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INFINITE-VALUED ASYMPTOTIC POINTS AND KOEBE ARCS

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In the following, C shall denote the unit circle $\{z : |z|=1\}$ and D the unit disc $\{z : |z|<1\}$.

Bagemihl and Seidel [2, Th. 3] have shown that if the set of *Fatou points* for a *normal holomorphic function* f in D has measure zero on some subarc Γ of C, then Γ contains a Fatou point for the value ∞. In this connection attention was called to the following problem: Does the assertion above remain true if it is only assumed that the measure of the set of Fatou points in every subarc γ of Γ is smaller than the length of γ?

Here we give some partial results in this direction. Thus e.g. the question is answered affirmatively if (in addition to the conditions above) the set of *critical points* (for definition, see section 5) is of the first category.

Most of our considerations concern *asymptotic points* for arbitrary holomorphic functions in D, the results on *Fatou points* for *normal* functions being consequences of fundamental properties of these functions.

Some preliminary results related to those in this paper are found in [7].

1.

In what follows, S^0, S^- and ∂S denote the interior, closure, and boundary of the set S. The linear measure of a set S is denoted by $m(S)$. By an *arc* Δ of C we mean an *open* arc whenever nothing is said to the contrary.

A simple, continuous curve γ in D, described by $z(t)$, $t \in [0,1)$, is called a *boundary path* if $|z(t)| \to 1$ as $t \to 1$ (cf. e.g. [1, p. 263]). The set $C \cap \partial\gamma$ shall be called the *end* of γ. It consists either of a single point or of a closed arc (cf. e.g. [3, p. 93–94]).

The boundary path γ is said to be an *asymptotic path* for the function φ (for the value α) if $\lim_{t\to 1}\varphi(z(t))$ exists (and equals α) (cf. e.g. [8, p. 48]). If γ has only one end point ζ — that is if $\gamma \cup \{\zeta\}$ is a Jordan arc — then γ is said to be an *asymptotic arc*. In this case ζ is an *asymptotic point* for φ (for the value α). The set of asymptotic points will be denoted by $A(\varphi)$,

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while the sets of asymptotic points for the value ∞ and for finite values with moduls greater than or equal to α are denoted by $A^{\infty}(\varphi)$ and $A^{\alpha}(\varphi)$, respectively. A point ζ may be an asymptotic point for several values; in particular $A^{\alpha}(\varphi)\cap A^{\infty}(\varphi)$ needs not be empty (though it is always countable; cf. [8, p. 39]). The concepts above may also be defined for a general domain G, and in this case we use the notation $A(\varphi,G)$, etc. Clearly $A(\varphi,G)$ consists of *accessible* boundary points for G (see e.g. 4, p. 29]).

If the ray from the origin to the point $\zeta=e^{i\theta}$ is an asymptotic arc for the value α, i.e. if $\lim_{r\to 1}\varphi(re^{i\theta})=\alpha$, then φ has the *radial limit* α at ζ. We shall simply denote this limit by $\varphi(\zeta)$ whenever it exists. It is well known that $\varphi(\zeta)$ exists almost everywhere on C if φ is holomorphic and *bounded* in D. In this case $\lim_{z\to\zeta}\varphi(z)$ even exists uniformly in every Stolz domain at ζ. (Fatou's theorem, see e.g. [9, p. 136]. A domain of the form

$$\{z\in D\ :\ |\arg(1-ze^{-i\theta})|\leqq \tfrac{1}{2}\pi-\delta\},\qquad \delta>0\,,$$

is called a *Stolz domain.*) Generally a point where φ satisfies this last condition is called a *Fatou point* for φ, and we shall denote the set of such points by $F(\varphi)$. If almost all points of a subarc Γ of C are Fatou points (that is, if $m(\Gamma\cap F(\varphi))=m(\Gamma)$), then Γ shall be called a *Fatou arc* for φ.

Obviously every Fatou point is an asymptotic point. A sufficient condition for every asymptotic point to be a Fatou point is that φ is holomorphic and *normal.* Normality of φ means that the family

$$\{f\circ S\ :\ S \text{ is a conformal mapping of } D \text{ onto itself}\}$$

is normal (see [5, p. 53]). In particular every *bounded* holomorphic function, and more generally: every holomorphic function which omits at least two finite values, is normal.

2.

Henceforth f shall denote a function which is holomorphic in D. For every positive number α, we define

$$U_\alpha = \{z\in D\ :\ |f(z)|>\alpha\},\qquad V_\alpha = \{z\in D\ :\ |f(z)|<\alpha\}\,.$$

U_α and V_α clearly consist of a countable number of components. On every component of V_α the function f is bounded; on a component of U_α it may be bounded or unbounded. The boundary of each component of U_α and of V_α is composed of a countable number of Jordan curves and boundary paths on which $|f(z)|=\alpha$, and a subset of C. For a component U

of U_α this subset of C always is non-empty, because otherwise $|f(z)| \leqq \alpha$ for all $z \in U$, according to the maximum principle.

Now let G be a subdomain of D. We define G^* in the following way: $z \in G^*$ if and only if either $z \in G$, or $z \in \gamma$ for some Jordan curve $\gamma \subset D \cap \partial G$, or z is contained in the Jordan domain bounded by such a curve γ. Clearly

$$D \cap \partial G^* \subset D \cap \partial G \quad \text{and} \quad C \cap \partial G^* \supset C \cap \partial G .$$

LEMMA 1. *Let G be a subdomain of D such that $|f(\zeta)| = \alpha$ for every $\zeta \in D \cap \partial G$. Then $|f(z)| \leqq \alpha$ for every $z \in G^* - G$.*

PROOF. If $z \in G^* - G$, then there is a Jordan curve $\gamma \subset D \cap \partial G$ such that z is contained in γ or in the Jordan domain bounded by γ. Now for every $\zeta \in \gamma$, $|f(\zeta)| = \alpha$, and so $|f(z)| \leqq \alpha$ according to the maximum principle.

We see that if V is a component of V_α, then $V^* = V$. If f is bounded on a component U of U_α, then it is also bounded on U^*, $|f|$ having the same supremum on U as on U^*.

We shall now prove a result which is a strengthening of Prop. 3 of [7]. ([7, Prop. 3] is essentially contained in [2, p. 16], though not explicitely stated).

PROPOSITION 1. *If f is bounded on some component U of U_α, then for some $\beta > \alpha$, $A^\beta(f, U) \cap C$ contains a set of positive measure.*

PROOF. It follows from the remarks succeeding Lemma 1 that f is bounded in the simply connected domain U^*. Let φ be a conformal mapping of the unit disc D_ζ in a ζ-plane onto U^*. We write

$$g = f \circ \varphi, \quad \text{and} \quad F^\beta = \{\omega \in F(\varphi) \cap F(g) \ : \ |g(\omega)| > \beta\} .$$

F^β is a Borel set, since φ and g (on C_ζ) are Baire functions. From Fatou's theorem follows that

$$m\big(C_\zeta - (F(\varphi) \cap F(g))\big) = 0 ,$$

since both φ and g are bounded. Thus if $m(F^\alpha) = 0$, then $|g(\omega)| \leqq \alpha$ almost everywhere on C_ζ. This implies that $|g(\zeta)| \leqq \alpha$ for every $\zeta \in D_\zeta$, which is a contradiction, since

$$|g(\zeta)| = |f(\varphi(\zeta))| > \alpha \quad \text{for } \varphi(\zeta) \in U .$$

Thus $m(F^\alpha) > 0$. Clearly $F^\alpha = \bigcup_{\beta > \alpha} F^\beta$, from which follows that $m(F^\beta) > 0$ for β small enough.

Since F^β is a Borel set, $\varphi(F^\beta)$ is measurable (see [10, p. 322]). A generalization of Loewner's lemma implies that if $m(F^\beta) > 0$, then $m(\varphi(F^\beta)) > 0$

(see e.g. [8, p. 34]. Cf. also [10, p. 322]). Clearly every point $\omega \in \varphi(F^\beta)$ belongs to $A^\beta(f, U^*) \cap C$. If γ is an asymptotic arc for f with respect to U^* but not with respect to U, then evidently the corresponding asymptotic value is α. Hence

$$C \cap A^\beta(f, U) = C \cap A^\beta(f, U^*) \quad \text{for} \quad \beta > \alpha .$$

Since $m(\varphi(F^\beta)) > 0$ and $\varphi(F^\beta) \subset C \cap A^\beta(f, U)$, we conclude that $C \cap A^\beta(f, U)$ contains a set of positive measure, for β small enough.

3.

Later on we shall classify the points of C in terms of the behaviour near C of the sets U_α and V_α. Now we proceed to relate the existence of asymptotic paths for ∞ to the behaviour of the sets U_α.

We shall say that the sequence $\{W_n\}_n$ is an (*asymptotic*) *tract* (for the value ∞) if W_n is a component of U_n, and $W_{n+1} \subset W_n$ for every n. (For this concept see [6, p. 142] and [7], where slightly different, though essentially equivalent definitions are given.) Clearly $\bigcap_n W_n = \emptyset$. The set $E = \bigcap_n E_n^-$ is called the *end* of the tract. If E consists of a single point, then the tract is called a *point tract*, otherwise it is an *arc tract*.

We say that a boundary path γ *belongs to the tract* $\{W_n\}$ if for every n, $z(t) \in W_n$ for t greater than some t_n. Clearly such a path is an asymptotic path for ∞, and its end is contained in the end of the tract. Thus a boundary path belonging to a point tract is an asymptotic arc for ∞. Naturally this may also be the case if the tract is an arc tract.

We formally state as a proposition an almost obvious result on asymptotic paths and tracts.

PROPOSITION 2. *To every tract for ∞ there belongs an asymptotic path for ∞.*

PROOF. Let $\{W_n\}_n$ be a tract for ∞. For every n there is a point $z_n \in W_n$ such that $|z_n| > 1 - n^{-1}$, since $C \cap \partial W_n \neq \emptyset$. Let γ_n be a Jordan arc in W_n joining z_n and z_{n+1}. We define $\gamma = \bigcup_n \gamma_n$. Clearly $\lim_\gamma f(z) = \infty$, and so γ is a boundary path. Thus γ is an asymptotic path for ∞, and the very construction of γ shows that it belongs to the tract.

PROPOSITION 3. *If $m(A^\alpha(f, U) \cap C) = 0$ for some component U of U_α, then U contains an asymptotic path for ∞.*

PROOF. Evidently $m(A^\beta(f, U) \cap C) = 0$ for every $\beta > \alpha$, and *a fortiori*

$$m(A^\beta(f, W) \cap C) = 0$$

for every $\beta > \alpha$, where W is a component of U_γ, $\alpha < \gamma < \beta$, contained in U. Hence it follows from Prop. 1 that f is unbounded on *every* component W_n of U_n contained in U, for $n > \alpha$. Consequently we may inductively build a tract $\{W_n\}_{n>\alpha}$ for ∞, all sets W_n being contained in U. Hence the announced result follows from Proposition 2.

Let Δ be a subarc of C. A *Koebe sequence* of arcs relative to Δ is a sequence $\{\Delta_n\}_n$ of Jordan arcs in D satisfying the following conditions:

1) For every $\varepsilon > 0$, $\Delta_n \subset \{z : |z| > 1-\varepsilon\}$ for all but finitely many n;
2) For $\zeta_1, \zeta_2 \in \Delta$, a subarc δ_n of Δ_n lies in the sector bounded by the rays from the origin to ζ_1 and ζ_2 and the subarc of Δ whose end points are ζ_1 and ζ_2, the end points of δ_n lying on each of the rays.

(For this definition, see [2, p. 9]).

Let c be a constant (finite or infinite). We shall say that Δ is a *Koebe arc for the value* c if there is a Koebe sequence $\{\Delta_n\}_n$ relative to Δ such that for every $\varepsilon > 0$, $|f(z)-c| < \varepsilon$ ($|f(z)| > \varepsilon^{-1}$ for $c = \infty$) for all $z \in \Delta_n$ and all but finitely many n. Clearly a subarc of a Koebe arc for c is a Koebe arc for c. A non-constant *normal* holomorphic function admits no Koebe arc for any value (cf. [3, p. 10]).

LEMMA 2. *Let γ be an asymptotic path for the value c. If γ is not an asymptotic arc, then every subarc of the end of γ is a Koebe arc for the value c.*

PROOF. Let ζ_1 and ζ_2 belong to the interior of $C \cap \partial\gamma$, and let γ_1 and γ_2 denote the rays from the origin to ζ_1 and ζ_2. Clearly γ intersects γ_1 and γ_2 an infinite number of times. Hence one easily sees that there is a sequence of subarcs of γ, which is a Koebe sequence relative to the arc $\Delta \subset C \cap \partial\gamma$ between ζ_1 and ζ_2. Clearly f tends to c along this sequence, which means that Δ is a Koebe arc for c.

LEMMA 3. *Let Γ be a subarc of C, and let $\{z_n\}_n$ be a sequence in D, all cluster points of which are contained in a closed subarc γ of Γ. Further let W_n be a component of U_n containing z_n. Then either there is an n_0 such that $C \cap \partial W_n \subset \Gamma$ for $n \geqq n_0$, or Γ contains a Koebe arc for ∞.*

PROOF. The assumption of no n_0 such that $C \cap \partial W_n \subset \Gamma$ for $n \geqq n_0$, implies the existence of a sequence $\{x_{n_i}\}_i$, $x_{n_i} \in W_{n_i}$, all cluster points of which are contained in $C - \Gamma$. Let γ_i be a Jordan arc in W_{n_i} joining z_{n_i} and x_{n_i} and let y_i be a point in γ_i such that $|y_i| = \inf\{|z| : z \in \gamma_i\}$. Here $|y_i| \to 1$, since $|f(y_i)| > n_i$. Hence $\{\gamma_i\}_i$ contains a Koebe sequence relative to some arc Δ, where Δ^- is contained in one of the arcs of which $\Gamma - \gamma$ is composed. Thus the existence of a Koebe subarc of Γ for ∞ is established.

We now give a generalization of Theorem 2 of [7]. (The essence of [7, Th. 2] is implicitely contained in [2]).

THEOREM 1. *Let f be holomorphic in D, and let Γ be a subarc of C such that $m(\Gamma \cap A^{\alpha}(f))=0$ for some finite α. Then Γ contains either a Fatou arc, or a Koebe arc for ∞, or a point of $A^{\infty}(f)$.*

PROOF. If f is bounded in a neighbourhood of a point $\zeta \in \Gamma$, then a subarc Δ of Γ is contained in a Jordan curve with rectifiable boundary, in whose interior region f is bounded. It follows that Δ is a Fatou arc (cf. e.g. [9, p. 129]). Thus if Γ contains no Fatou arc, then there exists a sequence $\{z_n\}_n$, where $|f(z_n)|>n$, such that $z_n \to \zeta$ for some $\zeta \in \Gamma$. Let W_n be that component of U_n which contains z_n. Assume that there is an n_0 such that $C \cap \partial W_n \subset \Gamma$ for $n \geqq n_0$. Then there is an $m>\alpha$ such that $C \cap \partial W_m \subset \Gamma$. Now

$$A^m(f, W_m) \cap C \subset A^m(f) \cap \Gamma \subset \Gamma \cap A^{\alpha}(f) .$$

From Prop. 3 then follows that W_m contains an asymptotic path for ∞, whose end is contained in Γ. Thus in this case application of Lemma 2 gives the announced result.

If there is no such n_0, then the existence of a Koebe arc for ∞ in Γ immediately follows from Lemma 3.

From Theorem 1 we deduce the following corollary, which contains one of the main results of [2] ([2, Th. 3]).

COROLLARY 1. *Let Γ be a subarc of C. If $m(\Gamma \cap A^{\circ}(f))=0$, then Γ contains either an asymptotic point for ∞ or a Koebe arc for ∞. If f is normal and $m(\Gamma \cap F(f))=0$, then Γ contains a Fatou point for ∞.*

PROOF. If $m(\Gamma \cap A^{\circ}(f))=0$, then Γ certainly contains no Fatou arc, and the Corollary follows immediately from Theorem 1. If f is normal, then — as mentioned above — Fatou points and asymptotic points coincide ([5, p. 53]), and f admits no Koebe arc ([2, p. 10]).

4.

In the proofs of the next results the concept of a *cross-path* will be useful. A cross-path is a simple, continuous curve γ in D, described by $z(t)$, $t \in (0,1)$ such that

$$|z(t)| \to 1 \text{ as } t \to 1, \qquad |z(t)| \to 1 \text{ as } t \to 0 .$$

Evidently γ is the union of two boundary paths, and the *end* of γ is defined to be the union of the ends of two constituting boundary paths (this clearly is independent of the decomposition of γ in boundary paths). If γ has only *two* end points ζ_1 and ζ_2 (i.e. if $\gamma \cup \{\zeta_1\} \cup \{\zeta_2\}$ is a Jordan arc), then γ is a *cross-cut* (cf. e.g. [4, p. 5]). (If in particular $\zeta_1 = \zeta_2 = \zeta$, then $\gamma \cup \{\zeta\}$ is a Jordan curve.)

We first give an auxiliary result for later use (cf. [7, Prop. 8]).

LEMMA 4. *Let H be a component of U_α, K a component of U_β, $\beta > \alpha$, and suppose that there is an* arc $\Gamma \subset C \cap \partial H \cap \partial K$. *Further suppose that if the end of a boundary path in $D \cap \partial H$ meets Γ, then it consists of a single point. Then $K \subset H$.*

PROOF. K is contained in some component of U_α, since $U_\beta \subset U_\alpha$. According to Lemma 1, all components of U_α except H are contained in $D - H^*$. Thus if $K \not\subset H$, then K is contained in some component of $D - H^*$. Now let G be a component of $D - H^*$. Then there is a cross-path $\gamma \subset D \cap \partial H^*$ such that G is one of the components of $D - \gamma$ (it is easily verified that $D - \gamma$ consists of exactly two components, cf. [7, Lemma 1]). If the end of γ is contained in $C - \Gamma$, then clearly G is that component whose boundary is disjoint from the arc $C - \Gamma$ (cf. [7, Lemma 1]), which means that $\Gamma \cap \partial G = \emptyset$. If γ has an end point $\zeta \in \Gamma$, then our assumptions combined with the fact that $C \cap \partial H^* \supset \Gamma$ imply that $\gamma \cup \{\zeta\}$ is a Jordan curve. Thus in both cases $C \cap \partial G \not\supset \Gamma$, and *a fortiori* $C \cap \partial K \not\supset \Gamma$, contrary to assumption. Hence we conclude that $K \subset H$.

We proceed to prove a lemma which is crucial for the development of our main results. (This lemma is closely related to [7, Prop. 5].)

LEMMA 5. *Let f be bounded in a domain $G \subset D$ such that $|f(z)| = \alpha$ for every $z \in D \cap \partial G$, and let $C \cap \partial G$ contain the arc Δ. Then either for some $\beta \geqq \alpha$ there is a component W of U_β such that $C \cap \partial W$ is contained in Δ and contains at most one accessible boundary point for W, or f is bounded in a Jordan domain H with rectifiable boundary containing a subarc of Δ, or Δ contains a Koebe arc for ∞.*

PROOF. Let Γ and γ be subarcs of Δ with $\gamma^- \subset \Gamma$, $\Gamma^- \subset \Delta$, and let γ_1 and γ_2 (resp. Γ_1 and Γ_2) denote the rays from the origin to the end points of γ (resp. Γ). The domain H bounded by $\gamma \cup \gamma_1 \cup \gamma_2$ and the domain K bounded by $\Gamma \cup \Gamma_1 \cup \Gamma_2$ are Jordan domains with rectifiable boundaries.

Assume that f is unbounded in H. Then there is a sequence $\{z_n\}_n$ of points in H, all cluster points of which are contained in γ^- and such that $|f(z_n)| > n$. Let W_n denote that component of U_n which contains z_n. Further let M be a number greater than α such that $|f(z)| \leqq M$ for $z \in G$.

Now two possibilities arise. Either $C \cap \partial W_n \not\subset \Gamma$ for all $n \geqq M$. In this case the existence of a Koebe arc for ∞ contained in $\Gamma \subset \Delta$ follows from Lemma 3. Or there is an $m \geqq M$ such that $C \cap \partial W_m \subset \Gamma$. Clearly $W_m \subset D-G$. Let δ be a cross-path in $D-G$, such that W_m is contained in one of the components F of $D-\delta$ (cf. [7, Lemma 1]). G then is contained in the other component. Certainly $\partial F \cap C$ contains a point of Δ. Since $C \cap \partial G \supset \Delta$, this is possible only if $C \cap \partial F \subset \Delta$ and $C \cap \partial F$ contains at most one accessible boundary point for F. Since $W_m \subset F$, the announced result immediately follows.

A simple consequence of Lemma 5 is the following proposition (which is a strengthening of [7, Prop. 6]).

PROPOSITION 4. *Let f be bounded in a domain $G \subset D$, where $|f(z)| = \alpha$ for every $z \in D \cap \partial G$, and let $C \cap \partial G$ contain the arc Δ. Then Δ either contains a Fatou arc, or a Koebe arc for ∞, or an asymptotic point for ∞.*

PROOF. If for some $\beta \geqq \alpha$ there is a component W of U_β such that $C \cap \partial W$ is contained in Δ and contains at most one accessible boundary point for W, then clearly $m(\Delta \cap A^\beta(f, W)) = 0$. Hence the announced result follows from Prop. 3 and Lemma 2.

If f is bounded in a Jordan domain H with rectifiable boundary ∂H, then almost every point of ∂H is a Fatou point for f (cf. [9, p. 129]). Thus if ∂H contains a subarc of Δ, then this subarc is a Fatou arc.

If neither of the above-mentioned situations occur, then the existence of a Koebe subarc of Δ for ∞ follows from Lemma 5.

5.

We now introduce the following classification of the points of C in terms of the behaviour of the sets U_α and V_α in the vicinity of the points. We shall term ζ an *upper ordinary point* if there are *arbitrarily great* α such that $\zeta \in U^-$ for some component U of U_α. If $\zeta \in U^-$, then clearly $\zeta \in W^-$ for some component W of U_β for every $\beta < \alpha$. Thus an equivalent definition is to require that $\zeta \in U^-$ for some component U of U_α for *every* α.

Similarily ζ is a *lower ordinary point* if there are *arbitrarily great* α such that $\zeta \in V^-$ for some component V of V_α. Clearly it is equivalent to require this condition to be satisfied for *some* α only.

Similarily ζ is a *critical point* if for some α, ζ is not a closure point for any component of U_β or V_β for any $\beta > \alpha$.

We shall denote the sets of upper ordinary, lower ordinary, and critical points by Ω_U, Ω_L and Ω_C (or $\Omega_U(f)$, $\Omega_L(f)$, $\Omega_C(f)$). Obviously

$$C = \Omega_U \cup \Omega_L \cup \Omega_C \quad \text{and} \quad (\Omega_U \cup \Omega_L) \cap \Omega_C = \emptyset ,$$

while $\Omega_U \cap \Omega_L$ may be non-empty. The following result is almost evident.

PROPOSITION 5. *If ζ is an end point for an asymptotic path γ for the value ∞, then ζ is an upper ordinary point. If ζ is an end point for a boundary path δ on which f is bounded, then ζ is a lower ordinary point.*

PROOF. In the former case, for arbitrary α there is a connected subset of γ contained in U_α, and so $\zeta \in U^-$ for some component U of U_α. In the latter case, for some β, V_β contains δ, and so ζ is a closure point for some component of V_β.

Let $\mathscr{U}_\alpha$ denote the set whose elements are the components of U_α, and let $\mathscr{V}_\alpha$ denote the set whose elements are the components of V_α. Clearly $\mathscr{U}_\alpha$ and $\mathscr{V}_\alpha$ are countable, since the components are open, non-empty and disjoint. We observe that we may write

$$(1)\qquad \Omega_L = \textstyle\bigcup_{n=1}^{\infty} \bigcup \{C \cap \partial V \,:\, V \in \mathscr{V}_n\},$$

$$(2)\qquad \Omega_U = \textstyle\bigcap_{n=1}^{\infty} \bigcup \{C \cap \partial U \,:\, U \in \mathscr{U}_n\}.$$

We now proceed to prove our main results.

THEOREM 2. *Let f be holomorphic in D, and let Γ be a subarc of C. If the set $\Omega_L(f)$ of lower ordinary points is of the second category in Γ, then Γ either contains a Fatou arc, or a Koebe arc for ∞, or an asymptotic point for ∞.*

PROOF. Since $\mathscr{V}_n$ is countable, it follows from (1) that for some component V of some V_n, $C \cap \partial V$ is not nowhere dense in Γ. But $C \cap \partial V$ is closed, and hence contains an arc Δ. It follows that application of Prop. 4 leads to the announced result.

REMARK. According to Theorem 1, Theorem 2 is trivially true without any hypothesis on $\Omega_L(f)$ if $m(\gamma \cap A^\circ(f)) = 0$ for some subarc γ of Γ. If there is no such subarc γ, then certainly $A^\circ(f)$, and hence $\Omega_L(f)$, is *dense* in Γ. We have not been able to show that this condition (denseness of $\Omega_L(f)$), is sufficient for the conclusion of Theorem 2 to hold. Our condition is (as stated above) that $\Omega_L(f)$ is of *the second category*.

Taking into account the facts that for *normal* functions, asymptotic points and Fatou points coincide and Koebe arcs for ∞ do not occur, we immediately get the following corollary of Theorem 2.

COROLLARY 2. *Let f be a normal holomorphic function in D, and let Γ be a subarc of C. If the set Ω_L is of the second category in Γ, then Γ either contains a Fatou arc or a Fatou point for ∞.*

We define the set $B(f)$ as follows:

$$B(f) = \left\{ \begin{array}{l} \zeta \in C \ : \ \text{there is a Jordan arc ending} \\ \qquad\qquad \text{at } \zeta \text{ on which } f \text{ is bounded} \end{array} \right\}.$$

Clearly $A^0(f) \subset B(f)$, and $B(f) \subset \Omega_L(f)$ (Prop. 5).

COROLLARY 3. *Let f be holomorphic in D, and let Γ be a subarc of C. If the set $B(f)$ is of the second category in every subarc of Γ (in particular if $B(f)$ is a residual set in Γ), then Γ either contains a Fatou arc or an asymptotic point for ∞.*

PROOF. $\Omega_L(f)$ is of the second category, hence Theorem 2 may be applied. Since $B(f)$ is of the second category in every subarc of Γ, it is dense in Γ. This excludes the possibility of Γ containing a Koebe arc for ∞.

THEOREM 3. *Let f be holomorphic in D, and let Γ be a subarc of C. If the set $\Omega_U(f)$ of upper ordinary points is residual in Γ, then Γ contains either a Fatou arc, or a Koebe arc for ∞, or an asymptotic point for ∞.*

PROOF. From (2) it follows that for every n, $\bigcup\{C \cap \partial U \, : \, U \in \mathscr{U}_n\}$ is a residual set. Hence there is a component W_1 of U_1 for which $\Gamma \cap \partial W_1$ is not nowhere dense. But $\Gamma \cap \partial W_1$ is closed, and hence contains an arc Γ_1. Further $\bigcup\{C \cap \partial U \, : \, U \in \mathscr{U}_2\}$ is a residual set in Γ_1, and hence there is a component W_2 of U_2 such that $\Gamma_1 \cap \partial W_2$ contains an arc Γ_2. Thus we may inductively construct sequences $\{W_n\}_n$, $\{\Gamma_n\}_n$, where W_n is a component of U_n, and $\Gamma_n \subset \Gamma_{n-1} \cap \partial W_n$.

Suppose that $D \cap \partial W_n$ for some n contains a boundary path γ such that $\Gamma_{n-1} \cap \partial \gamma$ contains an arc Δ. Then for some $m > n$, γ is contained in a component G of V_m. Consequently $\Delta \subset C \cap \partial G$. Application of Prop. 4 then leads to the announced result.

If there is no such boundary path γ for any n, then it follows from Lemma 4 that $W_{n+1} \subset W_n$, since

$$C \cap \partial W_n \cap \partial W_{n+1} \supset \Gamma_n \cap \Gamma_{n+1} = \Gamma_{n+1} \, .$$

Hence $\{W_n\}_n$ is a tract for ∞. Clearly we may choose a sequence of points $\{z_n\}_n$, $z_n \in W_n$, all cluster points of which are contained in some closed subarc of Γ. Then according to Lemma 3, Γ either contains a Koebe arc for ∞, or $C \cap \partial W_n \subset \Gamma$ for some n. In the latter case, the end of an asymptotic path belonging to the tract $\{W_n\}_n$ is contained in Γ. Hence in this case the announced result follows from Lemma 2.

The following corollary is immediate:

COROLLARY 4. *Let f be a normal holomorphic function in D, and let Γ be a subarc of C. If the set Ω_U is residual in Γ, then Γ either contains a Fatou arc, or a Fatou point for ∞.*

By combining Theorems 2 and 3 we get the following result (which is related to [7, Th. 4]).

THEOREM 4. *Let f be holomorphic in D, and let Γ be a subarc of C. If the set $\Omega_C(f)$ of critical points is of the first category in Γ, then Γ either contains a Fatou arc, or a Koebe arc for ∞, or an asymptotic point for ∞.*

PROOF. The set $\Omega_L \cup \Omega_U$ is residual. Hence, either Ω_L is of the second category, in which case the result follows from Theorem 2, or Ω_U is a residual set, in which case the result follows from Theorem 3.

The counterpart of Theorem 4 for normal functions is the following corollary.

COROLLARY 5. *Let f be a normal holomorphic function in D and let Γ be a subarc of C. If Ω_C is of the first category in Γ, then Γ contains either a Fatou arc or a Fatou point for ∞.*

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