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# MEASURE THEORY FOR $C^*$ ALGEBRAS

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## Introduction.

In measure theory two slightly different attitudes to the material can be found. Either one is interested in the measure of sets in the space, or one looks for the functions which can be integrated. The second attitude has certain technical advantages when the space $X$ is locally compact Hausdorff, since we can then by the Riesz representation theorem reduce the study of regular Borel measures on $X$ to the study of positive linear functionals on $K(X)$, the space of continuous functions with compact supports (throughout the paper we shall use "measure" synonymous with "positive measure").

Now, as is well known, a commutative $C^*$ algebra has the form $C(X)$ with $X$ compact Hausdorff if the algebra has an identity and the form $C_0(X)$, with $X$ locally compact Hausdorff otherwise. Thus $C^*$ algebras are the non-commutative analogues of function algebras, and we should expect to find a non-commutative analogue of measure theory in the Bourbaki sense by studying positive functionals on a $C^*$ algebra $A$ or on a suitable subset of $A$.

If $1 \in A$, the algebra resembles a $C(X)$ and consequently any operator should be integrable. Since a positive functional on all of $A$ is necessarily bounded, measure theory for $A$ is the study of continuous, positive functionals on $A$. As is well known, this theory exists and is vitally important for the study of the algebra.

If $1 \notin A$, the algebra resembles a $C_0(X)$. It is the aim of the present paper to show that in this case there is a reasonable non-commutative analogue of the notion of unbounded measure. In section 1 we find and discuss a two-sided dense ideal $K$ in $A$ which is the non-commutative analogue of $K(X)$ in $C_0(X)$. In section 2 we show that most of the relations between bounded positive functionals and representations carry over even when the functional is allowed to take on infinite values. In section 3 we study positive functionals on $K$ which satisfy a certain condition which compensates for the non-commutativity of $A$. Such a functional, which we call a $C^*$ integral, can be written as a sum of

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bounded positive functionals and therefore has a unique lower semi-continuous extension to all positive operators in $A$, a normal extension to the positive part of the enveloping von Neumann algebra, and a representation as a regular Borel measure on the weak* closure of the pure states. Finally we determine the set of $C^*$ integrals on the compact operators of a Hilbert space $H$ as isomorphic to the set of all positive bounded operators on $H$.

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The notation and terminology is more or less that of [2]. Throughout the paper, $A$ is our fixed $C^*$ algebra. It is assumed that $1 \notin A$ unless the contrary is explicitly stated. The $C^*$ algebra obtained by adjoining an identity to $A$ is denoted $\tilde{A}$. If $L$ and $M$ are subsets of $A$ we denote by $L^+$ the set of positive elements in $L$, by $L^*$ the set of adjoints of elements in $L$ and by $LM$ the (complex) linear span of all products $ab$ with $a \in L$, $b \in M$.

## 1. Operators with majorized supports.

A *subalgebra $B$ of $A$ is called *order-related* if $B^+$ is an order ideal in $A^+$ and $B$ is the linear span of $B^+$.

An order ideal $J$ of $A^+$ is called *invariant* if $a^* J a \subset J$ for all $a \in A$ or equivalently if $u^* J u = J$ for all unitary operators $u \in \tilde{A}$.

LEMMA 1.1. *If $J$ is an order ideal in $A^+$, define $I_1$ as the linear span of elements from $J$, and $I_2 = \{a \in A \mid a^* a \in J\}$. Then*

*$I_1$ is an order-related *algebra with $I_1{}^+ = J$,*

*$I_2$ is a left ideal ,*

*$I_2{}^* I_2 = I_1$ and $\overline{I_2{}^*} \cap \overline{I_2} = \overline{I_1}$ ,*

*where $\bar{I}$ denotes the uniform closure of $I$.*

*Moreover if $J$ is invariant, then $I_1$ is a two-sided ideal in $A$.*

PROOF. The inequalities

$$(a+b)^*(a+b) \leq 2(a^*a+b^*b)$$

and

$$(ba)^*(ba) \leq \|b\|^2 a^*a$$

show that $I_2$ is a left ideal. The equality

$$4b^*a = \sum_{n=0}^{3} i^n(a+i^n b)^*(a+i^n b)$$

shows that $I_2{}^*I_2 \subset I_1$, and since the reverse inclusion is obvious, $I_2{}^*I_2 = I_1$. Trivially $I_1{}^* = I_1$ and $I_1{}^+ = J$ and since

$$I_1 I_1 = (I_2{}^*I_2)(I_2{}^*I_2) = I_2{}^*(I_2 I_2{}^* I_2) \subset I_2{}^*I_2 = I_1,$$

we conclude that $I_1$ is an order-related *algebra.

Since the involution is continuous, $\overline{I_2{}^*} \cap \overline{I_2} = (\overline{I_2})^* \cap \overline{I_2}$ is a $C^*$ subalgebra of $A$ containing $I_1$. Therefore, if $a \in (\overline{I_2{}^*} \cap \overline{I_2})^+$, then $a^{\frac{1}{2}} \in (\overline{I_2{}^*} \cap \overline{I_2})^+$ and there exists $b \in I_2$ such that $b$ is near $a^{\frac{1}{2}}$ and therefore $b^*b$ is near $a$, that is, $a \in \overline{I_1}$.

Now suppose $J$ invariant. For any $a \in J$ and $b \in A$ we have

$$\begin{aligned} 4b^*a &= \sum_{n=0}^{3} i^n(a^{\frac{1}{2}} + i^n a^{\frac{1}{2}} b)^*(a^{\frac{1}{2}} + i^n a^{\frac{1}{2}} b) \\ &= \sum_{n=0}^{3} i^n(1+i^n b)^* a (1+i^n b) \in I_1 \end{aligned}$$

and hence $I_1$ is a two-sided ideal.

COROLLARY 1.2. *There is a one-to-one correspondence between order ideals in $A^+$ and order-related *subalgebras of $A$. The invariant order ideals correspond to the order-related two-sided ideals.*

We think of the elements of $A$ as operators on its universal Hilbert space and denote by $A''$ the weak closure of $A$. (See [2, § 12].) A projection $p \in A''$ is called *majorized* (relative $A$) if there exists $b \in A^+$ such that $p \leqq b$. We let $[a]$ denote the range projection of any operator $a$ on a Hilbert space and define

$$\begin{aligned} K_0{}^+ &= \{a \in A^+ \mid \exists b \in A^+, [a] \leqq b\}, \\ K^+ &= \{a \in A^+ \mid \exists a_i \in K_0{}^+, i = 1,2,\ldots,n, a \leqq \textstyle\sum a_i\}. \end{aligned}$$

$K_0{}^+$ is the set of operators in $A^+$ with majorized supports, and $K^+$ is the smallest order ideal containing $K_0{}^+$.

The definition of $K_0{}^+$ may be rephrased without reference to $A''$ as follows: $a \in K_0{}^+$ if there exists $b \in A^+$ such that for all functions $\varphi$, continuous on the spectrum of $a$, $0 \leqq \varphi \leqq 1$, we have $\varphi(a) \leqq b$. Since we can choose a sequence $\{\varphi_n\}$ of such functions such that $\{\varphi_n(a)\}$ converges strongly to $[a]$, and since the set of positive operators on a Hilbert space is strongly closed, the two definitions are equivalent.

If $K$ denotes the linear span of $K^+$ we have

THEOREM 1.3. *$K$ is a two-sided, dense, order-related ideal in $A$, minimal among all such.*

PROOF. Using the second definition of $K_0{}^+$ and the fact that *automorphisms of $A$ preserve spectral theory, we have in particular $u^* K_0{}^+ u = K_0{}^+$ for all unitary $u \in \tilde{A}$. It follows that also $K^+$ is unitarily invariant and hence an invariant order ideal, such that lemma 1.1 applies and $K$ is a two-sided order-related ideal in $A$.

Let $\varphi_n$ be the real function defined by

$$\varphi_n(t) = \begin{cases} 0 & \text{for } t \leqq n^{-1}, \\ 2(t-n^{-1}) & \text{for } n^{-1} < t \leqq 2n^{-1}, \\ t & \text{for } t > 2n^{-1}. \end{cases}$$

For any $a \in A^+$

$$\|a - \varphi_n(a)\| \leqq n^{-1} \quad \text{and} \quad [\varphi_n(a)] \leqq na$$

so that $\varphi_n(a) \in K_0{}^+$ and $\varphi_n(a)$ approaches $a$ from below as $n$ tends to infinity. It follows that $K$ is dense in $A$.

If $a \in K_0{}^+$ and $[a] \leqq b$, we have for any $c \in A$ with $c = c^*$, $\|c\| \leqq \frac{1}{2}$,

$$\tfrac{1}{2}a \leqq a^{\frac{1}{2}}(1+c)a^{\frac{1}{2}} \leqq a^{\frac{1}{2}}([a]+c)a^{\frac{1}{2}} \leqq a^{\frac{1}{2}}(b+c)a^{\frac{1}{2}}.$$

If $I$ is a dense order-related two-sided ideal in $A$, it contains elements arbitrarily near $b$, and therefore there exists $c$ as above so that $b+c \in I$. As $I$ is an ideal, $a^{\frac{1}{2}}(b+c)a^{\frac{1}{2}} \in I^+$, and since $I^+$ is an order ideal, $\frac{1}{2}a \in I^+$. It follows that $K \subset I$.

THEOREM 1.4. *If $\Phi$ is a *homomorphism from $A$ onto the $C^*$ algebra $B$, then $\Phi(K_A) = K_B$.*

PROOF. Since $\Phi$ preserves spectral theory, $\Phi(K_A) \subset K_B$. On the other hand, $\Phi(K_A)$ is a two-sided dense order-related ideal in $B$ and hence, by theorem 1.3, contains $K_B$.

If $A$ is commutative, it is of the form $C_0(X)$ and $K = K(X)$ the set of continuous functions with compact supports.

If $A = B_0(H)$, the set of compact operators on the Hilbert space $H$, then $K$ is the set of operators on $H$ with finite rank.

In these two examples $K_0{}^+ = K^+$, but this need not be true in general. If $a$ is a compact operator on the infinite dimensional Hilbert space $H$, $0 \leqq a \leqq 1$ and $[a] = 1$, define two projections on $H \oplus H$ by the matrices

$$p = \begin{pmatrix} a & (a-a^2)^{\frac{1}{2}} \\ (a-a^2)^{\frac{1}{2}} & 1-a \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $A$ be the $C^*$ algebra generated by $p$ and $q$. By definition $p$ and $q$ belong to $K_0{}^+$. If we had $K_0{}^+=K^+$, then also $(1-q)p(1-q)\in K_0{}^+$. But

$$(1-q)p(1-q) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{hence} \qquad [(1-q)p(1-q)] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since any polynomial in $p$ and $q$ will have a compact operator in the upper left corner of its matrix, we conclude that no element in $A$ will majorize $[(1-q)p(1-q)]$, and we have a contradiction.

The next theorem shows however that for a $C^*$ algebra with continuous trace (see [2, 4.5] for definition) we do have $K_0{}^+=K^+$.

THEOREM 1.5. *If $A$ has continuous trace, let $I^+$ denote the set of operators $a\in A^+$ such that*

$$\sup\{\dim\pi(a) \mid \pi\in\hat{A}\} < \infty$$

*and such that the function $\pi\to\operatorname{tr}(\pi(a))$ is continuous and vanishes outside a compact set of the structure space $\hat{A}$. Then $I^+=K_0{}^+$.*

PROOF. If $a\in K_0{}^+$ and $[a]\leqq b$, then the set $C=\{\pi\in\hat{A} \mid \|\pi(b)\|\geqq 1\}$ is compact ([2, 3.3.7]) and $\pi\notin C$ implies

$$\|[\pi(a)]\| \leqq \|\pi(b)\| < 1$$

and hence $\pi(a)=0$. For any $c\in A$ with $c=c^*$, $\|c\|\leqq\frac{1}{2}$, we have as in the proof of theorem 1.3

$$\tfrac{1}{2}[a] \leqq [a](b+c)[a]\,.$$

Since the linear combinations of elements $d\in A^+$ such that the function $\pi\to\operatorname{tr}(\pi(d))$ is continuous and finite on $\hat{A}$ are dense in $A$, there exist $c$ and $d$ such that $b+c\leqq d$, and thus

$$\begin{aligned}\tfrac{1}{2}\dim\pi(a) &\leqq \operatorname{tr}(\pi([a]d[a]))\\ &\leqq \sup\{\operatorname{tr}(\pi(d)) \mid \pi\in C\} < \infty\,.\end{aligned}$$

Hence $K_0{}^+\subset I^+$.

If on the other hand $a\in I^+$, then $\pi(a)$ vanishes outside a compact set $C$ in $\hat{A}$ and $\dim\pi(a)\leqq n<\infty$. For any $\pi_0\in C$, $\pi_0(a)$ has rank $\leqq n$ and since $\pi_0(A)$ is the set of compact operators on the representation space, there is a $b_0\in A^+$ such that $\pi_0(b_0)$ is a one-dimensional projection contained in an eigenspace of $\pi_0(a)$. We now proceed as in the proof of [2, 4.4.2]. We define $b_1=a^{\frac{1}{2}}b_0a^{\frac{1}{2}}$, and by cutting away the points of $\operatorname{sp}(b_1)$ near 0 we find an element $b\in A^+$ such that $0\leqq b\leqq 1$, $b\leqq a$ and $\pi(b)$ a one-dimensional projection for $\pi$ in a neighbourhood $O(\pi_0)$ of $\pi_0$. We have

$$a = (1-b+b)a(1-b+b) \leqq 2((1-b)a(1-b)+bab)$$

and for $\pi \in O(\pi_0)$

$$\dim\pi((1-b)a(1-b)) \leqq \dim\pi((1-b)[a](1-b)) \leqq \dim(\pi([a])-\pi(b)) \leqq n-1 .$$

We now repeat the process with $(1-b)a(1-b)$ instead of $a$, and in at most $n$ steps we find elements $c_1=b,c_2,\ldots,c_n$ in $A^+$ and a neighbourhood $O_0(\pi_0)$ such that $a \leqq \sum\alpha_i c_i$, $\alpha_i>0$, and $\{\pi(c_i)\}$ is a set of mutually orthogonal one-dimensional projections for $\pi \in O_0(\pi_0)$. We conclude that $c_0=\sum c_i$ satisfies $\pi([a]) \leqq \pi(c_0)$ for $\pi \in O_0(\pi_0)$.

Since $C$ is compact, a finite number of sets $O_i(\pi_i)$ will cover $C$, and by adding the respective elements $c_i$ we get a $c \in A^+$ such that $\pi([a]) \leqq \pi(c)$ for all $\pi \in C$ and hence for all $\pi \in \hat{A}$. We conclude that $[a] \leqq c$ and thus $I^+ \subset K_0{}^+$.

If $A$ is a separable $C^*$ algebra, there is a countable approximate identity $\{v_n\}$ such that $\{v_n\}$ converge strongly to 1 in $A''$. It follows that $[\sum 2^{-n}v_n]=1$ and thus $K=A$ if and only if $1 \in A$.

On the other hand if $H$ is a Hilbert space with uncountable dimension and $A$ is the $C^*$ algebra of operators on $H$ with countable dimensional range projections, then $1 \notin A$ but $K=A$. We thus have the same phenomena as in the commutative case.

## 2. Extended positive functionals.

Throughout this section $f$ denotes an *extended positive functional* on $A$, that is, a (not necessarily continuous) function $f\colon A^+ \to [0,\infty]$ satisfying

$$f(\alpha a+\beta b) = \alpha f(a)+\beta f(b)$$

for all $a,b \in A^+$ and $\alpha,\beta \in R^+$ (we agree that $0\cdot\infty=0$).

We define

$$L_1{}^+ = \{a \in A^+ \mid f(a)<\infty\} ,$$
$$N_1{}^+ = \{a \in A^+ \mid f(a)=0\} ,$$

and let $L_1$ and $N_1$ denote the linear span of $L_1{}^+$ and $N_1{}^+$, respectively. Finally we define

$$L_2 = \{a \in A \mid f(a^*a)<\infty\} ,$$
$$N_2 = \{a \in A \mid f(a^*a)=0\} .$$

Lemma 2.1. *$L_1$ and $N_1$ are *algebras, $L_2$ and $N_2$ are left ideals, and we have $L_2{}^*L_2=L_1$ and $N_2{}^*N_2=N_1$.*

PROOF. $L_1^+$ and $N_1^+$ are obviously order ideals, so lemma 1.1. applies.

We now extend $f$ by linearity to a linear functional on $L_1$. We call $f$ trivial if $L_1 = N_1$.

LEMMA 2.2. *$|f(b^*a)|^2 \leq f(a^*a)f(b^*b)$ for all $a,b \in L_2$.*

PROOF. Since $b^*a \in L_2{}^*L_2 = L_1$, $f(b^*a)$ is defined, and the usual proof of the Cauchy–Schwarz inequality applies.

THEOREM 2.3. *To any non-trivial extended positive functional $f$ on $A$ there corresponds a non-zero, non degenerated *representation $\pi_f$ of $A$ on a Hilbert space $H_f$.*

PROOF. The difference space $L_2 - N_2$ is a pre-Hilbert space under the product

$$(a+N_2 \mid b+N_2) = f(b^*a), \qquad a,b \in L_2 .$$

We define

$$\pi_f(a)(b+N_2) = ab+N_2$$

for all $a \in A$, $b \in L_2$. Since $L_2$ and $N_2$ are left ideals, the definition makes sense and we have

$$\begin{aligned}\|\pi_f(a)(b+N_2)\|^2 &= f((ab)^*(ab))\\ &\leqq \|a\|^2 f(b^*b) = \|a\|^2\|b+N_2\|^2 .\end{aligned}$$

Hence $\pi_f(a)$ has an extension (again denoted $\pi_f(a)$) as a bounded operator on the completion $H_f$ of $L_2 - N_2$. A routine inspection shows that $\pi_f$ is a *representation of $A$ on $H_f$. Since $f$ is non-trivial, $H_f \neq 0$. There exists an extension of $\pi_f$ to a normal representation of $A''$, and since $1 \in A''$, we have

$$\overline{\pi_f(A)H_f} = \overline{\pi_f(A'')H_f} = H_f ,$$

that is, $\pi_f$ is not degenerated.

We recall that in the ordinary construction with a continuous $f$ there is a cyclic vector $\xi \in H_f$ and we have $f(a) = (\pi_f(a)\xi \mid \xi)$ for all $a \in A$. In order to obtain a similar result in the unbounded case we impose restrictions on $f$.

THEOREM 2.4. *If $f$ is densely defined and there exists a set $\{f_i \mid i \in I\}$ of bounded positive functionals such that $f = \sum f_i$, then there is a set of vectors $\{\xi_i \mid i \in I\} \subset H_f$ such that*

$$f(a) = \sum (\pi_f(a)\xi_i \mid \xi_i)$$

*for all $a \in A^+$.*

PROOF. Since $f_i \leq f$, there is a well defined extension $T_i$: $H_f \to H_{fi}$, $\|T_i\| \leq 1$, of the map which sends $a + N_2$ into $a + N_2{}^i$, $a \in L_2$. The image $T_i(H_f)$ contains $L_2 - N_2{}^i$. Since $f$ is densely defined, $L_2$ is dense in $A$, and since $f_i$ is continuous, $L_2 - N_2{}^i$ is dense in $A - N_2{}^i$. It follows that if $T_i = U_i |T_i|$ is the polar decomposition of $T_i$, then $U_i U_i{}^*$ is the identity operator on $H_{fi}$. Now it is easy to see that $T_i \pi_f = \pi_{fi} T_i$, that is, $T_i$ is a coupling between $\pi_f$ and $\pi_{fi}$. Then also $U_i$ and $U_i{}^*$ are couplings. Since $f_i$ is bounded, there is a cyclic vector $\eta_i \in H_{fi}$ for the representation $\pi_{fi}$. We define $\xi_i = U_i{}^* \eta_i \in H_f$ and have for all $a \in A$

$$\begin{aligned}(\pi_f(a)\xi_i \mid \xi_i) &= (\pi_f(a) U_i{}^* \eta_i \mid U_i{}^* \eta_i) \\ &= (\pi_{fi}(a) U_i U_i{}^* \eta_i \mid \eta_i) = (\pi_{fi}(a) \eta_i \mid \eta_i) = f_i(a) .\end{aligned}$$

The theorem follows.

One might have hoped that in analogy with the commutative case $a \in L_1$ if and only if $|a| \in L_1{}^+$, but in general this is false. If $A$ is the $C^*$ algebra in the third example following theorem 1.4 and if $a$ is chosen so that $a$ is trace class but $a^{\frac{1}{2}}$ is not, then let $f$ be the extended positive functional on $A$ whose value at a matrix is the trace of the operator in the upper left corner. The functional $f$ is densely defined and can be obtained as a sum of vector states. We have $f(p) = \operatorname{tr}(a) < \infty$ and $f(q) = 0$ so that $p - q \in L_1$. However an easy computation shows that

$$(p-q)^2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad \text{hence} \qquad |p-q| = \begin{pmatrix} a^{\frac{1}{2}} & 0 \\ 0 & a^{\frac{1}{2}} \end{pmatrix}.$$

It follows that $f(|p-q|) = \operatorname{tr}(a^{\frac{1}{2}}) = \infty$.

We close this section with a theorem which states that any extended positive functional can be weakly approximated by a bounded one. Let $E$ denote the set of all extended positive functionals on $A$. We give $E$ the weakest topology which turns all $a \in A^+$ into continuous functions from $E$ to $[0, \infty]$. The restriction of this topology to the set $F$ of all bounded members of $E$ clearly gives $F$ the usual weak* topology and we have

THEOREM 2.5. *$E$ is a compact Hausdorff space with $F$ as a dense subset.*

PROOF. Clearly $E$ is a Hausdorff space. To prove compactness let $\{f_i \mid i \in I\}$ be a universal net in $E$. For any $a \in A^+$ the net $\{f_i(a) \mid i \in I\}$ is universal in $[0, \infty]$ and hence convergent. It is easily checked that the definition $f(a) = \lim f_i(a)$ gives an extended positive functional on $A$, such that $f_i$ converges to $f$. Since we have shown that any universal net is convergent in $E$, the space is compact.

In order to prove that $F$ is dense in $E$ we must, for each $f \in E$ and each finite set $\{a_n\} \subset A^+$ and $\varepsilon > 0$, find $g \in F$ such that

$$\begin{aligned} |f(a_n) - g(a_n)| &< \varepsilon \quad \text{for} \quad f(a_n) < \infty , \\ g(a_n) &> \varepsilon^{-1} \quad \text{for} \quad f(a_n) = \infty . \end{aligned}$$

Let $V$ be the finite dimensional real subspace of $A^R$, the self-adjoint part of $A$, spanned by the $a_n$ for which $f(a_n) < \infty$. The spaces $V$ and $A^R$ are both ordered vector spaces and so are their duals $V'$ and $A^{R'}$. If $\Phi$ is the injection of $V$ in $A^R$, then the dual map $\Phi'$ is order preserving and we have $\Phi'(F) \subset V'^+$. If $W$ is the closure of $\Phi'(F)$ in $V'$, then $W$ is a cone, and if $x \in V'^+ \setminus W$ there exists ([4, 1.3.8]) an element $a \in V''$ such that $a(W) \subset R^+$ but $a(x) < 0$. Since $V$ is finite dimensional, $V$ is also reflexive, so $a \in V$. We have

$$\Phi(a)(F) = a(\Phi'(F)) \subset a(W) \subset R^+$$

and hence $a \geqq 0$ in contradiction to $x(a) < 0$ and $x \in V'^+$. It follows that the restriction of $f$ to $V$ which is an element of $V'^+$ can be approximated by a $g_1 \in F$.

Setting $a_0 = \sum a_n$, $a_n \in V$, we can find $g_2 \in F$ such that $g_2(a_0) < \varepsilon$ and $g_2(a_n) > \varepsilon^{-1}$ for each $a_n \notin V$. Otherwise there would exist a constant $\alpha$ such that for an $a_n \notin V$

$$g(a_n) \leqq \alpha g(a_0) \quad \text{for all} \quad g \in F .$$

But that would imply $a_n \leqq \alpha a_0$, a contradiction.

As our approximating element we can now take $g = g_1 + g_2$.

## 3. $C^*$ integrals.

At first glance the natural non-commutative generalization of a regular Borel measure on a locally compact Hausdorff space would be any positive functional on the operators with majorized supports. However in order to get interesting theorems we need a condition (vacuous in the commutative case) which secures that operators which are unitarily equivalent in the algebra have nearly the same integral. To be more precise:

A linear functional $f$ on $K$ is called *unitarily bounded* if for all $a \in K$

$$\sup \{ |f(u^* a u)| \mid u \text{ unitary in } \tilde{A} \} < \infty .$$

We define a **$C^*$ *integral*** on $A$ to be any unitarily bounded positive functional on $K$.

A $C^*$ integral is called *invariant* if it is invariant under unitary transformations from $\tilde{A}$, that is, if it is the restriction of a trace on $A$ to $K$.

In view of lemma 2.2 and the fact that any operator in $A$ can be written as a linear combination of at most four unitary operators from $\tilde{A}$ we see that the term unitarily bounded is equivalent to the condition that, for any $a \in K$, $|f(b^*ac)|$ is bounded for $\|b\| \leqq 1$, $\|c\| \leqq 1$, $b, c \in A$. Clearly the $C^*$ integrals on $A$ form a cone in the algebraic dual of $K$, and any positive functional on $K$ majorized by a $C^*$ integral is itself a $C^*$ integral.

THEOREM 3.1. *Any $C^*$ integral $f$ can be written as a sum of continuous positive functionals.*

PROOF. For any $a \in \tilde{A}^+$ define

$$\varrho(a) = \inf\{f(s)+t \mid s \in K^+,\ t \in R^+,\ s+t \geqq a\}.$$

It is easily checked that the function $\varrho \colon \tilde{A}^+ \to R^+$ has the properties:

(1) $\varrho(\alpha a) = \alpha \varrho(a)$ for $\alpha \geqq 0$,
(2) $\varrho(a+b) \leqq \varrho(a) + \varrho(b)$,
(3) $\varrho(a) \leqq \varrho(b)$ for $a \leqq b$,
(4) $\varrho(a) \leqq \|a\|$,
(5) $\varrho(a) \leqq f(a)$ for $a \in K^+$.

Also $\varrho$ is the greatest function satisfying (1)–(5). Furthermore if $a \in \tilde{A}^+$ and $\alpha \geqq 0$, a majorization $s+t \geqq a+\alpha$, $s \in K^+$, $t \in R^+$, is possible only if $t \geqq \alpha$ in which case we have

$$f(s)+t = f(s)+(t-\alpha)+\alpha$$

and thus

(6) $\varrho(a+\alpha) = \varrho(a)+\alpha$.

Now suppose $\varrho(a)=0$, $a \in K^+$. Then there exist $s_n \in K^+$, $t_n \in R^+$, such that $s_n + t_n \geqq a$, $f(s_n) \leqq n^{-2}$ and $t_n \leqq n^{-1}$. Define $v_n = (n^{-1}+s_n)^{-1}s_n$. We have $0 \leqq v_n \leqq 1$ and $v_n \leqq n s_n$. Since the function $\varphi(\alpha) = (n^{-1}+\alpha)^{-2}\alpha$ has maximum $\frac{1}{4}n$ for $\alpha = n^{-1}$, we have

$$\begin{aligned}(1-v_n)a(1-v_n) &\leqq (1-v_n)s_n(1-v_n) + (1-v_n)t_n(1-v_n)\\ &\leqq n^{-2}(n^{-1}+s_n)^{-2}s_n + n^{-1} \leqq \tfrac{5}{4}n^{-1}\end{aligned}$$

so that

$$\|a^{\frac{1}{2}}(1-v_n)\|^2 \leqq \tfrac{5}{4}n^{-1}.$$

We also have

$$\begin{aligned}|f(a^2)-f(a^2v_n)|^2 &= |f(aa(1-v_n))|^2\\ &\leqq f(a^2)f((1-v_n)a^{\frac{1}{2}}aa^{\frac{1}{2}}(1-v_n)).\end{aligned}$$

Since $f$ is unitarily bounded we conclude that

$$|f(a^2)-f(a^2v_n)|^2 \to 0 .$$

But

$$|f(a^2v_n)|^2 \leqq f(a^4)f(v_n{}^2) \leqq f(a^4)f(v_n) \leqq nf(a^4)f(s_n) \leqq n^{-1}f(a^4),$$

and thus we have

(7) $\varrho(a)=0$ implies $f(a^2)=0$ for $a \in K^+$.

We now extend $\varrho$ to $\tilde{A}^R$ by the definition

$$\tilde{\varrho}(a) = \inf\{\varrho(b+c) \mid b,c \in \tilde{A}^+, b-c=a\} .$$

From $a=b-c \in \tilde{A}^+$ it follows that $a \leqq b+c$, and thus $\tilde{\varrho}(a)=\varrho(a)$ by (3). It is easy to see that $\tilde{\varrho}$ satisfies

$$\tilde{\varrho}(a+b) \leqq \tilde{\varrho}(a)+\tilde{\varrho}(b) ,$$
$$\tilde{\varrho}(\alpha a) = |\alpha|\tilde{\varrho}(a) \quad \text{for } \alpha \in R .$$

Thus $\tilde{\varrho}$ is a symmetric convex functional on $\tilde{A}^R$.

If $f$ is different from zero on $K$, then since $K_0{}^{+2}=K_0{}^+$, by (7) we can find $a \in K^+$ such that $\varrho(a) \neq 0$. We define a linear functional $g$ on the subspace of $\tilde{A}^R$ spanned by $a$ and 1 by

$$g(\alpha a+\beta) = \alpha\varrho(a)+\beta .$$

For $\beta<0$ we have

$$\tilde{\varrho}(a+\beta) \geqq \tilde{\varrho}(a)-\tilde{\varrho}(\beta) = \varrho(a)-|\beta| = g(a+\beta) = \tilde{\varrho}(a+\beta-\beta)-|\beta| \geqq -\varrho(a+\beta),$$

and for $\beta \geqq 0$ we have by (6)

$$g(a+\beta) = \varrho(a)+\beta = \varrho(a+\beta) .$$

Thus $|g(b)| \leqq \tilde{\varrho}(b)$ for all $b$ in the subspace, and by the Hahn–Banach theorem $g$ has an extension $f_1$, a linear functional on $\tilde{A}^R$, which satisfies

$$|f_1(b)| \leqq \tilde{\varrho}(b) \quad \text{for} \quad \text{all } b \in \tilde{A}^R .$$

Since by the definition of $\tilde{\varrho}$ and (3) and (6) we have

$$\tilde{\varrho}(b) \leqq \varrho(b_+ + b_-) = \varrho(|b|) \leqq \|b\|\varrho(1) = \|b\|$$

and since $f_1(1)=1$, we conclude that $f_1$ is a positive functional on $\tilde{A}$ and non-zero on $A$. For any $b \in K^+$ we have by (5)

$$f_1(b) \leqq \varrho(b) \leqq f(b) .$$

Thus the restriction of $f-f_1$ to $K$ is a $C^*$ integral. By a transfinite repetition of this procedure we get a set of bounded positive functionals on $A$ whose sum is $f$.

COROLLARY 3.2. *Any $C^*$ integral has a unique extension to a densely defined, lower semi-continuous extended positive functional on $A$.*

PROOF. By theorem 3.1 it follows that $f=\sum f_i$ on $K$, and since the functions $f_i$ are everywhere defined, the definition $\tilde{f}=\sum f_i$ clearly extends $f$ to $A^+$ and defines a lower semi-continuous extended positive functional. Since for all $a \in A^+$ we can find a sequence $\{a_n\} \subset K_0{}^+$ converging to $a$ from below, all lower semicontinuous extensions of $f$ coincide.

COROLLARY 3.3. *There is a one-to-one correspondence between densely defined lower semi-continuous traces on $A$ and invariant $C^*$ integrals on $A$.*

PROOF. If $f$ is an invariant $C^*$ integral, then by corollary 3.2 it has a unique extension as a densely defined lower semi-continuous trace. If on the other hand $f$ is such a trace, then its ideal of definition is a dense, two-sided, order-related ideal and hence by theorem 1.3 contains $K$, so that the restriction $f|K$ is an invariant $C^*$ integral. Since $f$ is a lower semi-continuous extension of $f|K$, it is the only one and the correspondence is one-to-one.

COROLLARY 3.4. *Any $C^*$ integral $f$ has an extension as a normal extended positive functional on $A''$.*

PROOF. If $f=\sum f_i$ each $f_i$ has a normal extension $\tilde{f}_i$ to $A''$, and the net consisting of finite sums of the functions $\tilde{f}_i$ clearly converges weakly to a normal extended positive functional on $A''$.

COROLLARY 3.5. *Any $C^*$ integral is a sum of vector functionals associated with its own representation.*

PROOF. Combine theorem 3.1 and theorem 2.4.

Let $P_A$ denote the weak* closure of the pure states of $A$ minus $\{0\}$. Then $P_A$ is a locally compact Hausdorff space and any positive functional on $A$ can be represented as a regular finite Borel measure on $P_A$. (For better results see [1]).

COROLLARY 3.6. *Any $C^*$ integral $f$ can be represented as a regular Borel measure on $P_A$.*

PROOF. $A^R$ admits an isometric linear injection in $C_0(P_A)$ and if $f=\sum f_i$, then by the Hahn–Banach theorem each $f_i$ can be extended to $\tilde{f}_i$ on $C_0(P_A)$.

For any $x \in P_A$ there exists an $a \in K^+$ such that $x(a) \neq 0$ and consequently $\{y \in P_A \mid y(a) > \frac{1}{2}x(a)\}$ and $\{y \in P_A \mid y(a) \geqq \frac{1}{2}x(a)\}$ are an open and a compact neighbourhood of $x$, respectively. It follows that any compact set in $P_A$ is contained in a compact set of the form

$$C = \{x \in P_A \mid x(a) \geqq 1\}, \qquad a \in K^+ .$$

If $\varphi \in C_0(P_A)^+$ has support in $C$, we have $\varphi \leqq \|\varphi\| a$ and thus

$$\sum \tilde{f}_i(\varphi) \leqq \|\varphi\| f(a) < \infty .$$

Consequently the functional $\sum \tilde{f}_i$ represents a regular Borel measure $\mu$ on $P_A$ and we have

$$f(a) = \int_{P_A} a(x)\, d\mu(x) \qquad \text{for all } a \in K .$$

Since all the pleasant properties of $C^*$ integrals are derived from theorem 3.1, it is interesting to know that the converse of this theorem also holds, so that we might have used the condition $f = \sum f_i$ as a definition of a $C^*$ integral. In fact we have a slightly sharper statement than the converse of theorem 3.1.

THEOREM 3.7. *Let $f$ be a positive functional on $K$ and suppose there is a net $\{f_i \mid i \in I\}$ of positive functionals on $A$ converging weakly to $f$ and such that $f_i \leqq f$ for all $i \in I$. Then $f$ is a $C^*$ integral.*

PROOF. To obtain a contradiction suppose that $f$ is not unitarily bounded at $a \in K^+$. We can then find $b \in A$ with arbitrary small norm such that $f(b^* a b)$ is arbitrarily great. Assume that for all numbers $i \leqq n$ we have chosen $b_i \in A$ and $f_i$ from the net such that, with the notations

$$c_i = \sum_{j=1}^{i} b_j \quad \text{and} \quad \alpha_i = \max\{\|f_j\|^{\frac{1}{2}} \mid j \leqq i\},$$

we have

(1) $\|b_i\| \leqq 2^{-i} \alpha_{i-1}^{-1}$ (for convenience $\alpha_0 = 1$),
(2) $f(c_i^* a c_i) \geqq i^2$,
(3) $f_i(c_i^* a c_i) \geqq \frac{1}{4} f(c_i^* a c_i)$.

Then since in general we have the inequality

$$f((b+c)^* a (b+c)) \geqq \left((f(b^* a b))^{\frac{1}{2}} - (f(c^* a c))^{\frac{1}{2}}\right)^2 ,$$

we can find $b_{n+1}$ satisfying (1) such that $c_{n+1} = b_{n+1} + c_n$ satisfies (2). Since the net $\{f_i \mid i \in I\}$ converges weakly to $f$, we can then find $f_{n+1}$ satisfying (3).

Now define $c=\sum b_n$. We have $c^*ac \in K^+$ but

$$\begin{aligned} f(c^*ac) \geqq f_n(c^*ac) &= f_n((c_n+c-c_n)^*a(c_n+c-c_n)) \\ &\geqq \Big((f_n(c_n{}^*ac_n))^{\frac{1}{2}} - \|f_n\|^{\frac{1}{2}} \sum_{i=n+1}^{\infty} \|b_i\|\Big)^2 \\ &\geqq \Big(\tfrac{1}{2}(f(c_n{}^*ac_n))^{\frac{1}{2}} - \sum_{i=n+1}^{\infty} 2^{-i}\Big)^2 \geqq (\tfrac{1}{2}n-2^{-n})^2 . \end{aligned}$$

This is a contradiction and shows that $f$ is indeed unitarily bounded.

We now turn our attention to the $C^*$ algebra $B_0(H)$, the set of the compact operators on $H$. As mentioned before $K$ is then the set of the operators of finite rank. Since the trace is a faithful invariant $C^*$ integral on $B_0(H)$, any other $C^*$ integral $f$ is absolutely continuous (in fact bounded) with respect to tr. We show that $f$ admits a Radon–Nikodym derivative with respect to tr which is a bounded operator on $H$ so that we may regard $B(H)$ as a generalization of an $L_1$ (loc.).

THEOREM 3.8. *There is a one-to-one correspondence between $C^*$ integrals on $B_0(H)$ and $B(H)^+$. If $b \in B(H)^+$, then the corresponding $C^*$ integral $f$ is given by $f(a)=\operatorname{tr}(ba)$ for all $a \in K$.*

PROOF. If $\xi,\eta \in H$, let $(\xi\otimes\eta)$ denote the operator in $K$ such that $(\xi\otimes\eta)\psi=(\psi\,|\,\eta)\xi$ for all $\psi \in H$. Let $f=\sum f_i$ be a $C^*$ integral on $B_0(H)$. With each $f_i$ is associated a unique positive operator $b_i$ of trace class such that $f_i(a)=\operatorname{tr}(b_ia)$ for all $a \in B_0(H)$. In particular $f_i(\xi\otimes\eta)=(b_i\xi\,|\,\eta)$. For each finite set $\{b_i\}$ we have

$$\begin{aligned} \|\sum b_i\| &= \sup \sum (b_i\xi\,|\,\xi)\ |\ \|\xi\| \leqq 1,\ \xi \in H\} \\ &= \sup\{\sum f_i(\xi\otimes\xi)\ |\ \|\xi\| \leqq 1,\ \xi \in H\} \\ &\leqq \sup\{f(\xi\otimes\xi)\ |\ \|\xi\| = 1,\ \xi \in H\} \leqq \alpha < \infty \end{aligned}$$

because $f$ is bounded when $(\xi\otimes\xi)$ runs through the one-dimensional projections. It follows that the net of finite sums of operators $b_i$ converges strongly to a positive operator $b \in B(H)$. We have

$$f(a) = \sum \operatorname{tr}(b_ia) = \operatorname{tr}(ba)$$

for all $a$ with finite rank.

Conversely if $b \in B(H)^+$, the definition $f(a)=\operatorname{tr}(ba)$ gives a positive functional on $K$ and since $|f(a)| \leqq \|b\| \operatorname{tr}(|a|)$ for $a \in K$, we have $f \leqq \|b\|$ tr, and hence $f$ is a $C^*$ integral.

The following example shows that in order to get well behaved positive functionals on $K$ the condition of unitary boundedness is necessary.

If $\{\xi_i \mid i \in I\}$ is an algebraic basis for the Hilbert space $H$ consisting of unit vectors, then clearly the set $\{(\xi_i \otimes \xi_j) \mid i,j \in I \times I\}$ is a set of generators for $K$ in $B_0(H)$. If $H$ is infinite dimensional we may refine this set of generators to a basis for $K$ with infinitely many projections $(\xi_i \otimes \xi_i)$ and if $a \geqq 0$, then $\alpha_{ii} \geqq 0$. If namely $\eta$ is a vector orthogonal to the (finite dimensional) space spanned by the vectors $\xi_j$, $j \neq i$, in the expression for $a$, but $(\xi \mid \xi_i) \neq 0$ (all $\xi_j$ are independent), then

$$(a\eta \mid \eta) = \alpha_{ii} |(\eta \mid \xi_i)|^2 \geqq 0 .$$

Let $(\xi_n \otimes \xi_n)$ be a sequence of projections in the basis and define

$$f(a) = \sum n \alpha_{nn} \qquad \text{for} \qquad a = \sum \alpha_{ij} (\xi_i \otimes \xi_j) .$$

Clearly $f$ is a linear, positive functional on the operators of finite rank, but $f$ is not unitarily bounded and thus is not a $C^*$ integral.

As mentioned in the introduction, this paper is an attempt to extend measure theory to non-commutative systems in terms of integrals of functions. There is also the interesting problem whether it is possible to give a treatment in the set-theoretical mood. In such a treatment the non-commutative equivalent to the measurable sets would be the projections in $A''$, the compact sets would be the range projections of elements in $A$ with majorized supports and a measure would be a positive function on the projections, completely additive on orthogonal projections and finite and unitarily bounded on majorized projections. It should be noted that the example following theorem 2.4 shows that a $C^*$ integral need not be subadditive on projections.

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