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A NOTE ON UNIVERSAL HOMOGENEOUS MODELS

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The title refers to Bjarni Jónsson [4] as generalized by M. Morley and R. Vaught [5]. The purpose is to reformulate the central argument by eliminating superfluous aspects in setting and hypothesis.

Let C be a class of sets whose members, to conform to one of the "intended interpretations", will be called *models*. Their structure, if any, need not be further specified; but we assume a class of (one-to-one, onto) mappings between them called *isomorphisms*. The latter are, as usual, required to include all identity maps and to be closed under inversion and composition. Thus the isomorphism of models is an equivalence relation whose classes are called *isomorphism types*.

It will be useful to recall the definition of the direct limit: If $\{M_\alpha\}$ are models indexed by a directed set with, for each $\alpha<\beta$, specification of an isomorphism $f_\beta{}^\alpha$ of M_α into M_β, such that for $\alpha<\beta<\gamma$, $f_\gamma{}^\alpha=f_\gamma{}^\beta f_\beta{}^\alpha$, then a *direct limit* is a model M_∞ equipped with isomorphisms $f_\infty{}^\alpha$ of M_α into M_∞ for which $f_\infty{}^\alpha=f_\infty{}^\beta f_\beta{}^\alpha$ when $\alpha<\beta$; and satisfying, moreover: if g^α are isomorphisms of M_α into any model M such that $g^\alpha=g^\beta f_\alpha{}^\beta$ when $\alpha<\beta$, then there exists a unique isomorphism g^∞ of M_∞ into M for which $g^\alpha=g^\infty f_\infty{}^\alpha$. An example of a direct limit is a directed union of models; but the limit can exist even when the union fails to be a model as, for example, in the case of complete metric spaces. A direct limit is, up to isomorphism, unaffected by suppression of all but a cofinal set of indices. Since only direct limits of chains will be needed, we may restrict the index sets to regular cardinals, these being the types of minimal well-ordered cofinal subsets.

Let C′ be a subclass of C closed under isomorphism. A model U having submodels isomorphic to every member of C′ is called C′ *universal*; U is C′ *injective* if every isomorphism into U whose domain is in C′ can be extended to any overmodel also in C′. A C′ injective model which shares with each C′ model an isomorphically common C′ submodel is C′ universal; indeed, it is even universal for direct limits of chains of C′ models. However, since the embedding cannot be shown to be onto, this is,

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unfortunately, of no help in establishing uniqueness of a C′ injective direct limit.

Suppose U and V are C′ injective models not in C′, each of which is a direct limit of $\aleph$ C′ models, and contains, for every pair of its C′ submodels, a C′ overmodel of the first containing an isomorphic image of the second. Let, finally, the direct limit of fewer than $\aleph$ C′ models be in C′. Then every isomorphism between C′ submodels of U and V can be extended to an isomorphism of U with V. The proof can be taken over from [5].

A C model is called C′ *homogeneous* if every isomorphism between two of its C′ submodels is the restriction of an automorphism. The U and V of the preceding paragraph are clearly such.

This leads to the existence of C′ universal C′ homogeneous models under the following assumptions: There are exactly $\aleph$ isomorphism types in C′; every pair of C′ models is isomorphically embeddable in a third; the direct limit of $\aleph$ or fewer C′ models exists, and if fewer than $\aleph$ belongs to C′ while if exactly $\aleph$ does not (unless, of course, the f_β^α are terminally onto).

It should be noted that every C′ universal C′ homogeneous model is C′ injective: An isomorphism of a C′ submodel of the C′ model A into U can be extended by any isomorphism of A into U composed with an automorphism of U correcting it on the submodel. It follows, under the assumptions of the preceding paragraph, that two C′ universal C′ homogeneous models which are direct limits of $\aleph$ C′ models are isomorphic.

In [5] C consists of relational systems of a specified type and C′ of those whose underlying set has cardinality less than some K. Actually, as there formulated universality includes models of power K as well, but since the axioms gurantee that these are always unions of chains of lower powered submodels, this comes free of charge, as remarked above.

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