MATH. SCAND. 19 (1966), 127–130

SETS OF LOCAL UNIFORM CONVERGENCE

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Let $\sum_n a_n z^n$ be a Taylor series with radius of convergence one, $\lim_n a_n = 0$, and $\sum_n |a_n| = +\infty$. The author [3] defined sets of local uniform convergence and sets of non-uniform convergence of such series. A point z on the unit C, $|z| = 1$, is a *point of local uniform convergence* if the Taylor series converges uniformly on some neighborhood of z, that is, some open arc of C that contains z; the set of all such points is called the *set of local uniform convergence.* The complement of the set of local uniform convergence with respect to the set of convergence is called the *set of non-uniform convergence.*

This note proves theorems about sets of local uniform convergence similar to those in [3]; but the proofs use geometric polynomials instead of Fejér polynomials as building blocks. The author feels that it is of interest that these results can be accomplished with geometric polynomials.

THEOREM. *If G is any open set on C, then there exists a function $f(z) = \sum_n a_n z^n$ whose Taylor series converges everywhere on C and has G as its set of local uniform convergence.*

The case where G is the entire unit circle C was treated by Hardy [1] and Herzog and Piranian [2], who each gave an example of a Taylor series which converges uniformly, but not absolutely on C.

Assume that G is not the entire unit circle C. Let $\{A_q\}$ be any denumerable set of disjoint open arcs on C such that

(i) δ_q represents the angular length of A_q;
(ii) α_q, $\alpha_q \neq 1$, represents the midpoint of A_q;
(iii) the sequence $\{\alpha_q\}$ monotonically approaches 1.

Let $\{\omega_q\}$ be a dense subset of $C - G$, and let $\alpha_{q,p} = \alpha_q \omega_p$. Let

$$G_t(z) = 1 + z + z^2 + \ldots + z^t \tag{1}$$

Received March 28, 1966.

This paper is part of Chapter III of a doctoral dissertation written at Michigan State University under Professor F. Herzog.

and

(2) $$C_{q,p}(z) = z^{N_{q,p}} G_{t_q}(z/\alpha_{q,p})/t_q ,$$

where t_q and $N_{q,p}$ are positive integers; the $N_{q,p}$ are chosen to ensure non-overlapping of the polynomials (2) if they are summed as follows: $C_{1,1}+C_{2,1}+C_{1,2}+C_{3,1}+\ldots$. A series of polynomials

(3) $$\sum_{q,p} 2^{-p} C_{q,p}(z)$$

will constitute a Taylor series. Let the sequence $\{t_q\}$ satisfy the conditions that

$$t_q \geqq 2^q \quad \text{and} \quad \sum_q \pi/(\delta_q t_q) = T < +\infty .$$

Let

$$A_{q,p} = \{e^{i\varphi}\omega_p : \; e^{i\varphi} \in A_q\} .$$

Then for each p, any $e^{i\theta}$ on C can belong to at most one $A_{q,p}$. If $e^{i\theta}$ belongs to $A_{q_0,p}$, then

$$|C_{q_0,p}(e^{i\theta})| \leqq 1 ,$$

while

$$|C_{q,p}(e^{i\theta})| \leqq \pi/(\delta_q t_q) \quad \text{if} \quad q \neq q_0 .$$

If $e^{i\theta}$ does not belong to $A_{q,p}$ for all q, then

$$|C_{q,p}(e^{i\theta})| \leqq \pi/(\delta_q t_q) .$$

In either case, the series (3) converges absolutely and

$$\sum_{q,p} 2^{-p} |C_{q,p}(e^{i\theta})| \leqq T+1 .$$

Let $^*C_{q,p}(z)$ denote any block of consecutive terms from the beginning of $C_{q,p}(z)$. Let $C^*_{q,p}(z)$ denote $C_{q,p}(z) - {}^*C_{q,p}(z)$. In order to show that the Taylor series defined by the series of polynomials (3) converges everywhere on C, let z be any point on C and let $\varepsilon > 0$. Choose P such that $2^{-n} < \varepsilon$ when $n > P$. By the way the $C_{q,p}(z)$ were defined in (2),

$$|{}^*C_{q,p}(z)| \leqq 1 ,$$

and so for $p > P$,

$$2^{-p} \, |{}^*C_{q,p}(z)| < \varepsilon .$$

There exists $Q > 0$ such that

$$|{}^*C_{q,p}(z)| < \varepsilon \qquad \text{when } q > Q, \; p \leqq P .$$

Hence if $p+q > P+Q$, then

$$2^{-p} \, |{}^*C_{q,p}(z)| < \varepsilon$$

and the Taylor series converges at z.

To show that every point of $C-G$ is a point of non-uniform convergence of the Taylor series defined by the series (3), let z be any point of $C-G$. Let N be any neighborhood of z. Then there exists an integer P such that $\omega_P \in N$. Also, there exists an integer Q such that $\alpha_{q,P} \in N$ when $q > Q$. But, $2^{-P}\,|C_{q,P}(\alpha_{q,P})| = 2^{-P}$ and so the Taylor series does not converge uniformly on N. Hence, z is a point of non-uniform convergence of the Taylor series.

On the other hand, suppose that z is any point of G. Let

$$\Delta(z, C-G) \;=\; 3\delta\,,$$

where $\Delta(z,F)$ denotes the angular distance between the point z and the set F. Let

$$N \;=\; \{e^{i\theta}:\ \Delta(z,e^{i\theta}) < \delta\}\,.$$

Let ζ be any point of N and let $S_K(z)$ denote the remainder of the Taylor series after the K-th power of z. For each K there exist integers s and M such that

$$S_K(z) \;=\; 2^{-s}\,C^*_{M-s,s}(z) + \sum_{k=s+1}^{M-1} 2^{-k}\,C_{M-k,k}(z) + \sum_{m=M+1}^{\infty}\ \sum_{p+q=m} 2^{-p}\,C_{q,p}(z)\,.$$

Since the series of polynomials (3) converges absolutely,

$$\begin{aligned}
|S_K(\zeta)| \;&\leqq\; 2^{-s}\,|C^*_{M-s,s}(\zeta)| + \sum_{k=s+1}^{M-1} 2^{-k}\,|C_{M-k,k}(\zeta)| + \sum_{m=M+1}^{\infty}\ \sum_{p+q=m} 2^{-p}\,|C_{q,p}(\zeta)| \\
&\leqq\; 2^{-s}\,|C^*_{M-s,s}(\zeta)| + \sum_{m=M}^{\infty}\ \sum_{p+q=m} 2^{-p}\,|C_{q,p}(\zeta)| \\
&\leqq\; 2^{-s}\,|C^*_{M-s,s}(\zeta)| + \sum_{p=1}^{P} 2^{-p} \sum_{q=M-p}^{\infty} |C_{q,p}(\zeta)| + \sum_{p=P+1}^{\infty} 2^{-p} \sum_{q=1}^{\infty} |C_{q,p}(\zeta)|\,.
\end{aligned}$$

Let $\varepsilon > 0$ and let P be chosen large enough so that $2^{-P}(T+1) < \frac{1}{3}\varepsilon$. Then there exists Q such that $\Delta(\alpha_{q,p},N) > \delta$ when $q > Q$ and $p \leqq P$. Since

$$\sum_{q=1}^{\infty} |C_{q,p}(\zeta)| \;\leqq\; T+1, \qquad \sum_{p=P+1}^{\infty} 2^{-p} \sum_{q=1}^{\infty} |C_{q,p}(\zeta)| \;\leqq\; 2^{-P}(T+1) \;<\; \tfrac{1}{3}\varepsilon\,.$$

Since

$$|C^*_{q,p}(\zeta)| \;<\; \pi/(\delta t_q) \;\leqq\; 2^{-q}(\pi/\delta)$$

when $q > Q$ and $p \leqq P$,

$$2^{-s}\,|C^*_{M-s,s}(\zeta)| \;<\; 2^{-s}\,2^{-M+s}\,(\pi/\delta) \;=\; 2^{-M}(\pi/\delta) \qquad \text{if } s \leqq P\,.$$

Also, $|C^*_{q,p}(\zeta)| \leqq 1$, so that

$$2^{-s}\,|C^*_{M-s,s}(\zeta)| \;\leqq\; 2^{-s} \;<\; 2^{-P} \;<\; \tfrac{1}{3}\varepsilon \qquad \text{if } s > P\,.$$

In either case, if M is chosen sufficiently large,

$$2^{-s}\,|C^*_{M-s,s}(\zeta)| < \tfrac{1}{3}\varepsilon\,.$$

Moreover, if $M > P+Q$,

$$\sum_{p=1}^{P} 2^{-p} \sum_{q=M-p}^{\infty} |C_{q,p}(\zeta)| \leqq \sum_{p=1}^{P} 2^{-p} \sum_{q=M-p}^{\infty} 2^{-q}(\pi/\delta)$$
$$= \sum_{p=1}^{P} 2^{-M+1}(\pi/\delta) = 2^{-M+1}(\pi/\delta)P\,,$$

since in each case $q > Q$ and $p \leqq P$. Again if M is chosen sufficiently large,

$$\sum_{p=1}^{P} 2^{-p} \sum_{q=M-p}^{\infty} |C_{q,p}(\zeta)| < \tfrac{1}{3}\varepsilon\,.$$

Hence $|S_K(\zeta)| < \varepsilon$ when K is sufficiently large and the Taylor series converges uniformly on N. Hence z is a point of local uniform convergence. Therefore G is the set of local uniform convergence of the Taylor series.

If G is taken to be empty, then every point of C is a point of non-uniform convergence, and one has the following corollary.

COROLLARY. *There exists a function $f(z)$ whose Taylor series converges everywhere on C, but not uniformly on any arc of C.*

The function $f(z)$ in the above corollary is not necessarily continuous in $|z| \leqq 1$, but is radially continuous on every radius of $|z| \leqq 1$.

Corollaries similar to those given in [3] follow by replacing the Fejér polynomials $Q_{q,p}(z)$ in [3] by the geometric polynomials $C_{q,p}(z)$ defined in (2).

COROLLARY. *The characteristic property of the set of non-uniform convergence of a Taylor series that converges everywhere on C is that the set is closed.*

REFERENCES

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