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EXCISION AND COFIBRATIONS

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An *excision map* is an inclusion of pairs of spaces $(X,A) \hookrightarrow (Y,B)$ such that $X-A=Y-B$. As is well known, an excision map needs not induce isomorphisms in homology unless fairly strict conditions (depending on the homology theory) are placed on the pairs. Even less it is true that general relative homeomorphisms $f:(X,A) \to (X',A')$ give rise to isomorphisms. The purpose of this note is to prove the following.

THEOREM 1. *Any map $f:(X,A) \to (X',A')$ between closed cofibered pairs that induce an isomorphism $(f/A)_.:H_.(X/A)\approx H_.(X'/A')$ induces an isomorphism $f_.:\ H_.(X,A)\approx H_.(X',A')$ and conversely.*

COROLLARY. *Let $f:(X,A) \to (X',A')$ be a relative homeomorphism between closed cofibered pairs which maps neighbourhoods of A to neighbourhoods of A' (e.g. f may be closed). Then f induces an isomorphism $f_.: H_.(X,A)\approx H_.(X',A')$.*

Here $H_.$ stands for an arbitrary generalized homology theory, not necessarily satisfying the dimension axiom, defined on a suitable category of topological spaces and maps. For convenience, though, we assume in the proof that $H_.$ is actually defined on the category of all topological spaces and (continuous) maps. The argument also works for cohomology. We are particularly interested in the case where $H_.$ is the singular homology functor and f is a relative homeomorphism (as in the corollary). One might be tempted to conjecture that in this case $(f/A)_.$ is always an isomorphism. There are simple counterexamples, however, to disprove the conjecture. We return briefly to this question at the end of the paper.

In this paper a cofibered pair (X,A) is a pair of spaces having the absolute homotopy extension property with respect to any space. Thus given a homotopy $G:A\times I \to Z$ and a map $h:X \to Z$ such that $G(x,0)=h(x)$ for $x\in A$, there is a homotopy $H:X\times I \to Z$ such that $H(x,0)=h(x)$ for $x \in X$ which is an extension of G.

Clearly a closed pair (X,A) is cofibered if and only if every map

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$(X \times 0) \cup (A \times I) \to Z$ has an extension to $X \times I$. Cofibered pairs (X,A) are usually closed. For instance if X is Hausdorff, this is automatically true. (For this and other interesting point-set topological properties of cofibrations, see (3).)

If (X,A) is a closed cofibered pair, then $(X/A,*)$ is a closed cofibered pair (where $*$ is the collapsed subset A), and the collapsing map $k:(X,A) \to (X/A,*)$ is a relative homeomorphism. This follows from the commutativity of the diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\quad k \times \mathrm{id} \quad} & X/A \times I \\ \cup & & \cup \\ (X \times 0) \cup (A \times I) & \longrightarrow & (X/A \times 0) \cup (* \times I) \end{array}$$

and the fact that (because I is compact) $k \times \mathrm{id}$ is an identification map. Therefore a map $f:(X,A) \to (X',A')$ between closed cofibered pairs splits up into a commutative rectangle

$$\begin{array}{ccc} (X,A) & \xrightarrow{f} & (X',A') \\ k \downarrow & & \downarrow k' \\ (X/A,*) & \xrightarrow{\tilde{f}} & (X'/A',*) \end{array}$$

of closed cofibered pairs and maps. Clearly the theorem follows if we show that collapsings $(X,A) \to (X/A,*)$ always induce isomorphisms in homology, $H.(X,A) \approx H.(X/A,*)$. Conversely this is a trivial consequence of the theorem itself.

The following property of closed cofibered pairs is actually characteristic, but we don't need that (cf. [1, p. 111], and [2, exc. 1. E. 6]).

LEMMA 1. *Let* (X,A) *be a closed cofibered pair. There is a function* $\varphi:X \to I$ *such that* $A = \varphi^{-1}(1)$ *and a deformation* $D:X \times I \to X$ *relative to* A *such that* $D(\varphi^{-1}(0,1] \times 1) \subset A$.

PROOF. The identity map on $(X \times 0) \cup (A \times I)$ extends to a retraction $r:X \times I \to (X \times 0) \cup (A \times I)$. Let $\varphi_n, \varphi'_n:X \to I$ be the functions

$$\varphi'_n(x) = \mathrm{pr}_2\big(r(x, 1/n)\big), \qquad \varphi_n(x) = n \inf\,(\varphi'_n(x), 1/n)$$

for $n = 1, 2, \ldots$. Then φ_n maps X into the unit interval and takes the value 1 on A for all n. Moreover, if $x \in X - A$, then $\mathrm{pr}_1\big(r(x,0)\big) \in X - A$ and by continuity $\mathrm{pr}_1\big(r(x, 1/n)\big) \in X - A$ for n sufficiently large. Therefore eventually $r(x, 1/n) \in X \times 0$ and $\varphi_n(x) = \varphi'_n(x) = 0$. It follows that the function $\sum_{n=1}^{\infty}\varphi_n/2^n : X \to I$ takes the value 1 exactly on A and so does the function $\varphi = \inf(\varphi_1, \sum_{n=1}^{\infty}\varphi_n/2^n)$. Let $D: X \times I \to X$ be defined by $D(x,t) = \mathrm{pr}_1(r(x,t))$. Then D is a deformation relative to A, and if $\varphi(x) > 0$ for some x, so is $\varphi_1(x)$ and $\varphi'_1(x)$, and therefore $r(x,1) \in A \times I$ showing that $D(x,1) \in A$.

Given a pair (Y,B), then a *neighbourhood retraction* to B in Y is a map $\sigma: Y \to Y$ which retracts some neighbourhood of B onto B. If $C \subset \sigma^{-1}B$, let $\sigma_C: (Y,C) \to (Y,B)$ be the map of pairs defined by σ.

Corollary 1. *Let (X,A) be a closed cofibered pair. Then there is a neighbourhood retraction $\varrho: X \to X$ to A in X with the property: For any neighbourhood V of A there is a neighbourhood $W \subset V \cap \varrho^{-1}A$ such that the inclusion $i_{WV}: (X,W) \hookrightarrow (X,V)$ is homotopic relative to A to the composite*

$$i_{AV} \circ \varrho_W : (X,W) \to (X,A) \hookrightarrow (X,V)\,.$$

Proof. With the notations of lemma 1 let $U = \varphi^{-1}(0,1]$ and let ϱ be the map $D(\cdot,1)$. Given $V \supset A$, for each $x \in A$ there is a neighbourhood W_x of x, in X, contained in $V \cap U$, such that $D(W_x \times I) \subset V$. Let $W = \bigcup W_x$. Then W is a neighbourhood of A in X contained in $V \cap \varrho^{-1}A$ and $D(W \times I) \subset V$. Therefore D defines a homotopy $i_{WV} \simeq i_{AW} \circ \varrho_W$ rel. A.

Corollary 2. *Let (X,A) be a closed cofibered pair. Then*

$$H_\cdot(A) \approx \lim_{\leftarrow} H_\cdot(V), \qquad H_\cdot(X,A) \approx \lim_{\leftarrow} H_\cdot(X,V)\,,$$

V varying over a fundamental system of neighbourhoods of A.

Proof. For sufficiently small neighbourhoods V of A the composite map $\varrho_V \circ i_{AV}: (X,A) \hookrightarrow (X,V) \to (X,A)$ is homotopic to the identity, and therefore $i_{AV\cdot}: H_\cdot(X,A) \to H_\cdot(X,V)$ is a monomorphism. It follows that the canonical map $H_\cdot(X,A) \to \lim H_\cdot(X,V)$ is a monomorphism. On the other hand, for V and W as in Corollary 1 there is a commutative diagram

$$\begin{array}{ccc} H_\cdot(X,W) & \xrightarrow{\;i_{WV\cdot}\;} & H_\cdot(X,V) \\ & {\scriptstyle \varrho_{W\cdot}}\searrow \quad \nearrow{\scriptstyle i_{AV\cdot}} & \\ & H_\cdot(X,A) & \end{array}$$

showing that any element in $H_\cdot(X,V)$ which is the image of an element in $H_\cdot(X,W)$, is already the image of an element in $H_\cdot(X,A)$. It follows that the canonical map $H_\cdot(X,A) \to \lim_{\leftarrow} H_\cdot(X,V)$ is an epimorphism as well. The same argument applied to the single spaces A, V, W rather than to pairs gives the isomorphism between the absolute groups. Cohomology isomorphisms can be proved similarly.

LEMMA 2. *Let* (X,A) *be a closed cofibered pair. Then the collapsing* $k:(X,A) \to (X/A,*)$ *induce isomorphisms in homology.*

PROOF. For any neighbourhood V of A in X kV is a neighbourhood of $*$ in X/A, and this correspondence is bijective. Let

$$k_V : (X,V) \to (X/A, kV/A)$$

and

$$k'_V : (X-A, V-A) \to (X/A-*,\ kV/A-*)\,,$$

be the maps defined by k. Then k'_V is a homeomorphism and so induces isomorphisms in homology and cohomology. By the excision property k_V also induces isomorphisms. Passing to limits and using Corollary 2 we get that $k. = \lim_{\leftarrow} k_{V}.$ is an isomorphism.

REMARK. Corollary 2 and the proof of lemma 2 are not well adapted to the case of an arbitrary generalized homology theory defined on some admissible category. The neighbourhoods V can be considered open and then a pair (X,V) may not be an admissible pair in the category. Even so the excision map $(X-A, V-A) \hookrightarrow (X,V)$ may not be an admissible map. This can be fixed in several ways. Perhaps the easiest way is to consider the commutative diagram

$$\begin{array}{ccccc} H.(X,A) & \xrightarrow{i_{AF\cdot}} & H.(X,F) & \xrightarrow{\varrho_{F\cdot}} & H.(X,A) \\ \downarrow k. & & \downarrow k_{F\cdot} & & \downarrow k. \\ H.(X/A,*) & \xrightarrow{i_{*F/A\cdot}} & H.(X/A,F/A) & \xrightarrow{\varrho_{F/A\cdot}} & H.(X/A,*) \end{array}$$

where $F = \Phi^{-1}[t,1]$ for some t in $(0,1)$. Then F is a *closed* neighbourhood of A, and if $U' = \Phi^{-1}(t/2,1]$, then $(X-U', F-U') \hookrightarrow (X,F)$ is an admissible map in any reasonable category. Since $\overline{U'} \subset \mathring{F}$, it follows now from the excision axiom like in Lemma 2 that $k_{F}.$ is an isomorphism. Since the composites of the horizontal maps are the identity maps, $k.$ is monic on the left and epic on the right, hence an isomorphism.

This method avoids the limiting process. On the other hand it does not give the extra information of Corollary 2.

Finally, consider the case where $H.$ is the singular homology functor (any coefficients) and $f:(X,A) \to (X',A')$ is an arbitrary relative homeomorphism. We should like to conclude that $f.:H.(X,A) \to (H.(X',A')$ is an isomorphism. If (X,A) and (X',A') are compact pairs, then this conclusion follows from the corollary of Theorem 1. The compactness condition is much too restrictive though. On the other hand, if (X,A) and (X',A') are any two closed cofibered pairs, then the conclusion may fail. Let $\tilde{I}$ be the sum $\{0\} \cup (I - \{0\})$. Then $(\tilde{I},0)$ is a closed cofibered pair, and there is a canonical relative homeomorphism $(\tilde{I},0) \to (I,0)$. However, $H_0(\tilde{I}) \approx Z \oplus Z$ and $H_0(I) \approx Z$.

An even better counterexample one gets by considering the closed unit disk in the plane and the space included which is the open unit disk together with two points on the boundary. Collapsing the boundary gives two based spaces $(X,*)$, $(S^2,*)$ and a relative homeomorphism $i':(X,*) \to (S^2,*)$ induced from the inclusion map. Since $(S^1,*)$ may be considered contained in and a strong deformation retract of $(X,*)$,the map i'. is not an isomorphism. Notice that in this example the spaces are both connected and locally path connected.

REFERENCES

1. C. H. Dowker, *Homotopy extension theorems*, Proc. Lond. Math. Soc. 6 (1956), 100–117.
2. E. H. Spanier, *Algebraic topology*, New York · London · Toronto, 1966.
3. A. Strøm, *Note on cofibrations*, Math. Scand. 19 (1966), 11–14.

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