MATH. SCAND. 19 (1966), 151–160

EXISTENCE AND PROPERTIES OF RIESZ POTENTIALS SATISFYING LIPSCHITZ CONDITIONS

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1. Introduction.

Let F be a compact set in the m-dimensional Euclidean space. According to a well-known result there exists a non-trivial, positive measure supported by F and having a bounded Riesz potential of order α, $0<\alpha<m$ (that is, a bounded potential formed with the kernel $r^{-(m-\alpha)}$; for definitions, compare section 2), if and only if F has α-capacity larger than zero. The main purpose of this note is to investigate the corresponding problem with the bounded potential replaced by a potential of order α belonging to the Lipschitz class of a given order β, $0<\beta<1$. It turns out that the right condition on F is that F shall have positive Hausdorff measure of order $m+\beta-\alpha$, if $\beta<\alpha$. (See Theorem 2. The theorems are stated in section 2.) This result is obtained as a consequence of two theorems, the Theorems A and B, by Carleson and Frostman, respectively, and a new theorem, Theorem 1, which is a converse of Theorem A and gives a kind of Lipschitz condition on those measures which have a potential of a given order α in a given Lipschitz class β. The Theorems 1 and A essentially solve also the following problem (see Theorem 3): Assume that μ is a positive measure with compact support such that the Riesz potential of order α_1 of μ belongs to the Lipschitz class of order β_1 for certain given values of α_1 and β_1. For which values of α_2 and β_2 is it then true that the Riesz potential of order α_2 of the same measure μ belongs to the Lipschitz class of order β_2?

To prove Theorem 1 we use some inversion formulas giving the measure μ in terms of the Riesz potential of μ. These inversion formulas are given in section 3. In section 4 we prove Theorem 1.

2. Statement of the theorems.

In the sequel μ denotes a positive measure, that is, a non-negative, completely additive set function defined at least for Borel sets. Further-

Received August 3, 1966.

more, the support of μ, suppμ, shall be compact. If $r>0$ and $x=(x_1,\ldots,x_m)$ is a point in the Euclidean space R^m, $m\geqq 1$, then $\mu(r,x)$ denotes the value of μ for the closed sphere $\{y \mid |y-x|\leqq r\}$. Here $|y-x|$ is the Euclidean distance between y and x in R^m.

Let α denote a number satisfying $0<\alpha<m$. We denote by u_α^μ *the Riesz potential of order* α *of* μ, *also called the* α*-potential of* μ, defined by

$$u_\alpha^\mu(x) = \int \frac{d\mu(y)}{|x-y|^{m-\alpha}}.$$

Here and elsewhere, the integration is extended over the whole space R^m, if no limits of integration are indicated.

Let E be any set. *The Hausdorff measure of order* α *of* $E, \Lambda_\alpha(E)$, is

$$\Lambda_\alpha(E) = \lim_{\varepsilon\to 0} \Lambda_\alpha^{(\varepsilon)}(E), \qquad \varepsilon>0,$$

with

$$\Lambda_\alpha^{(\varepsilon)}(E) = \inf \sum_\nu r_\nu^\alpha ,$$

where the infimum is taken over all coverings of E by denumerably many spheres with radii $r_\nu\leqq\varepsilon$.

For $0\leqq\beta\leqq 1$ we denote by Lipβ the class of all functions f satisfying a *Lipschitz condition of order* β *in* R^m, that is, such that for a certain constant, *the Lipschitz constant,*

$$|f(x)-f(y)| \leqq \text{const.}\,|x-y|^\beta \qquad \text{for all } x,y\in R^m .$$

We shall prove

THEOREM 1. *Let* α *and* β *be any numbers such that* $0<\alpha<m$ *and* $0\leqq\beta\leqq 1$. *Assume that* μ *is a positive measure with compact support such that* $u_\alpha^\mu\in$ Lipβ. *Then there is a constant depending only on* α, β, m *and* u_α^μ *such that for all* $x\in R^m$ *and all* $r>0$

$$\mu(x,r) \leqq \text{const.}\,r^{m+\beta-\alpha} \text{ if } \alpha>\beta , \tag{2.1}$$

$$\mu(x,r) \leqq \text{const.}\,r^{m}\log 1/r \text{ if } \alpha=\beta , \tag{2.2}$$

$$\mu(x,r) \leqq \text{const.}\,r^{m} \text{ if } \alpha<\beta . \tag{2.3}$$

This theorem which will be proved in section 4, is a converse of the following result by Carleson.

THEOREM A (Carleson). *Let* α *and* β *be any numbers such that* $0<\alpha<m$ *and* $0<\beta<1$. *Assume that* μ *is a positive measure with compact support such that for some constant*

$$\mu(x,r) \leqq \text{const.}\, r^{m+\beta-\alpha} \qquad \textit{for all } x \in R^m \textit{ and all } r > 0 \, .$$

Then $u_\alpha^\mu \in \mathrm{Lip}\,\beta$ *with a Lipschitz constant only depending on* α, β, m *and* μ.

A proof of Theorem A is found in [1, pp. 15–16] for a special choice of α. The same method of proof, however, is applicable with obvious modifications for a general value of α.

The above theorems have a close connection with the problem (essentially solved by Theorem 2 below) concerning the existence of a non-trivial positive measure supported by a given compact set and having a Riesz potential of a given order belonging to a given Lipschitz class. To obtain this connection we use the following result by Frostman (see [3, pp. 87–89] or [2, pp. 5–6]).

THEOREM B (Frostman). *Let* $0<\gamma<m$. *Let* F *be a compact set in* R^m. *Then* $\Lambda_\gamma(F)>0$ *if and only if there exists a positive measure* μ *with* $\mathrm{supp}\,\mu \subset F$ *and* $\mu(F)>0$ *such that*

$$\mu(x,r) \leqq r^\gamma \qquad \textit{for all } x \in R^m \textit{ and all } r>0 \, .$$

The Theorems 1, A and B immediately give the following theorem:

THEOREM 2. *Let* α *and* β *be any fixed numbers such that* $0<\alpha<m$, $0<\beta<1$ *and* $\alpha>\beta$. *Let* F *be a compact set. A necessary and sufficient condition for the existence of a positive measure* μ *with*

$$\mathrm{supp}\,\mu \subset F \quad \textit{and} \quad \mu(F) > 0 \quad \textit{such that} \quad u_\alpha^\mu \in \mathrm{Lip}\,\beta \, ,$$

is that

$$\Lambda_{m+\beta-\alpha}(F) > 0 \, .$$

As an immediate consequence of the Theorems 1 and A we clearly also obtain

THEOREM 3. *Let* α_i *and* β_i, $i=1,2$, *be any numbers such that* $0<\alpha_i<m$, $0<\beta_i<1$, $\alpha_i>\beta_i$ *for* $i=1$ *and* $i=2$, *and, finally,*

$$\alpha_1-\alpha_2 = \beta_1-\beta_2 \, .$$

Let μ *be a positive measure with compact support. Then*

$$u_{\alpha_1}^\mu \in \mathrm{Lip}\,\beta_1 \qquad \textit{if and only if} \qquad u_{\alpha_2}^\mu \in \mathrm{Lip}\,\beta_2 \, .$$

3. Some inversion formulas.

Let C_0^∞ be the class of complex functions which are infinitely differentiable and have compact supports. μ denotes as usual a positive measure with compact support and α a number satisfying $0<\alpha<m$. Some refer-

ences and formulas connected with the inversion formulas in this section are found in [4, pp. 74–77].

We shall use the Fourier transformation. Let $\hat{T} = \mathscr{F}T$ be the Fourier transform of a tempered distribution T normed so that

$$\hat{f}(\xi) = \int e^{-2\pi i(x,\xi)} f(x)\,dx, \qquad (x,\xi) = \sum_1^m x_i \xi_i\,,$$

if f is in the Lebesgue class $L^1(R^m)$. Since

$$\mathscr{F}|x|^{-(m-\alpha)} = A_1(\alpha,m)\,|x|^{-\alpha}, \quad A_1(\alpha,m) = \frac{\pi^{\frac{1}{2}m-\alpha}\cdot\Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}(m-\alpha))},$$

the Fourier transform of the convolution u_α^μ of $|x|^{-(m-\alpha)}$ and μ is

$$\hat{u}_\alpha^\mu = A_1(\alpha,m)\,|x|^{-\alpha}\,\hat{\mu}\,. \tag{3.1}$$

We start by stating an inversion formula giving the measure μ in terms of the α-potential of μ, u_α^μ, in the particularly simple case when α is an even integer, $\alpha = 2h$.

If $\varphi \in C_0^\infty$ and h is a natural number such that $0 < 2h < m$, then

$$\int \varphi(x)\,d\mu(x) = A_2\cdot\int u_{2h}^\mu(x)\,\Delta^h\varphi(x)\,dx\,, \tag{3.2}$$

where

$$A_2 = \{(-4\pi^2)^h\,A_1(2h,m)\}^{-1}\,,$$

$A_1(\alpha,m)$ is the constant in (3.1), and Δ^h is the Laplace operator iterated h times.

To prove (3.2) we use the definition of the Fourier transform of the tempered distribution u_{2h}^μ combined with (3.1) and the formula

$$\mathscr{F}(\Delta^h\varphi)(\xi) = (-4\pi^2)^h\,|\xi|^{2h}\,\hat{\varphi}(\xi)\,. \tag{3.3}$$

This gives

$$\int u_{2h}^\mu(x)\,\Delta^h\overline{\varphi(x)}\,dx = (-4\pi^2)^h\int \hat{u}_{2h}^\mu(\xi)\,|\xi|^{2h}\,\overline{\hat{\varphi}(\xi)}\,d\xi = A_2^{-1}\int \hat{\mu}(\xi)\,\overline{\hat{\varphi}(\xi)}\,d\xi\,,$$

which proves (3.2), since

$$\int \overline{\varphi(x)}\,d\mu(x) = \int \overline{\hat{\varphi}(\xi)}\,\hat{\mu}(\xi)\,d\xi\,.$$

We obtain an analogue of (3.2) for a given Riesz potential of order α with an arbitrary α, $0 < \alpha < m$, and $m > 1$ if in the right member of (3.2) we replace $\Delta^h\varphi(x)$ by a certain Riesz potential generated by a measure of the form $\Delta^k\varphi(t)dt$. In fact, let

$$v(x) = \int \frac{\Delta^k \varphi(t)}{|x-t|^\gamma} dt ,$$

where $\varphi \in C_0^\infty$, *k is a natural number, and* $0<\gamma<m$. *Assume that* $0<\alpha<m$ *and* $m>1$ *and let k and* γ *be chosen so that*

$$m+\alpha = 2k+\gamma . \tag{3.4}$$

Then

$$\int \varphi(x)\, d\mu(x) = A_3 \int u_\alpha^\mu(x)\, v(x)\, dx , \tag{3.5}$$

where $A_3 = \{(-4\pi^2)^k A_1(m-\gamma,m)\, A_1(\alpha,m)\}^{-1}$, *and* $A_1(\alpha,m)$ *is the constant in* (3.1).

We observe that as $m>1$ it is clearly possible for any α to choose k and γ as indicated. We also observe that the inversion formula (3.2) is only a special case of (3.5) that is, the case when $\alpha=2h$, $k=h+1$ and $\gamma=m-2$, because in the case $v(x)=\text{const.}\Delta^h\varphi(x)$ which is easily proved. Now we turn to the proof of (3.5). A *formal* application of the Parseval relation leads to

$$\int u_\alpha^\mu(x)\, \overline{v(x)}\, dx = \int \hat{u}_\alpha^\mu(\xi)\, \overline{\hat{v}(\xi)}\, d\xi . \tag{3.6}$$

Using (3.1) and the analogous formula for v, that is,

$$\hat{v}(\xi) = A_4\, |\xi|^{2k-m+\gamma}\, \hat{\varphi}(\xi) ,$$

where

$$A_4 = A_1(m-\gamma,m)\,(-4\pi^2)^k ,$$

we would get (3.5) from (3.6) and (3.4). To prove (3.6) we introduce the auxiliary function χ_ε defined by

$$\chi_\varepsilon(x) = \exp\{-\varepsilon|x|^2\}, \qquad \varepsilon>0 ,$$

with Fourier transform

$$\hat{\chi}_\varepsilon(\xi) = (\pi/\varepsilon)^{\frac{1}{2}m} \exp(-\pi^2|\xi|^2/\varepsilon) .$$

Since v is infinitely differentiable and all the derivatives of v are bounded, $v\chi_\varepsilon$ belongs to Schwartz's class of infinitely differentiable rapidly decreasing functions. In view of this, we can now use the definition of the Fourier transform of the tempered distribution u_α^μ to conclude

$$\int u_\alpha^\mu(x)\, \overline{v(x)}\, \chi_\varepsilon(x)\, dx = \int \hat{u}_\alpha^\mu(\xi)\, (\bar{\hat{v}} * \hat{\chi}_\varepsilon)(\xi)\, d\xi . \tag{3.7}$$

The left member of (3.7) clearly tends to the left member of (3.6) when $\varepsilon \to 0$, since $u_\alpha^\mu v$ is in the Lebesgue class $L^1(R^m)$ [compare the formulas (4.3) and (4.5)]. By using the facts that $\hat{u}_\alpha^\mu$ is bounded outside every neighborhood of the origin and locally Lebesgue integrable, and that $(\hat{v}*\hat{\chi}_\varepsilon)(\xi) \to \hat{v}(\xi)$ when $\varepsilon \to 0$, it may be proved by straightforward calculations that the right member of (3.7) tends to the right member of (3.6) when $\varepsilon \to 0$. We omit the details. The proof of (3.6) is complete.

Finally, we give another kind of inversion formula for μ which is valid also for $m=1$.

Assume that α is not an even integer. Let h be the non-negative integer such that $2h < \alpha < 2h+2$. If $\varphi \in C_0^\infty$, then

$$(3.8)\quad \int \varphi(x)\, d\mu(x) = A_5 \int \frac{1}{|y|^{m+\alpha-2h}} \left\{ \int (u_\alpha^\mu(x+y) - u_\alpha^\mu(x))(\Delta^h\varphi(x+y) - \Delta^h\varphi(x))\, dx \right\} dy\,,$$

where A_5 is a constant depending only on m, α and h and where the outer integral in the right member of (3.8), *as well as the inner integral, is absolutely convergent.*

(3.8) was proved in [4, pp. 76–77] in a little different form for the case $h=0$. Using (3.3) the proof proceeds along the same lines for a general h including the explicit calculation of the constant A_5.

4. Proof of Theorem 1.

We shall use the inversion formulas (3.2) and (3.5) to prove Theorem 1 when $m > 1$. When $m=1$ the formula (3.8) gives the simplest proof. However, we shall sketch a proof of Theorem 1 by means of (3.8) for any m and any α which is not an even integer.

Let μ be a positive measure with compact support such that $u_\alpha^\mu \in \operatorname{Lip}\beta$ where $0 < \alpha < m$ and $0 \leqq \beta \leqq 1$. We shall prove that the inequalities (2.1), (2.2) and (2.3), respectively, hold for $x=0$ with certain constants and it will appear from the proofs that the inequalities hold with the same constants for all $x \in R^m$.

Let ψ be a fixed function in C_0^∞ such that $0 \leqq \psi(x) \leqq 1$ for all $x, \psi(x)=1$ for $|x| \leqq 1$ and $\psi(x)=0$ for $|x| \geqq 2$. We define the function ψ_δ by $\psi_\delta(x) = \psi(x/\delta)$ for all $x \in R^m$ and all $\delta > 0$. Then $\psi_\delta \in C_0^\infty$ and $0 \leqq \psi_\delta(x) \leqq 1$ for all $x, \psi_\delta(x)=1$ for $|x| \leqq \delta$ and $\psi_\delta(x)=0$ for $|x| \geqq 2\delta$. Furthermore, for any natural number s,

$$(4.1)\qquad |\Delta^s \psi_\delta(x)| \leqq \text{const.}\,\delta^{-2s} \qquad \text{for all } x \in R^m \text{ and all } \delta > 0\,,$$

where the constant depends only on ψ and s.

Using (3.2) we first prove Theorem 1 in the particularly simple case when $\alpha = 2h$, h a natural number, $0 < 2h < m$. As $h \geqq 1$ we have

$$\int \Delta^h \psi_\delta(x)\, dx = 0\,, \tag{4.2}$$

which is an immediate consequence for instance of Green's formula [compare also (4.6)]. If we use (3.2) with φ replaced by ψ_δ we get, due to the properties of ψ_δ and the fact that $u_{2h}^\mu \in \operatorname{Lip}\beta$,

$$\begin{aligned}\mu(0,\delta) &\leqq \int \psi_\delta(x)\, d\mu(x) = \text{const.} \int u_{2h}^\mu(x)\, \Delta^h\psi_\delta(x)\, dx \\ &= \text{const.} \int \left(u_{2h}^\mu(x) - u_{2h}^\mu(0)\right) \Delta^h \psi_\delta(x)\, dx \\ &\leqq \text{const.} \int\limits_{|x|\leqq 2\delta} \delta^\beta\, \delta^{-2h}\, dx = \text{const.}\, \delta^{m+\beta-2h}\,.\end{aligned}$$

This proves Theorem 1 for $\alpha = 2h$.

We now turn to the proof of Theorem 1 for a general α and the dimension $m > 1$ using the inversion formula (3.5). Let ψ_δ be the same function as above and define, for fixed numbers k and γ, k a natural number, $0 < \gamma < m$, v_δ, $\delta > 0$, by

$$v_\delta(x) = \int \frac{\Delta^k \psi_\delta(t)}{|x-t|^\gamma}\, dt, \qquad \text{where } 2k+\gamma = m+\alpha,\ m > 1\,. \tag{4.3}$$

We observe that the integration in (4.3) is an integration over $\{t \mid |t| \leqq 2\delta\}$ only, since $\psi_\delta(t) = 0$ for $|t| \geqq 2\delta$. We shall first prove the following inequalities:

$$|v_\delta(x)| \leqq \text{const.}\, \delta^{-\alpha} \qquad \text{for } x \in R^m \text{ and } \delta > 0\,, \tag{4.4}$$

and

$$|v_\delta(x)| \leqq \text{const.}\, \delta^m\, |x|^{-m-\alpha} \qquad \text{for } |x| \geqq 4\delta \text{ and } \delta > 0\,, \tag{4.5}$$

where the constants depend only on ψ, k, γ and m. (4.4) follows from the estimate

$$|v_\delta(x)| \leqq \text{const.} \int_0^{2\delta} \delta^{-2k}\, r^{m-1-\gamma}\, dr$$

and the relation $\alpha = 2k+\gamma-m$. To prove (4.5) we observe that as a consequence of Green's formula we have

$$v_\delta(x) = \text{const.} \int\limits_{|t|\leqq 2\delta} \frac{\psi_\delta(t)}{|x-t|^{\gamma+2k}}\, dt \qquad \text{for } |x| > 2\delta\,.$$

If we combine this formula with the inequality $|x-t| \geqq \frac{1}{2}|x|$ which is valid if $|x| \geqq 4\delta$ and $|t| \leqq 2\delta$ we get (4.5) since $m+\alpha = 2k+\gamma$.

Clearly v_δ is Lebesgueintegrable over the whole space R^m and since $\int v_\delta(x)\,dx = \hat{v}_\delta(0)$, the formula

$$\hat{v}_\delta(\xi) = \text{const.}\,|\xi|^\alpha\,\hat{\psi}_\delta(\xi)$$

proves the following analogue of (4.2):

$$\int v_\delta(x)\,dx = 0\,. \tag{4.6}$$

If we now use the formula (3.5) with φ and v replaced by ψ_δ and v_δ, respectively, we obtain, due to (4.6),

$$\begin{aligned}\mu(0,\delta) \leqq \int \psi_\delta(x)\,d\mu(x) &= \text{const.}\int u_\alpha^\mu(x)\,v_\delta(x)\,dx \\ &= \text{const.}\int \big(u_\alpha^\mu(x)-u_\alpha^\mu(0)\big)\,v_\delta(x)\,dx \\ &= \int\limits_{|x|\leqq 4\delta} + \int\limits_{|x|>4\delta} = \mathrm{I}+\mathrm{II}\,.\end{aligned}$$

The first integral, I, is, according to (4.4) and the assumption that $u_\alpha^\mu \in \operatorname{Lip}\beta$, majorized by

$$\text{const.}\,\delta^{\beta-\alpha}\int\limits_{|x|\leqq 4\delta} dx = \text{const.}\,\delta^{m+\beta-\alpha}\,.$$

The second integral, II, is estimated by means of (4.5). We consider first the case when $\alpha > \beta$ and obtain

$$|\mathrm{II}| \leqq \text{const.}\,\delta^m \int\limits_{|x|\geqq 4\delta} |x|^{-m-\alpha+\beta}\,dx = \text{const.}\,\delta^{m+\beta-\alpha}, \qquad \alpha > \beta\,.$$

Hence

$$\mu(0,\delta) \leqq \text{const.}\,\delta^{m+\beta-\alpha} \qquad \text{for } \alpha > \beta \text{ and } m > 1$$

with a constant which depends only on α, β, m and u_α^μ. We clearly get the formula (2.1) of Theorem 1 for $m > 1$ with the same constant for a general $x \in R^m$ if we repeat the calculations above with the function ψ replaced by the function ψ_x defined by $\psi_x(y) = \psi(y-x)$.

If $\alpha \leqq \beta$ we have to estimate the integral II in a different manner. We split the integration in II into two parts, $1 \geqq |x| \geqq 4\delta$ and $|x| > 1$. In the first of these parts we use (4.5) and the fact that $u_\alpha^\mu \in \operatorname{Lip}\beta$ and in the second (4.5) and the fact that u_α^μ is bounded. Straightforward calculations then prove the formulas (2.2) and (2.3) of Theorem 1.

Finally we turn to the proof of Theorem 1 — in particular for $m = 1$ — by means of the inversion formula (3.8). Assume that α is not an even integer. We define the functions ψ and ψ_δ as above and use (3.8) with φ replaced by ψ_δ. Hence

$$\mu(0,\delta) \leqq \int \psi_\delta(x)\,d\mu(x) \leqq \text{const.} \iint \frac{|u_\alpha^\mu(x)-u_\alpha^\mu(y)|\;|\Delta^h(\psi_\delta(x)-\psi_\delta(y))|}{|x-y|^{m+\alpha-2h}}\,dx\,dy\,.$$

Denoting the last integrand by J we have, since $\operatorname{supp}\psi_\delta \subset \{x \mid |x| \leqq 2\delta\}$,

$$\mu(0,\delta) \leqq \text{const.}\left\{\int\limits_{|y|\leqq 2\delta} dy \int\limits_{|x|\leqq 3\delta} J\,dx + \int\limits_{|y|\leqq 2\delta} dy \int\limits_{|x|>3\delta} J\,dx\right\} = \mathrm{I}+\mathrm{II}\,.$$

From the mean value theorem and (4.1) we deduce

$$|\Delta^h(\psi_\delta(x)-\psi_\delta(y))| \leqq \text{const.}\,\delta^{-(2h+1)}\,|x-y|\,,$$

and hence

$$\mathrm{I} \leqq \text{const.} \int\limits_{|y|\leqq 2\delta} dy \int\limits_{|x|\leqq 3\delta} |x-y|^{\beta-m-\alpha+2h+1}\,\delta^{-(2h+1)}\,dx$$

which gives

(4.7) $$\mathrm{I} \leqq \text{const.}\,\delta^{m+\beta-\alpha} \quad \text{if} \quad \alpha < 2h+\beta+1\,.$$

To estimate II we observe that by (4.1)

$$|\Delta^h(\psi_\delta(x)-\psi_\delta(y))| \leqq \text{const.}\,\delta^{-2h}\,,$$

and so

$$\mathrm{II} \leqq \text{const.} \int\limits_{|y|\leqq 2\delta} dy \int\limits_{|x|>3\delta} |x-y|^{\beta-m-\alpha+2h}\,\delta^{-2h}\,dx$$

which gives

$$\mathrm{II} \leqq \text{const.}\,\delta^{m+\beta-\alpha} \quad \text{if} \quad \alpha > 2h+\beta\,.$$

Combining the estimates of I and II we obtain (2.1) for $x=0$ and so for all $x \in R^m$, if

(4.8) $$2h+\beta < \alpha < 2h+\beta+1\,,$$

where h is the non-negative integer such that

(4.9) $$2h < \alpha < 2h+2\,.$$

This obviously proves (2.1) in particular for $m=1$. If $m>1$ and h is such that (4.9) is satisfied, (4.8) is not true for all $\alpha>\beta$. In order to prove (2.1) by means of an inversion formula of the type of (3.8) for those α which do not satisfy (4.8) we must use the analogue of (3.8) not with φ replaced by ψ_δ but by a potential of the form (4.3). We omit the details since they become more complicated than in the proof of (2.1) by means of (3.5).

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