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THE SINGULAR SPECTRUM OF ELLIPTIC DIFFERENTIAL OPERATORS IN $L^p(R_n)$

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Introduction.

The essential spectrum of a self-adjoint operator A in a Hilbert space can be defined as the spectrum with the exception of isolated eigenvalues of A of finite multiplicity. The essential spectrum of A is stable under perturbations, which are in some sense compact relative to A. The subject has been treated by Gokhberg and Krein [7], Kato [9], Wolf [12], Birman [4], Rejto [10], and others. In an earlier paper [3] the author applied results from [7] and [9] to elliptic differential operators in $L^p(R_n)$, $1<p<\infty$. In the present paper, we improve the results of [3], building on a theorem of Birman [4] in the case of $p=2$ (see Theorem 1.1). He considered operators A and B bounded from below, making use of the theory of Friedrichs (see [6]) on quadratic forms, and obtained conditions on the difference between the corresponding forms A and B, under which $\mathsf{A}^{-1}-\mathsf{B}^{-1}$ is compact, and consequently A and B have the same essential spectrum.

Birman applied this theory to a perturbed Δ^k-operator in $L^2(R_n)$. The main problem in this connection is to find conditions on a function b, in order that the embedding of $W_{2k}(R_n)$ (see Definition 2.1) into $L^2(R_n;b)$ with b as weight function be compact. In Birman [4] such conditions are given and combined with the abstract results to determine the essential spectrum of the perturbed Δ^k-operator.

We derive a stronger result on compactness, which is valid in the more general case $1<p<\infty$ and improves a condition given in [3, Lemma 2.5]. The proof is based on a known condition for boundedness of embedding operators, quoted as Lemma 2.4.

We apply these results for $p=2$ and the theorem of Birman to treat a perturbed elliptic constant coefficient operator in $L^2(R_n)$. We determine a rather large class of operators V, whose resolvent V^{-1} differs from the resolvent L^{-1} of a constant coefficient operator L by a compact operator.

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Since the essential spectrum of the constant coefficient operator is easy to compute, using Fourier transform, we obtain an expression for the essential spectrum of the perturbed operator V. In our case, this coincides with the continuous spectrum $\sigma_c(\mathsf{V})$, being the set of numbers λ such that $R(\mathsf{V}-\lambda)$ is not closed. The main result is given in Theorem 5.3. In Theorem 5.4, Theorem 5.3 is illustrated in the case of the Schrödinger operator, and we show in 5.5 that our theorem contains earlier results of Agudo and Wolf [1], Birman [4], and the author [3]. The recent work of Rejto treats exclusively this operator, and he obtains a stronger result, quoted in Remark 5.5, under the assumption that $q(x)$ is bounded from below ([10, Theorem 3.1]).

Theorem 6.1 is the simpler version of Theorem 5.3 for ordinary differential operators, and in Theorem 6.2 we give an application to the Euler operator in $L^2(0,\infty)$. Section 6 is an improvement in the case $p=2$ of results obtained in [2] for $1<p<\infty$.

Finally, in Section 7 we consider a uniformly elliptic differential operator in $L^p(R_n)$ and apply the compactness result to obtain an expression for the singular spectrum (Definition 7.1) of a perturbed constant coefficient operator. This is also based on results of [3] and gives an improvement of the main result of [3].

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1. Quadratic forms.

Let $\mathscr{H}$ be a Hilbert space with the inner product of $u \in \mathscr{H}$ and $v \in \mathscr{H}$ denoted by (u,v), and the norm of $u \in \mathscr{H}$ denoted by $|u|$.

We consider symmetric, bilinear forms $A'[u,v]$, $B'[u,v]$ etc. with domains $D[A']$, $D[B']$ etc. dense in $\mathscr{H}$. To simplify the notation we set $A'[u,u]=A'[u]$ etc.

All forms A' are assumed bounded below, i.e. there exists $K>0$, such that

$$A'[u]+K|u|^2 > 0 \quad \text{for} \quad u \in D[A'], \quad u \neq 0 . \tag{1}$$

The domain $D[A']$ is a pre-Hilbert space with the inner product

$$(u,v)_{A',K} = A'[u,v]+K(u,v) . \tag{2}$$

A form A' is said to be closed, if $D[A']$ is complete with respect to the norm corresponding to (2) for some (and hence all except possibly the smallest) K for which (1) holds.

In what follows we denote closed forms by A, B etc. If A is a closed form, all the A,K-norms are equivalent except possibly for the smallest K. It is understood that $D[A]$ is provided with one of these norms.

According to the theory of Friedrichs (see [6]), there is a 1–1 correspondence between closed, symmetric, bilinear forms A, bounded from below, and self-adjoint operators A, bounded from below. The connection between A and A is given by

$$A[u,v] = (\mathsf{A}u,v) \quad \text{for} \quad u \in D(\mathsf{A}),\ v \in D[A]\,.$$

Let A_0 be a symmetric densely defined linear operator, bounded below. The Friedrichs extension A of A_0 is the self-adjoint operator corresponding to the closure A of the form (A_0u,v).

The bilinear form $A[u,v]$ is completely determined by the quadratic form $A[u]$. In what follows we consider for simplicity only the quadratic form.

The essential spectrum $\sigma_e(\mathsf{A})$ of a self-adjoint operator A is the set of real numbers λ, such that either the range of $\mathsf{A}-\lambda$ is not closed or the null space of $\mathsf{A}-\lambda$ is infinite-dimensional. The continuous spectrum $\sigma_c(\mathsf{A})$ is the set of real numbers λ, such that the range of $\mathsf{A}-\lambda$ is not closed. The resolvent set of A is denoted by $\varrho(\mathsf{A})$.

1.1 Theorem. (Birman). *Let the closed positive definite quadratic forms $A[u]$ and $C[u]$ satisfy the following conditions:*

a) $D[A]=D[C]$.
b) *There exists a positive quadratic form $F[u]$ on $D[A]$ such that*
   i) $|A[u]-C[u]| \leqq F[u] \leqq K\,A[u]$,
   ii) *the operator A^{-1} is compact from $\mathscr{H}$ into $D[F]$.*

Then the operator $\mathsf{A}^{-1}-\mathsf{C}^{-1}$ is compact from $\mathscr{H}$ into $D[A]$. Consequently,

$$\sigma_e(\mathsf{A}) = \sigma_e(\mathsf{C})\,,$$

and hence $\varrho(\mathsf{A})=\varrho(\mathsf{C})$, except for an at most countable set of isolated eigenvalues of either A or C of finite multiplicity.

Proof. We refer to Birman [4] for the first statement. The second statement follows from the corresponding statement for A^{-1} and C^{-1}, which in turn is a well known consequence of the compactness of $\mathsf{A}^{-1}-\mathsf{C}^{-1}$ (cf. Wolf [12]).

2. Embedding operators.

2.1. Definition. R_n is the n-dimensional real and C_n the n-dimensional complex space. All functions considered in what follows are complex valued and defined on R_n, and we omit for simplicity explicit reference to R_n in our notation, writing L^p for $L^p(R_n)$ etc. Here $1 < p < \infty$.

We set

$$S_{x,R} = \{y \in R_n \mid |x-y| \leqq R\}; \quad S_x = S_{x,1};$$
$$S' = \{x \in R_n \mid |x| = 1\}.$$

The elementary differential operators D_j (in the sense of distributions) are defined by

$$D_j = -i\frac{\partial}{\partial x_j}, \qquad 1 \leqq j \leqq n.$$

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is any n-tuple of non-negative integers, then we set

$$D^\alpha = \prod_{j=1}^n D_j^{\alpha_j}; \qquad \xi^\alpha = \prod_{j=1}^n \xi_j^{\alpha_j}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in C_n,$$

$$W_k{}^p = \{f \mid D^\alpha f \in L^p \text{ for } 0 \leqq |\alpha| \leqq k\}, \qquad \text{where } |\alpha| = \sum_{j=1}^n \alpha_j.$$

Since we shall deal mainly with the case $p=2$, we introduce the notation

$$W_k = W_k{}^2 = \{f \mid D^\alpha f \in L^2 \text{ for } 0 \leqq |\alpha| \leqq k\}$$

for simplicity.

Let $b(x)$ be a measurable function on R_n. Then the operator $\mathsf{B}_\alpha{}^p$ is defined by

$$D(\mathsf{B}_\alpha{}^p) = \{f \in L^p \mid D^\alpha f \in L^1_{\mathrm{loc}},\ b D^\alpha f \in L^p\}$$

and

$$\mathsf{B}_\alpha{}^p f = b D^\alpha f \text{ for } f \in D(\mathsf{B}_\alpha{}^p).$$

We denote by $\mathsf{B}^p_{\alpha,R}$ the operator thus defined corresponding to the function $b\chi_{S_{o,R}}$, where χ_M denotes the characteristic function of the set M.

The operator $\mathsf{B}_\alpha{}^p$ is said to be $W_k{}^p$*-ε-bounded* if $D(\mathsf{B}_\alpha{}^p) \supseteq \mathrm{W}_k{}^p$ and for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that for $u \in W_k{}^p$

$$|\mathsf{B}_\alpha{}^p u| \leqq \varepsilon |u|_{W_k{}^p} + K(\varepsilon)\,|u|.$$

The operator $\mathsf{B}_\alpha{}^p$ is W_k*-compact* if $D(\mathsf{B}_\alpha{}^p) \supseteq W_k{}^p$ and its restriction $\mathsf{B}_\alpha \mid W_k{}^p$ is a compact operator from $W_k{}^p$ into L^p.

2.2. Definition. We introduce the following subspaces of L^p_{loc}:

$$E^p = \left\{f \mid \sup_{x \in R_n} \int_{S_x} |f(y)|^p \, dy < \infty\right\};$$

$$F^p = \left\{ f \mid \int_{S_x} |f(y)|^p \, dy \to 0 \text{ as } |x| \to \infty \right\};$$

$$I_h^p = \left\{ f \mid \exists\, a > 0 \text{ such that } \sup_{x \in R_n} \int_{S_x} |f(y)|^p \, |x-y|^{ph-n-a} dy < \infty \right\};$$

$$N_h^p = I_h^p \cap F^p .$$

If $ph > n$, then $I_h^p = E^p$ and $N_h^p = F^p$.

2.3. Lemma. *Suppose that there exists a function $\beta(r)$ such that $|b(x)|^p \leqq \beta(|x|)$ for all $x \in R_n$, and suppose that β satisfies the following conditions for some $h > p^{-1}$:*

(i) $\displaystyle\int_0^1 \beta(r) r^{ph-1-a} \, dr < \infty$ *for some* $a > 0$,

(ii) $\displaystyle\sup_{1 \leqq R < \infty} \int_R^{R+1} \beta(r) \, dr < \infty$,

then $b \in I_h^p$. If, moreover,

(iii) $\displaystyle\int_R^{R+1} \beta(r) \, dr \to 0$ *as* $R \to \infty$,

then $b \in N_h^p$.

We omit the proof of Lemma 2.3 which is due to E. Thue Poulsen. The restriction $h > p^{-1}$ makes no difficulties for our purposes, since we need the spaces I_h^p and N_h^p only for positive integers h.

2.4. Lemma. (a) *If $b \in I^p_{k-|\alpha|}$, $|\alpha| < k$, then for every $\varepsilon > 0$ there exists a $K(\varepsilon)$, such that for $u \in W_k^p$ and $R \geqq R_0 > 0$*

$$|\mathsf{B}^p_\alpha u|_{L^p(S_{0,R})} \leqq \varepsilon |u|_{W_k^p(S_{0,R})} + K(\varepsilon) |u|_{L^p(S_{0,R})} .$$

(b) *Suppose that $b \in I^p_{k-|\alpha|}$, and denote by $\|\mathsf{B}^p_\alpha\|_k$ the norm of B^p_α as an operator from W_k^p into L^p. Then for $R \geqq 1$ and $a > 0$*

$$\|\mathsf{B}^p_\alpha - \mathsf{B}^p_{\alpha, R}\|_k{}^p \leqq K(a) \sup_{|x| \geqq R} \int_{S_x} |b(y)|^p \, |x-y|^{p(k-|\alpha|)-n-a} \, dy .$$

Proof. We refer to [5, (5.2) on p. 86] for the proof of (a) and to [3, Lemma 2.8 (a)] for the proof of (b).

2.5. Lemma. *If $b \in N^p_{k-|\alpha|}$, $|\alpha| < k$, then $\mathsf{B}_\alpha{}^p$ is $W_k{}^p$-compact.*

Proof. By Lemma 2.4 (a), $\mathsf{B}_\alpha{}^p$ is W_k^p-ε-bounded. We shall prove that $\mathsf{B}_\alpha{}^p$ is W_k^p-compact by proving

1) $\|\mathsf{B}_\alpha{}^p - \mathsf{B}_\alpha{}^p{}_{,R}\|_k \to 0$ as $R \to \infty$,

2) $\mathsf{B}_\alpha{}^p{}_{,R}$ is W_k^p-compact for every $R > 0$,

separately.

1) Fix $a_0 > 0$ such that

$$\int\limits_{S_x} |b(y)|^p \, |x-y|^{p(k-|\alpha|)-n-a_0} \, dx \; < \; K \quad \text{for} \quad x \in R_n \, .$$

By a simple application of Hölder's inequality there exist $a_1 > 0$ and $p^0, q^0 > 1$ with $1/p^0 + 1/q^0 = 1$, such that

$$(1) \qquad \int\limits_{S_x} |b(y)|^p \, |x-y|^{p(k-|\alpha|)-n-a_1} \, dy$$
$$\leqq \left\{ \int\limits_{S_x} |b(y)|^p \, dy \right\}^{1/p^0} \left\{ \int\limits_{S_x} |b(y)|^p \, |x-y|^{p(k-|\alpha|)-n-a_0} \, dy \right\}^{1/q^0} ,$$

as proved in [3, Lemma 2.6 (b)]. It follows from (1) and the assumptions on $b(x)$ that

$$(2) \qquad \int\limits_{S_x} |b(y)|^p \, |x-y|^{p(k-|\alpha|)-n-a_1} \, dy \to 0 \quad \text{as} \quad |x| \to \infty \, .$$

By Lemma 2.4 (b) and (2) it follows that 1) holds.

2) It is clear that it suffices to consider $|\alpha| = 0$. Let $\{u_n\}$ be a bounded sequence in $W_k{}^p$. Then it is well known, that for $|\alpha| < k$ the sequence $\{D^\alpha u_n\}$ is compact in $L^p(S_{0,R})$, so we can assume, by passing to a subsequence,

$$(1) \qquad |u_n - u_m|_{L^p(S_{0,R})} \to 0 \quad \text{as } n, m \to \infty \, .$$

By Lemma 2.4 (a) we have

$$(2) \quad |\,\mathsf{B}^p_{0,R}(u_n - u_m)|_{L^p(S_{0,R})} \leqq \varepsilon |u_n - u_m|_{W_k{}^p(S_{0,R})} + K(\varepsilon) |u_n - u_m|_{L^p(S_{0,R})} \, .$$

It follows from (1) and (2), that the sequence $\{\mathsf{B}^p_{0,R} u_n\}$ is convergent in L^p, and the operator $\mathsf{B}^p_{0,R}$ is $W_k{}^p$-compact.

3. Formal differential operators.

For simplicity we introduce the notations

$$\Sigma_1 = \sum_{|\alpha|=m} \sum_{|\beta|=m} \; ; \quad \Sigma_2 = \sum_{|\alpha|=m} \sum_{|\beta|\leqq m-1} + \sum_{|\alpha|\leqq m-1} \sum_{|\beta|=m} + \sum_{|\alpha|\leqq m-1} \sum_{|\beta|\leqq m-1} \; ;$$
$$\Sigma_{12} = \Sigma_1 + \Sigma_2 = \sum_{|\alpha|\leqq m} \sum_{|\beta|\leqq m} \, .$$

Let P be a polynomial of order $2m$ in n complex variables,

$$P(\xi) = \sum_{12} \xi^\alpha a_{\alpha\beta} \xi^\beta ,$$

where the $a_{\alpha\beta}$ are constants satisfying

(i) $a_{\beta\alpha} = \bar{a}_{\alpha\beta}$,

(ii) $\sum_1 a_{\alpha\beta} \xi^\beta \bar{\xi}^\alpha \geqq K |\xi|^{2m}$ for $\xi \in C_n$, $K > 0$.

We set

$$\mathscr{R}(P) = \{P(\xi) \mid \xi \in R_n\} ,$$
$$\complement \mathscr{R}(P) = \{\lambda \in C_1 \mid \lambda \notin \mathscr{R}(P)\}$$

The formal differential operator l is obtained by substitution of D_j for ξ_j in P, that is,

$$l = \sum_{12} D^\alpha a_{\alpha\beta} D^\beta .$$

In the rest of this section we consider *operators with variable coefficients.*

The formal differential operators r, q and v are defined by

$$\begin{aligned} r &= \sum_1 D^\alpha (a_{\alpha\beta} + b_{\alpha\beta}(x)) D^\beta + \sum_2 D^\alpha a_{\alpha\beta} D^\beta, \\ q &= \sum_2 D^\alpha b_{\alpha\beta}(x) D^\beta, \\ v &= r + q . \end{aligned}$$

Here $b_{\alpha\beta}(x)$ are functions in L^1_{loc} satisfying the following conditions:

I. $b_{\beta\alpha}(x) = \overline{b_{\alpha\beta}}(x)$.

II. $\sum_1 (a_{\alpha\beta} + b_{\alpha\beta}(x)) \xi^\beta \bar{\xi}^\alpha \geqq K |\xi|^{2m}$ for $\xi \in C_n$, $K > 0$,

that is, v is uniformly elliptic.

III. 1) $b_{\alpha\beta} \in L^\infty$, $b_{\alpha\beta} \in F^1$ for $|\alpha| = |\beta| = m$,

2) $b_{\alpha\beta} \in I^1_{m-|\beta|}$, $b_{\alpha\beta} \in F^1$ for $|\alpha| = m$, $|\beta| < m$,

3) $b_{\alpha\beta} = b'_{\alpha\beta} b''_{\alpha\beta}$, where

$b'_{\alpha\beta} \in N^2_{m-|\alpha|}$, $b''_{\alpha\beta} \in N^2_{m-|\beta|}$, for $|\alpha|, |\beta| < m$.

(Note that $I^2_{m-k} = E^2$ and $N^2_{m-k} = F^2$ for $k < m - \frac{1}{2}n$.)

4. Differential quadratic forms in L^2.

4.1. Definition. We consider the following quadratic forms with domain $C_0{}^m$ corresponding to the formal differential operators defined in Section 3:

$$L'[u] = \int_{R_n} \sum_{12} a_{\alpha\beta} D^\beta u D^\alpha \bar{u} \, dx ,$$

$$R'[u] = L'[u] + \int_{R_n} \Sigma_1 b_{\alpha\beta}(x) D^\beta u D^\alpha \bar{u}\, dx\,,$$

$$Q'[u] = \int_{R_n} \Sigma_2 b_{\alpha\beta}(x) D^\beta u D^\alpha \bar{u}\, dx\,,$$

$$V'[u] = R'[u] + Q'[u].$$

4.2. Lemma. *There exist constants K_1, K_2, $K_3 > 0$, such that*

(i) $K_1 |u|^2_{W_m} \leqq L'[u] + K_2 |u|^2 \leqq K_3 |u|^2_{W_m}$,
(ii) $K_1 |u|^2_{W_m} \leqq R'[u] + K_2 |u|^2 \leqq K_3 |u|^2_{W_m}$.

Proof. The left inequalities follow easily from (ii) and II of Section 3. The right inequalities are trivial, since, by III 1) of Section 3, $b_{\alpha\beta} \in L^\infty$.

4.3. Lemma. *We can assume that L' and R' are positive definite and that* 4.2 (i) *and* (ii) *hold with $K_2 = 0$ by adding, if necessary, a term Ku to l and r. The forms L' and R' are closable. Denote by L the closure of L' and by R the closure of R'. Then*

$$D[L] = D[R] = W_m\,,$$

and for $u \in W_m$,

(i) $K_1 |u|^2_{W_m} \leqq L[u] \leqq K_3 |u|^2_{W_m}$,
(ii) $K_1 |u|^2_{W_m} \leqq R[u] \leqq K_3 |u|^2_{W_m}$.

L and R are given explicitly by the same expression as L' and R' for all $u \in W_m$.

Proof. This follows immediately from Lemma 4.2.

4.4. Lemma. 1) *For every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$, such that for $u \in C_0{}^m$*

$$|Q'[u]| \leqq \varepsilon |u|^2_{W_m} + K(\varepsilon)|u|^2\,.$$

2) *There exist constants $K_1, K_2, K_3 > 0$, such that for $u \in C_0{}^m$*

$$K_1 |u|^2_{W_m} \leqq V'[u] + K_2 |u|^2 \leqq K_3 |u|^2_{W_m}\,.$$

Proof. We have the following estimate of the terms in $Q'[u]$ for $u \in C_0{}^m$:

(1)

$$\left| \int_{R_n} b_{\alpha\beta}(x)\, D^\beta u\, D^\alpha \bar{u}\, dx \right| \leqq \frac{c}{2} \int_{R_n} |b'_{\alpha\beta}(x)|^2\, |D^\alpha u|^2\, dx + \frac{1}{2c} \int_{R_n} |b''_{\alpha\beta}(x)|^2\, |D^\beta u|^2\, dx$$
$$= T_{\alpha\beta, c}[u]\,.$$

It follows from the conditions III 2) and 3) of section 3 by Lemma 2.4 that for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that

$$(2) \qquad T_{\alpha\beta, c}[u] < \varepsilon |u|^2_{W_m} + K(\varepsilon)|u|^2 .$$

When $|\alpha| = m$ ($|\beta| = m$), we set $b'_{\alpha\beta} = 1$ and $b''_{\alpha\beta} = b_{\alpha\beta}$ ($b'_{\alpha\beta} = b_{\alpha\beta}$ and $b''_{\alpha\beta} = 1$) and apply Lemma 2.4 after suitable choice of c. By addition of the inequalities (2) we obtain 1).

2) Since $V'[u] = R'[u] + Q'[u]$, the assertion follows from Lemma 4.2 (ii) and 4.4, 1).

4.5. Lemma. *We can assume, by adding a term Ku to l, if necessary, that V' is positive definite and that* 4.4 2) *holds with $K_2 = 0$. The form V' is closable. Denote by V the closure of V'. Then*

$$D[V] = W_m ,$$

and for $u \in D[V]$,

$$K_1 |u|^2_{W_m} \leqq V[u] \leqq K_3 |u|^2_{W_m} .$$

Proof. This follows immediately from Lemma 4.4.

4.6. Definition. We use the notations $T_{\alpha\beta, c}[u]$ introduced in the proof of Lemma 4.4, 1), setting $T_{\alpha\beta} = T_{\alpha\beta, 1}$. Then the form G' is defined for $u \in C_0{}^m$ by

$$G'[u] = \Sigma_{12} T_{\alpha\beta}[u] .$$

4.7. Lemma. *For $u \in C_0{}^m$,*

$$|V'[u] - L'[u]| \leqq G'[u] \leqq K |u|^2_{W_m} .$$

Proof. The left inequality is obtained by addition of the inequalities (1) for $c = 1$ of the proof of Lemma 4.4, 1).

From the condition III 1) of Section 3 it follows for $u \in C_0{}^m$ that

$$T_{\alpha\beta}[u] \leqq K |u|^2_{W_m} \quad \text{for } |\alpha| = |\beta| = m .$$

By addition of this and the inequalities

$$T_{\alpha\beta}[u] \leqq K |u|^2_{W_m} \quad \text{for } (|\alpha|, |\beta|) \neq (m, m),$$

established in (2) of the proof of Lemma 4.4, 1), we arrive at the right inequality of the lemma.

4.8. Lemma. *The form G' is densely defined and bounded on $D[L]$. Let G be the closure of G' with respect to the L-metric. Then for $u \in D[L]$,*

$$(1) \qquad |V[u] - L[u]| \leqq G[u] \leqq K L[u] .$$

PROOF. By Lemma 4.2 (i) the W_m- and L-metrics are equivalent, and it follows by Lemma 4.7 that G' is densely defined and bounded on $D[L]$. Then the form G can be defined as the closure of G' with respect to the L-metric, and the result follows from Lemma 4.7.

4.9. REMARK. G is a positive quadratic form. It is not necessarily positive definite, so that the pseudo-metric defined by G need not be a metric, but that is unessential for what follows.

5. Differential operators in L^2.

Operators with constant coefficients.

The self-adjoint, positive definite operator L corresponding to the form L is the unique self-adjoint operator associated with l. Furthermore, L is the closure of the operator with domain C_0^{2m} defined by l. The domain of L coincides with W_{2m}, and the L-norm and the W_{2m}-norm are equivalent, so that L^{-1} is a bounded operator from L^2 onto W_{2m}.

5.1. LEMMA. *The essential spectrum and the resolvent set of* L *are*

$$\sigma_e(\mathsf{L}) = \mathscr{R}(P), \qquad \varrho(\mathsf{L}) = \complement\mathscr{R}(P) ,$$

respectively, where $\mathscr{R}(P)$ *and* $\complement\mathscr{R}(P)$ *are as defined in Section* 3.

PROOF. L is unitarily equivalent via the Fourier–Plancherel transform to the maximal operator P in L^2 corresponding to multiplication by P. It is clear that $\sigma_e(\mathsf{P}) = \mathscr{R}(P)$, and $\varrho(\mathsf{P}) = \complement\mathscr{R}(P)$, and the lemma is proved.

Operators with variable coefficients.

The self-adjoint, positive definite operator V corresponding to the form V can be defined by

$$D(\mathsf{V}) = \{u \in D[V] \mid \exists v \in L^2,\ V[u,w] = (v,w) \text{ for } w \in D[V]\}$$

and

$$\mathsf{V}u = v \quad \text{for} \quad u \in D(\mathsf{V}) .$$

Without further assumptions on the functions $b_{\alpha\beta}(x)$ a simpler definition of V is not available; in particular, it is not known whether V is the only self-adjoint operator associated with v.

If $b_{\alpha\beta} \in W_{|\alpha|,\,\mathrm{loo}}$, then $v(u) \in L^2$ for $u \in C_0^{2m}$, and V is the Friedrichs extension of the operator V_0 with domain C_0^{2m} defined by v. Even if the functions $b_{\alpha\beta}$ have stronger isolated singularities, the set

$$D_0 = \{u \in C_0^{2m} \mid v(u) \in L^2\}$$

can still be dense in W_m; in that case V is the Friedrichs extension of the operator with domain D_0 defined by v. By Lemma 2.4 (a) it is easy to prove, that the quadratic form V cannot be extended to a larger domain lying in $W^2_{1,\,\mathrm{loc}}$. Hence the operator V is the only self-adjoint operator defined by v, and such that the domain of the corresponding quadratic form is contained in $W^2_{1,\,\mathrm{loc}}$.

5.2. LEMMA. L^{-1} *is a compact operator from* L^2 *into* $D[G]$.

PROOF. If either $|\alpha|=m$ or $|\beta|=m$, we set $b_{\alpha\beta}=b'_{\alpha\beta}b''_{\alpha\beta}$, where

$$b'_{\alpha\beta} = \operatorname{sgn} b_{\alpha\beta}\,|b_{\alpha\beta}|^{\frac{1}{2}}, \qquad b''_{\alpha\beta} = |b_{\alpha\beta}|^{\frac{1}{2}}.$$

It is clear, that conditions III of Section 3 imply

$$b'_{\alpha\beta} \in N_{2m-|\alpha|},\quad b''_{\alpha\beta} \in N_{2m-|\beta|} \qquad \text{for}\quad 0 \leqq |\alpha|,\ |\beta| \leqq m\ .$$

From this it follows by Lemma 2.5, that the operators B_α and B_β as defined in 2.1 corresponding to the functions $b'_{\alpha\beta}$ and $b''_{\alpha\beta}$, respectively, are W_{2m}-compact. Hence the embedding of W_{2m} in $D[G]$ is compact, and since L^{-1} is bounded from L^2 onto W_{2m}, Lemma 5.2 is proved.

5.3. THEOREM. *Let* L *and* V *be the operators defined above. Then* $\mathsf{V}^{-1}-\mathsf{L}^{-1}$ *is a compact operator, and hence*

$$\sigma_c(\mathsf{V}) = \mathscr{R}(P)\ ,$$

while $\varrho(\mathsf{V}) = \complement\mathscr{R}(P)$ *except for an at most countable set of isolated eigenvalues of* V *of finite multiplicity.*

PROOF. We apply Theorem 1.1 with $\mathscr{H}=L^2$, $A=L$, $C=V$, $F=G$, $\mathsf{A}=\mathsf{L}$, $\mathsf{C}=\mathsf{V}$.

L and V are closed, positive definite quadratic forms with $D[L]=D[V]$ by Lemmas 4.3 and 4.5.

The condition 1.1 b) (i) is satisfied by Lemma 4.8, and condition 1.1 b) (ii) is satisfied by Lemma 5.2.

Then it follows from Lemma 5.1 and Theorem 1.1 that, $\mathsf{V}^{-1}-\mathsf{L}^{-1}$ is a compact operator, and $\sigma_e(\mathsf{V})=\mathscr{R}(P)$. Since $\sigma_c(\mathsf{V})$ is a closed subset of the half-line $\sigma_e(\mathsf{V})$, and V has at most countably many eigenvalues, it follows, that

$$\sigma_c(\mathsf{V}) = \sigma_e(\mathsf{V}) = \mathscr{R}(P)\ ,$$

and the theorem is proved.

For the sake of illustration we apply the preceding result to the important special case $v=-\Delta+q(x)$.

5.4. THEOREM. *Set*

$$v = -\Delta + q(x),$$

where $q(x)$ is a real-valued function in N_2^1.

Let V *be the operator defined above corresponding to v. Then*

$$\sigma_c(\mathsf{V}) = [0,\infty),$$

while $\varrho(\mathsf{V}) = \complement[0,\infty)$, except for an at most countable set of isolated eigenvalues of V *of finite multiplicity.*

If $D_0 = \{u \in C_0^2 \mid qu \in L^2\}$ is dense in W_1, in particular, if $q \in L^2_{\mathrm{loc}}$, then V *is the Friedrichs extension of the operator with domain D_0 defined by v.*

Moreover, V is the only self-adjoint operator in L^2 defined by v, such that the domain of the corresponding form is contained in $W^2_{1,\mathrm{loc}}$. If $q(x)$ is continuous except for $x \in S$, where S is a manifold of dimension $\leqq n-1$ (set of measure 0) satisfying certain weak conditions, as it is generally the case for physically interesting potentials, then it follows from Lemma 2.4 (a), that the domain of the quadratic form is contained in $W^2_{1,\mathrm{loc}}$, and V is the closure of v on D_0.

5.5. REMARK. The statement of Theorem 5.4 concerning the essential spectrum of V has previously been shown to follow from various conditions on $q(x)$ as follows:

1) [1], q is a bounded function in L^2, $n=3$.
2) [4, Theorem 3.4],

$$\text{(i)} \quad \sup_{x\in R_n} \int_{S_{x,a}} |q(y)|\,|x-y|^{1-n}\,dy \to 0 \quad \text{as} \quad a \to 0,$$

$$\text{(ii)} \quad \int_{S_x} |q(y)|\,dy \to 0 \quad \text{as} \quad |x| \to \infty.$$

3) [3, Theorem 3.9 and Lemma 2.6 (b)],

$$\text{(i)} \quad \int_{S_{0,R}} |q(x)|^{\frac{1}{2}n+\alpha(R)}\,dx < K(R), \quad \alpha(R) > 0, \ 0 < R < \infty,$$

(ii) there exists $a > 0$, such that

$$\int_{S_x} |q(y)|^2\,|x-y|^{4-n-a}\,dy < K \quad \text{for} \quad x \in R_n,$$

$$\text{(iii)} \quad \int_{S_x} |q(y)|^2\,dy \to 0 \quad \text{as} \quad |x| \to \infty.$$

4) [10, Theorem 2.2],

(i) there exists $a > 0$, such that

$$\int_{S_x} |q(y)|^2 \, |x-y|^{4-n-a} \, dy < K \quad \text{for} \quad x \in R_n ,$$

(ii) there exists $\mu > 0$, such that

$$\int_{S_x} |q(y)|^{\mu} \, dy \to 0 \quad \text{as} \quad |x| \to \infty .$$

5) [10, Theorem 3.1],

(a) $q(x) \geqq C$,
(b) $q(x) = q_1(x) \, q_2(x)$,

where $q_1(x)$ satisfies

(i) there exists $a > 0$, such that

$$\int_{S_x} |q_1(y)|^2 \, |x-y|^{4-n-a} \, dy < K \quad \text{for} \quad x \in R_n ,$$

(ii) $\displaystyle\int_{S_x} |q_1(y)|^2 \, dy \to 0 \quad \text{as} \quad |x| \to \infty ,$

while $q_2(x)$ satisfies

(iii) $\displaystyle\int_{S_x} |q_2(y)|^2 \, |x-y|^{2-n} \, dy < K \quad \text{for} \quad x \in R_n .$

It is clear that 1) is contained in 3), and 3) is contained in 4). It follows by an application of Hölder's inequality that the conditions of 4) imply that $q \in N_2^1$. Obviously, the conditions of 2) imply that $q \in N_2^1$.

If $q \in N_2^1$, then $q(x)$ satisfies 5) (b) with $q_1(x) = |q(x)|^{\frac{1}{2}}$ and $q_2(x) = |q(x)|^{\frac{1}{2}}$ $\operatorname{sgn} q(x)$. Thus, for $q(x) > C$, the result of Theorem 5.4 is contained in [10, Theorem 3.1].

We conclude that all results up to now are contained in Theorem 5.4 and [10, Theorem 3.1], Rejto's result being the strongest when $q(x)$ is bounded from below and ours being the strongest when $q(x)$ is not bounded from below.

6. Ordinary differential operators.

6.1. THEOREM. *Let v be the formal differential operator defined by*

$$v = \sum_{k=0}^{m} D^k(a_k + b_k(x))D^k + \sum_{k=1}^{m} D^k(c_k + d_k(x))D^{k-1} + D^{k-1}(c_k + d_k(x))D^k ,$$

where $D = -id/dx$, a_k *and* c_k *are real constants, and* $b_k(x)$ *and* $d_k(x)$ *are real functions satisfying*

(i) $b_m(x) \in L^\infty$, *and* $a_m + b_m(x) \geqq K > 0$ *for* $x \in R_1$,

(ii) *the functions* $b_k(x)$ *are locally integrable and satisfy for* $0 \leqq k \leqq m$ *the condition*

$$\int_x^{x+1} |b_k(t)|\, dt \to 0 \quad as \quad |x| \to \infty ,$$

(iii) *the functions* $d_k(x)$ *are locally integrable and satisfy for* $0 \leqq k \leqq m$ *the condition*

$$\int_x^{x+1} |d_k(t)|\, dt \to 0 \quad as \quad |x| \to \infty ,$$

(iv)
$$\sup_{x \in R_1} \int_x^{x+1} |d_m(t)|^2 dt < \infty .$$

Let $\tilde{\mathsf{V}}$ *be any self-adjoint operator associated with* v, *and let* P *be the polynomial in one variable*

$$P(\xi) = \sum_{k=0}^{m} a_k \xi^{2k} + \sum_{k=1}^{m} c_k \xi^{2k-1} .$$

Then

$$\sigma_c(\tilde{\mathsf{V}}) = \mathscr{R}(P) ,$$

while $\complement\mathscr{R}(P)$ *is resolvent set of* $\tilde{\mathsf{V}}$, *except for an at most countable set of isolated eigenvalues of* $\tilde{\mathsf{V}}$ *of finite multiplicity.*

Proof. It suffices to consider the operator V defined in Section 5, since $\tilde{\mathsf{V}}$ and V are extensions of their restrictions to $D(\mathsf{V}) \cap D(\tilde{\mathsf{V}})$ by the same finite dimension and therefore have the same essential spectrum and index. Since the conditions 3, I–III take the simple form (i)–(iv) in the case $n=1$, Theorem 6.1 for $\tilde{\mathsf{V}} = \mathsf{V}$ is a special case of Theorem 5.3.

6.2. Theorem. *Let* z *be the formal differential operator defined by*

$$z = \sum_{k=0}^{m} D^k\big(a_k + b_k(t)\big)t^{2k} D^k + \sum_{k=1}^{m} D^k\big(c_k + d_k(t)\big)t^{2k-1} D^{k-1} + \\ + D^{k-1}\big(c_k + d_k(t)\big)t^{2k-1} D^k ,$$

where a_k *and* c_k *are real constants, and* $b_k(t)$ *and* $d_k(t)$ *are real functions on* $(0, \infty)$ *satisfying*

(i) $b_m(t) \in L^\infty$, *and* $a_m + b_m(t) \geqq K > 0$ *for* $0 < t < \infty$,

(ii) *the functions* $b_k(t)$ *are locally integrable and satisfy for some* $K > 1$, $0 \leqq k \leqq m$, *the condition*

$$\int_{s}^{Ks} |b_k(t)|\, t^{-1}\, dt \to 0 \quad as \quad s \to \begin{cases} 0 \\ \infty \end{cases},$$

(iii) *the functions $d_k(t)$ are locally integrable and satisfy for some $K > 1$, $0 \leqq k \leqq m$, the condition*

$$\int_{s}^{Ks} |d_k(t)|\, t^{-1}\, dt \to 0 \quad as \ s \to \begin{cases} 0 \\ \infty \end{cases},$$

(iv) *there exists $K > 1$, such that*

$$\sup_{0<s<\infty} \int_{s}^{Ks} |d_m(t)|^2 t^{-1} dt \; < \; \infty \, .$$

Let $\tilde{\mathsf{Z}}$ be any self-adjoint operator associated with z. Let P be the polynomial in one variable

$$P(\xi) = \sum_{k=0}^{m} a_k \prod_{j=0}^{k-1} |\xi - (\tfrac{1}{2}+j)i|^2 + \sum_{k=1}^{m} 2c_k \operatorname{Re}\left\{\prod_{j=0}^{k-1} (\xi - (\tfrac{1}{2}+j)i) \prod_{j=0}^{k-2} (\xi + (\tfrac{1}{2}+j)i)\right\}.$$

Then

$$\sigma_c(\tilde{\mathsf{Z}}) = \mathscr{R}(P)\, ,$$

while $\complement \mathscr{R}(P)$ is resolvent set of $\tilde{\mathsf{Z}}$, except for an at most countable set of isolated eigenvalues of $\tilde{\mathsf{Z}}$ of finite multiplicity.

PROOF. Let τ be the unitary operator from $L^2(0,\infty)$ onto $L^2(-\infty,\infty)$ defined for $f \in L^2(0,\infty)$ by

$$\tau f(x) = e^{\frac{1}{2}x} f(e^x), \qquad -\infty < x < \infty \, .$$

Let the formal differential operators l_k, $\overline{l_k}$ and v be defined by

$$l_k = \prod_{j=0}^{k-1} \left(D + (\tfrac{1}{2}+j)i\right),$$

$$\overline{l_k} = \prod_{j=0}^{k-1} \left(D - (\tfrac{1}{2}+j)i\right),$$

$$v = \sum_{k=0}^{m} \overline{l_k}\left(a_k + b_k(e^x)\right) l_k + \sum_{k=1}^{m} \overline{l_k}\left(c_k + d_k(e^x)\right) l_{k-1} + \overline{l_{k-1}}\left(c_k + d_k(e^x)\right) l_k \, .$$

Let V be the operator associated with v as defined in Section 5. Then it is easy to check that the operator

$$\mathsf{Z} = \tau^{-1} \mathsf{V} \tau$$

corresponds to z (cf. [2]).

It is clear that the self-adjoint differential expression v can be written in the form

$$v = \sum_{k=0}^{m} D^k(\alpha_k+\beta_k(x))D^k + \sum_{k=1}^{m} D^k(\gamma_k+\delta_k(x))D^{k-1} + D^{k-1}(\gamma_k+\delta_k(x))D^k ,$$

where the α_k and γ_k are real constants with $\alpha_m=a_m$, and the $\beta_k(x)$ and $\delta_k(x)$ are linear combinations of the functions $b_k(e^x)$ and $d_k(e^x)$ with $\beta_m(x)=b_m(e^x)$ and $\delta_m(x)=d_m(e^x)$. Obviously, 6.2 (i) implies 6.1 (i). Also, it easily follows from 6.2 (ii) that the functions $b_k(e^x)$, and hence the functions $\beta_k(x)$ satisfy 6.1 (ii). Similarly, it follows from 6.2 (iii) and (iv) that the functions $d_k(e^x)$ and hence the functions $\delta_k(x)$ satisfy 6.1 (iii) and (iv). Hence Theorem 6.1 applies to V, and since Z is unitarily equivalent to V, and P is the polynomial associated with the constant coefficient part of v, Theorem 6.2 is proved for $\tilde{\mathsf{Z}}=\mathsf{Z}$ and consequently for any $\tilde{\mathsf{Z}}$ (cf. 6.1).

7. Operators in L^p.

7.1. Definition. Let $P(\xi)$ be a polynomial with complex coefficients of degree $k \geqq 1$ defined for $\xi \in R_n$, $n \geqq 1$, and let $\mathscr{R}(P)$ and $\complement\mathscr{R}(P)$ denote the range of $P(\xi)$ and its complement, respectively. We assume, that $P(\xi)$ is uniformly elliptic, i.e. that its principal part $Q(\xi)$ satisfies the inequality

$$|Q(\xi)| \geqq K|\xi|^k, \qquad K>0,\ \xi \in R_n .$$

The complex valued functions $b_\alpha(x)$ are supposed to belong to $N^p_{k-|\alpha|}$ for $|\alpha|<k$, cf. Definition 2.2.

The differential expression m is defined by

$$m = P(D_1,\ldots,D_n) + \sum_{|\alpha|<k} b_\alpha(x)D^\alpha .$$

The operator in L^p with domain C_0^∞ defined by m is denoted by M'.

The singular spectrum $\sigma_s(\mathsf{A})$ of a closed, densely defined operator A in a Banach space $\mathscr{B}$ is defined as the set of complex numbers λ, such that either the range of $\mathsf{A}-\lambda$ is not closed or

$$\dim \mathscr{N}(\mathsf{A}-\lambda) = \dim \mathscr{N}((\mathsf{A}-\lambda)^*) = \infty .$$

7.2. Theorem. *The operator* M' *has a closure* M, *which is the only closed extension of* M' *with domain contained in* $W^p_{k,\mathrm{loc}}$. *The singular spectrum* $\sigma_s(\mathsf{M})$ *is given by*

$$\sigma_s(\mathsf{M}) = \mathscr{R}(P) ,$$

and the resolvent set of M *equals* $\complement\mathscr{R}(P)$ *except for at most a sequence of isolated points* λ_i, *such that*

$$0 < \dim \mathscr{N}(\mathsf{M}-\lambda) = \dim \mathscr{N}((\mathsf{M}-\lambda)^*) < \infty .$$

Proof. It is well known that the operator L' defined by $P(D_1,\ldots,D_n)$ with domain C_0^∞ has a closure L with domain $W_k{}^p$, and L is the only closed, densely defined operator in L^p associated with $P(D_1,\ldots,D_n)$. Also the L-norm is equivalent to the $W_k{}^p$-norm. It was proved in [3, Theorem 3.5] that

$$\sigma_s(\mathsf{L}) = \mathscr{R}(P) \qquad \text{and} \qquad \varrho(\mathsf{L}) = \complement\mathscr{R}(P) .$$

By Lemma 2.4 (a) the M'-norm and the L'-norm are equivalent on C_0^∞. It follows, that M' has a closure M with domain $D(\mathsf{M}) = D(\mathsf{L}) = W_k{}^p$. A further application of Lemma 2.4 (a) shows, that

$$\{u \in L^p \cap W^p_{k,\mathrm{loc}} \mid mu \in L^p\} \subseteqq W_k{}^p ,$$

so that M is unique as a closed extension of M' with domain contained in $W^p_{k,\mathrm{loc}}$.

Since the L-norm is equivalent to the $W_k{}^p$-norm, it follows by Lemma 2.4 (b), since $b(x) \in N^p_{k-|\alpha|}$, that $\mathsf{L}-\mathsf{M}$ is L-compact. Then the rest of the theorem follows by application of a perturbation result of Gokhberg and Krein (cf. [7] and [3, Theorem 1.4]).

Added in proof: Schechter [11] has obtained results which essentially generalize the results of [10] and the present paper for $p=2$.

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