\partial x_k}\left(a_{jk}(x)\frac{\partial u}{\partial x_j}\right)+2i\sum_{j=1}^{n}b_j(x)\frac{\partial u}{\partial x_j}+i\sum_{j=1}^{n}\frac{\partial}{\partial x_j}b_j(x)u+q(x)u\right\},$$

$x=(x_1,\ldots,x_n)\in G\subseteqq \mathsf{R}^n$. We shall always assume that $a_{jk}=a_{kj}$ and that the coefficients a_{jk}, b_j, q and k are real and measurable and $k(x)>0$ a.e.

If the coefficients are sufficiently regular, we can define an operator S_0 in the Hilbert space $H=L^2(\mathsf{R}^n,k\,dx)$ in the following way:

$$D(S_0) = C_0^\infty(\mathsf{R}^n) = \text{the infinitely often differentiable functions with compact support};$$
$$S_0u = Su \quad\text{for}\quad u\in D(S_0)\,.$$

Obviously, S_0 is a symmetric operator.

We ask for conditions on the coefficients to ensure that S_0 is essentially selfadjoint. Wienholtz proved the following theorem (Satz 1, p. 59 in [4]):

THEOREM (Wienholtz). *If the coefficients a_{jk} and b_j are three times continuously differentiable, if q is continuous, if $k\equiv 1$, if there exists a constant $C_0\in\mathsf{R}$ such that*

$$0 < \sum_{j,k=1}^{n} a_{jk}(x)\,\xi_j\bar{\xi}_k \leqq C_0\sum_{j=1}^{n}|\xi_j|^2$$

for every $\xi=(\xi_1,\ldots,\xi_n)\in\mathsf{C}^n$, $\xi\neq 0$, and every $x\in\mathsf{R}^n$, and if, furthermore, the operator S_0 is bounded below, then S_0 is essentially selfadjoint.

Wienholtz also has a result for annular domains (Satz 4, S. 65 in [4]).

The essential content of the present paper is the observation that the proof of Wienholtz' theorem can be generalized to arbitrary domains G under suitable assumptions on the principal part of S. Furthermore, we

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weaken the regularity assumptions on the coefficients, and finally we allow a weightfunction k.

Before we state the regularity assumptions, we define $H_2(G)$ as the completion of $C_0^\infty(G)$ with respect to the Dirichlet norm

$$\|u\|_2 = \left\{ \sum_{j,k=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_k} \right\|^2 + \sum_{j=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^2 + \|u\|^2 \right\}^{\frac{1}{2}},$$

and let $H_{2,\mathrm{loc}}(G)$ denote the set of all locally $H_2(G)$ functions. In what follows all differentiations are made in the sense of distributions.

The assumptions on regularity of the coefficients are most adequately formulated as follows:

(A)
(i) a_{jk} and $\partial a_{jk}/\partial x_k$ are locally essentially bounded;
(ii) b_j and $(\partial b_j/\partial x_j) \in L^1_{\mathrm{loc}}(G)$;
(iii) the map $u \to Su$ is continuous from $H_2(K)$ into H when K is a compact subset of G;
(iv) if $S_0{}^*$ is the adjoint operator to S_0 in H, then

$$D(S_0{}^*) = \{u \in H \cap H_{2,\mathrm{loc}}(G) \mid Su \in H\}$$

and

$$S_0{}^* u = Su \quad \text{for} \quad u \in D(S_0{}^*).$$

These conditions are for instance satisfied if

(A′)
(i′) $a_{jk} \in C^2(G)$, $b_i \in C^1(G)$ and $q \in Q_{\alpha,\mathrm{loc}}(G)$;
(ii′) the matrix $\{a_{jk}(x)\}$ is strictly positively definit for every $x \in G$;
(iii′) k and k^{-1} are locally essentially bounded.

The space $Q_{\alpha\ \mathrm{loc}}(G)$ is defined on p. 8 in Jörgens [2]. Theorem 2, p. 80, in Ikebe–Kato [1] shows that, in addition to all the continuous functions $Q_{\alpha,\mathrm{loc}}(G)$ contains functions with such singularities as for instance the Coulomb potential. That (A′) implies (A) is shown in [1] and [2] except for the fact that these authors have $k \equiv 1$; but this does not change the proof apart from trivialities.

Wienholtz' condition on boundedness of the largest eigenvalue of the matrix $\{a_{jk}(x)\}$ is replaced by the following condition, partly due to Jörgens ([2, p. 7]):

(B)
There exists a real valued function ϱ, defined in G, such that
(1) $\varrho(x) \geqq 0$ and $\varrho(x) \to \infty$ as $|x| \to \infty$ or as $x \to \partial G =$ boundary of G;
(2) ϱ satisfies a uniform Lipschitz condition in every compact subdomain of G;
(3) $\sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial \varrho}{\partial x_j} \frac{\partial \varrho}{\partial x_k} \leqq k(x)$ a.e. in G.

Note that (2) implies that the derivatives $\partial\varrho/\partial x_i$ exist a.e. so that (3) makes sense (cf. Rademacher [3]).

The condition (B) may be split up into conditions (B′) on the coefficients a_{jk} in a neighbourhood of infinity and conditions (B″) near the boundary of G:

(B′) There exists a real-valued function ϱ, defined in G, such that

(1′) $\varrho(x) \geqq 0$ and $\varrho(x) \to \infty$ as $|x| \to \infty$;

(2′) ϱ satisfies a uniform Lipschitz-condition in every compact subdomain of G;

(3′) there exists a continuous function ψ: $[0,\infty[\to]0,\infty[$ such that

$$\sum_{i,k=1} a_{jk}(x) \frac{\partial\varrho}{\partial x_j} \frac{\partial\varrho}{\partial x_k} \leqq \psi^2(\varrho(x))k(x) \text{ a.e. in } G,$$

$$\int_k^\infty dt/\psi(t) = \infty \text{ for every } k > 0.$$

(B″) There exists a real-valued function σ, defined in G, such that

(1″) $\sigma(x) > 0$ and $\sigma(x) \to 0$ as $x \to \partial G$;

(2″) σ satisfies a uniform Lipschitz-condition in every compact subdomain of G;

(3″) there exists a continuous function φ: $]0,\infty[\to]0,\infty[$ such

$$\sum_{j,k} a_{jk}(x) \frac{\partial\sigma}{\partial x_j} \frac{\partial\sigma}{\partial x_k} \leqq \varphi^2(\sigma(x))\, k(x) \text{ a.e. in } G\,,$$

$$\int_0^\varepsilon dt/\varphi(t) = \infty \text{ for every } \varepsilon > 0.$$

The condition (B) and the union of conditions (B′), (B″) are equivalent. To prove that (B′), (B″) imply (B), define

$$\varrho'(x) = \tfrac{1}{2}\int_0^{\varrho(x)} dt/\psi(t) + \tfrac{1}{2}\int_{\sigma(x)}^{\sigma_0} dt/\varphi(t)\,,$$

where $\sigma_0 = \max \sigma(x)$.

Then (B) holds with ϱ' in place of ϱ.

It is easily seen that these conditions are weaker than Wienholtz', both in the case of R^n and in the case of annular domains. Consider for example the case $G = \mathsf{R}^n$, $k \equiv 1$, and choose $\varrho(x) = |x|$. It is seen that the largest eigenvalue of $\{a_{jk}(x)\}$ is allowed to grow as $\psi(|x|)^2$, where $\int^\infty \psi(t)^{-1} dt = \infty$, in particular, it may grow as $|x|^2$.

THEOREM. *If* (A) *and* (B) *are satisfied and* S_0 *is bounded below, then* S_0 *is essentially selfadjoint.*

EXAMPLE. If the operator S_0 defined by the differential expression

$$Su = -\Delta u + q(x)u, \qquad x \in \mathsf{R}^n, \quad q \in Q_{\alpha,\mathrm{loc}}(\mathsf{R}^n),$$

is bounded below, then it is essentially selfadjoint.

PROOF OF THE THEOREM. We may and do assume that S_0 is bounded below by 1. By a well-known theorem it is then enough to show that $R(S_0)$ is dense in H. Let $h \in H$ be orthogonal to $R(S_0)$. We shall show that $h=0$. We find in the same way as Wienholtz in his proof of Satz 1, p. 59, in [4] that

$$\int_G |h|^2 \sum_{j,k} a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} dx \geqq \int_G |h|^2 k u^2 \, dx$$

for all real-valued $u \in C_0^\infty(G)$.

By regularization this inequality holds for every real-valued function u which has compact support in G and which satisfies a uniform Lipschitz-condition in G. Especially it is valid for all u of the form $u(x) = f(\varrho(x))$, where

$$f\colon [0,\infty[\to [0,\infty[$$

is a continuous function with a piecewise continuous derivative. It follows from (B (3)) that

$$\int_G |h|^2 k |f'(\varrho(x))|^2 \, dx \geqq \int_G |h|^2 k |f(\varrho(x))|^2 \, dx .$$

We choose $f = f_R$, where

$$f_R(t) = \begin{cases} 1 & \text{for } t \leqq R \\ 0 & \text{for } t \geqq R+1 \\ \text{linear} & \text{for } R \leqq t \leqq R+1 , \end{cases}$$

and the inequality yields

$$\int_{\{x \in G \,|\, R \leqq \varrho(x) \leqq R+1\}} |h|^2 k \, dx \geqq \int_{\{x \in G \,|\, \varrho(x) \leqq R+1\}} |h|^2 k \, dx .$$

The left hand side converges to 0 as $R \to \infty$ and the right hand side to $\|h\|^2$, and therefore $h=0$. This proves the theorem.

If, in particular, the dimension $n=1$, this gives the following, apparently unknown result for Sturm–Liouville operators. Let us consider the differential expression

$$Lu = \frac{1}{k(x)}\{-(p(x)u')' + q(x)u\} \qquad \text{for} \quad x \in I ,$$

where I is an open interval on the real line, and let us for simplicity assume that the coefficients satisfy the conditions:

$$\begin{cases} p,\ q \text{ and } k \text{ are real-valued measurable functions.} \\ p \text{ is locally Lizschitzian and } p(x) > 0,\ \forall x \in I. \\ q \in L^2_{\text{loc}}(I). \\ k \in C(I) \text{ and } k(x) > 0 \text{ for every } x \in I. \end{cases}$$

The minimal Sturm–Liouville operator L_0 in the Hilbert space $H = L^2(I, kdx)$ is defined by

$$D(L_0) = C_0^\infty(I) \quad \text{and} \quad L_0 u = Lu \text{ for } u \in D(L_0) .$$

We then have the following

COROLLARY. *If L_0 is bounded below as an operator in H and the integral of $(k/p)^{\frac{1}{2}}$ diverges at both endpoints of the interval, then L_0 is essentially selfadjoint, i.e. there is limit point case in both endpoints.*

In particular: If the operator L_0, defined by the differential expression

$$Lu = -u'' + q(x)u, \qquad x \in \mathsf{R}, \qquad q \in L^2_{\text{loc}}(\mathsf{R}) ,$$

is bounded below, then there is limit point case in $\pm\infty$.

REFERENCES

1. T. Ikebe and T. Kato, *Uniqueness of the selfadjoint extensions of singular elliptic differential operators*, Arch. Rational Mech. Anal. 9 (1962), 77–92.
2. K. Jörgens, *Wesentliche Selbstadjungiertheit singulärer elliptischer Differentialoperatoren zweiter Ordnung in $C_0^\infty(G)$*, Math. Scand. 15 (1964), 5–17.
3. H. Rademacher, *Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale*, Math. Ann. 79 (1919), 340–359.
4. E. Wienholtz, *Halbbeschränkte partielle Differentialoperatoren zweiter Ordnung vom elliptischen Typus*, Math. Ann. 135 (1958), 50–80.

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